Power series representations for complex bosonic effective actions. I. A small field renormalization group step

Tadeusz Balaban,[1,a)] Joel Feldman,[2,b)] Horst Knörrer,[3,c)] and Eugene Trubowitz[3,d)]
[1]*Department of Mathematics, Rutgers, The State University of New Jersey, 110 Frelinghuysen Rd., Piscataway, New Jersey 08854-8019, USA*
[2]*Department of Mathematics, University of British Columbia, Vancouver, British Columbia V6T 1Z2, Canada*
[3]*Mathematik, ETH-Zentrum, CH-8092 Zürich, Switzerland*

(Received 27 October 2009; accepted 27 January 2010; published online 21 May 2010)

We develop a power series representation and estimates for an effective action of the form: $\ln[\int e^{f(\alpha_1,\ldots,\alpha_s;z^*,z)}d\mu(z^*,z)/\int e^{f(0,\ldots,0;z^*,z)}d\mu(z^*,z)]$. Here, $f(\alpha_1,\ldots,\alpha_s;z_*,z)$ is an analytic function of the complex fields $\alpha_1(\mathbf{x}),\ldots,\alpha_s(\mathbf{x})$, $z_*(\mathbf{x})$, $z(\mathbf{x})$ indexed by $\mathbf{x}$ in a finite set X, and $d\mu(z^*,z)$ is a compactly supported product measure. Such effective actions occur in the small field region for a renormalization group analysis. Using methods similar to a polymer expansion, we estimate the power series of the effective action. © *2010 American Institute of Physics.* [doi:10.1063/1.3329425]

I. INTRODUCTION

Let X be a finite set and $\mathbb{C}^X$ the space of complex valued bosonic fields (i.e., functions) on X. Furthermore, let $d\mu(z^*,z)$ be a product measure on $\mathbb{C}^X$ of the form

$$d\mu(z^*,z)=\prod_{\mathbf{x}\in X} d\mu_0(z^*(\mathbf{x}),z(\mathbf{x})), \tag{1.1}$$

where $d\mu_0(\zeta^*,\zeta)$ is a normalized measure on $\mathbb{C}$ that is supported in $|\zeta|\le r$ for some constant r. That is,

$$\int |\zeta|^k d\mu_0(\zeta^*,\zeta)\le r^k \quad \text{for all } k\in\mathbb{N}. \tag{1.2}$$

For an analytic function $f(\alpha_1,\ldots,\alpha_s;z_*,z)$ of fields $\alpha_1,\ldots,\alpha_s,z_*,z$, we develop criteria under which

$$g(\alpha_1,\ldots,\alpha_s)=\ln\frac{\int e^{f(\alpha_1,\ldots,\alpha_s;z^*,z)}d\mu(z^*,z)}{\int e^{f(0,\ldots,0;z^*,z)}d\mu(z^*,z)} \tag{1.3}$$

exists. This is done using norms for f which are defined in terms of the expansion of f in powers of the fields. The construction also gives estimates on the corresponding norms of g.

In Ref 2, we described the analogous construction for real valued fields, which is technically simpler. As pointed out there, expression like (1.3) occur during the course of each iteration step in a Wilson style renormalization group flow. Here (z_*,z) are the fluctuation fields integrated out

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[a)]Electronic mail: tbalaban@math.rutgers.edu.
[b)]Electronic mail: feldman@math.ubc.ca. URL: http://www.math.ubc.ca/~feldman/.
[c)]Electronic mail: knoerrer@math.ethz.ch. URL: http://www.math.ethz.ch/~knoerrer/.
[d)]Electronic mail: trub@math.ethz.ch.

at the current scale, while $\alpha_1,\ldots,\alpha_s$ can be fields that are to be integrated out in future scales or can be source fields that are used to generate and control correlation functions and are never integrated out. We shall use the methods developed here to control the ultraviolet limit in the coherent state functional integral representation for many boson systems described in Ref. 1 (Theorems 2.2 and 3.7).

To give an example of the norms used to control, (1.3), assume for simplicity that $s=1$ and write $\alpha_1=\alpha$. Then, f has a power series expansion

$$f(\alpha;z^*,z)=\sum_{n_1,n_2,n_3\geq 0}\sum_{\substack{\vec{\mathbf{x}}\in X^{n_1}\\ \vec{\mathbf{y}}_*\in X^{n_2}\\ \vec{\mathbf{y}}\in X^{n_3}}} a(\vec{\mathbf{x}};\vec{\mathbf{y}}_*,\vec{\mathbf{y}})\alpha(\mathbf{x}_1)\cdots\alpha(\mathbf{x}_{n_1})z_*(\mathbf{y}_{*1})\cdots z_*(\mathbf{y}_{*n_2})z(\mathbf{y}_1)\cdots z(\mathbf{y}_{n_3})$$

with coefficients $a(\vec{\mathbf{x}};\vec{\mathbf{y}}_*,\vec{\mathbf{y}})$ that are symmetric under permutations of the components of the vectors $\vec{\mathbf{x}}$, $\vec{\mathbf{y}}_*$, and $\vec{\mathbf{y}}$, respectively.

Assume that X is a metric space with metric d. Fix a parameter κ. An example of a norm that we use is

$$\|f\|=|f(0;0,0)|+\sum_{n_1+n_2+n_3\geq 1}\sup_{\mathbf{x}\in X}\max_{1\leq\ell\leq n_1+n_2+n_3}\sum_{\substack{(\vec{\mathbf{x}},\vec{\mathbf{y}}_*,\vec{\mathbf{y}})\in X^{n_1}\times X^{n_2}\times X^{n_3}\\ (\vec{\mathbf{x}},\vec{\mathbf{y}}_*,\vec{\mathbf{y}})_\ell=\mathbf{x}}} w(\vec{\mathbf{x}};\vec{\mathbf{y}}_*,\vec{\mathbf{y}})|a(\vec{\mathbf{x}};\vec{\mathbf{y}}_*,\vec{\mathbf{y}})|,$$

where the weight system w is defined as

$$w(\vec{\mathbf{x}};\vec{\mathbf{y}}_*,\vec{\mathbf{y}})=\kappa^{n_1}(4r)^{n_2+n_3}e^{\tau_d(\vec{\mathbf{x}},\vec{\mathbf{y}}_*,\vec{\mathbf{y}})}\quad\text{for } (\vec{\mathbf{x}},\vec{\mathbf{y}}_*,\vec{\mathbf{y}})\in X^{n_1}\times X^{n_2}\times X^{n_3} \tag{1.4}$$

and $\tau_d(\vec{\mathbf{x}},\vec{\mathbf{y}}_*,\vec{\mathbf{y}})$ is the minimal length of a tree whose set of vertices contains the set $\{\mathbf{x}_1,\ldots,\mathbf{x}_{n_1},\mathbf{y}_{*1},\ldots,\mathbf{y}_{*n_2},\mathbf{y}_1,\ldots,\mathbf{y}_{n_3}\}$. For this norm, our main result, Theorem 3.4, states that

$$g(\alpha)=\ln\frac{\int e^{f(\alpha;z^*,z)}d\mu(z^*,z)}{\int e^{f(0;z^*,z)}d\mu(z^*,z)}$$

exists provided that $\|f\|<\frac{1}{16}$, and that, in this case,

$$\|g\|\leq\frac{\|f\|}{1-16\|f\|}. \tag{1.5}$$

Theorem 3.4 applies to more general norms than those described above (see Definitions 2.3, 2.6, and 3.1.) To reflect the geometry and scale structure of a "large field/small field" decomposition of X, one can replace the constant κ by a "weight factor" $\kappa:X\to(0,\infty]$ and the factor κ^{n_1} in (1.4) by $\kappa(\mathbf{x}_1)\cdots\kappa(\mathbf{x}_{n_1})$ [see Example 2.4, part (i)]. Another variation in the norms comes from the fact that one is often led to bound source fields by their sup norm rather than by their L^1 norm (see Definition 2.6).

If the measure $d\mu_0(\zeta^*,\zeta)$ is rotation invariant and there are no nontrivial monomials of the form $a(\vec{\mathbf{x}};\vec{\mathbf{y}},\vec{\mathbf{y}})\alpha(\mathbf{x}_1)\cdots\alpha(\mathbf{x}_{n_1})(z_*(\mathbf{y}_1)z(\mathbf{y}_1))\cdots(z_*(\mathbf{y}_{n_2})z(\mathbf{y}_{n_2}))$ in the power series of f, then (1.5) can be improved to a quadratic bound (see Corollary 3.5). This situation occurs in our analysis of many boson systems. There we also need information on how the g of (1.3) varies with f. This is provided by Corollary 3.6.

In the analysis of an infrared limit, one often has an increasing sequence

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$$

of subsets of a lattice in $\mathbb{R}^d$ that exhausts the lattice. For each index n, there will be a function f_n that is (close to) the restriction of a function on the whole lattice to X_n. Theorem 3.4 can possibly be used to construct

$$g_n = \log \frac{\int e^{f_n(\alpha_1,\ldots,\alpha_s;z^*,z)} d\mu(z^*,z)}{\int e^{f_n(0,\ldots,0;z^*,z)} d\mu(z^*,z)}.$$

To take the limit $n \to \infty$ of the functions g_n, we need to compare g_n and the "restriction" of g_m to X_n when $n < m$. In the context of this paper, think of $X = X_m$ and of X_n as a subset X' of X.

The problem described above amounts to the following: with the notation of (1.3), set

$$f'(\alpha_1, \ldots, \alpha_s; z_*, z) = f(\alpha_1, \ldots, \alpha_s; z_*, z)\Big|_{\substack{\alpha_k(\mathbf{x})=z^*(\mathbf{x})=z(\mathbf{x})=0 \\ \text{for } \mathbf{x} \in X \setminus X', k=1,\ldots,s}}.$$

Then, we want to compare

$$g = \log \frac{\int e^{f(\alpha_1,\ldots,\alpha_s;z^*,z)} d\mu(z^*,z)}{\int e^{f(0,\ldots,0;z^*,z)} d\mu(z^*,z)} \quad \text{and} \quad g' = \log \frac{\int e^{f'(\alpha_1,\ldots,\alpha_s;z^*,z)} d\mu(z^*,z)}{\int e^{f'(0,\ldots,0;z^*,z)} d\mu(z^*,z)}.$$

Actually, in the multiscale analysis even more complicated comparisons arise.

To facilitate such comparisons, we introduce an auxiliary real valued "history field" $\mathfrak{h}$ on X. It will only be evaluated with $\mathfrak{h}(\mathbf{x}) \in \{0,1\}$ so that $\mathfrak{h}^2 = \mathfrak{h}$. Set

$$\tilde{f}(\alpha_1, \ldots, \alpha_s; z_*, z; \mathfrak{h}) = f(\alpha_1 \mathfrak{h}, \ldots, \alpha_s \mathfrak{h}; z_* \mathfrak{h}, z\mathfrak{h}).$$

That is, we replace $\alpha_i(\mathbf{x})$ by $\alpha_i(\mathbf{x})\mathfrak{h}(\mathbf{x})$ everywhere in the power series expansion for f, and the same for $z^*(\mathbf{x})$ and $z(\mathbf{x})$. Clearly,

$$f = \tilde{f}\Big|_{\mathfrak{h}(\mathbf{x})=1 \text{ for all } \mathbf{x} \in X}, \quad f' = \tilde{f}\Big|_{\substack{\mathfrak{h}(\mathbf{x})=1 \text{ for all } \mathbf{x} \in X' \\ \mathfrak{h}(\mathbf{x})=0 \text{ for all } \mathbf{x} \in X \setminus X'}}.$$

Theorem 3.4 can be applied to construct

$$\tilde{g} = \log \frac{\int e^{\tilde{f}(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})} d\mu(z^*,z)}{\int e^{\tilde{f}(0,\ldots,0;z^*,z;\mathfrak{h})} d\mu(z^*,z)}.$$

Clearly,

$$g = \tilde{g}\Big|_{\mathfrak{h}(\mathbf{x})=1 \text{ for all } \mathbf{x} \in X}, \quad g' = \tilde{g}\Big|_{\substack{\mathfrak{h}(\mathbf{x})=1 \text{ for all } \mathbf{x} \in X' \\ \mathfrak{h}(\mathbf{x})=0 \text{ for all } \mathbf{x} \in X \setminus X'}}.$$

The measures that typically arise in renormalization group steps are rarely product measures. To apply the results of this paper, one must first perform a change in variables so as to diagonalize the (essential part) of the covariance of the measure. Linear changes in variables, as well as substitutions that typically occur in renormalization group steps, are controlled in Sec. IV and Appendix A.

II. NORMS

To get a general setup for the norms that we shall use, we need a number of definitions.

Definition 2.1: (n-tuples)

(i) Let $n \in \mathbb{Z}$ with $n \geqslant 0$ and $\vec{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in X^n$ be an ordered n-tuple of points of X. We denote by $n(\vec{\mathbf{x}}) = n$ the number of components of $\vec{\mathbf{x}}$. Set

$$\phi(\vec{\mathbf{x}}) = \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n).$$

If $n(\vec{\mathbf{x}})=0$, then $\phi(\vec{\mathbf{x}})=1$. The support of $\vec{\mathbf{x}}$ is defined to be

$$\text{supp}\,\vec{\mathbf{x}} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset X.$$

(ii) For each $s \in \mathbb{N}$, we denote

$$\mathbf{X}^{(s)} = \bigcup_{n_1,\ldots,n_s \geqslant 0} X^{n_1} \times \cdots \times X^{n_s}.$$

[We distinguish between $X^{n_1}\times\cdots\times X^{n_s}$ and $X^{n_1+\cdots+n_s}$. We use $X^{n_1}\times\cdots\times X^{n_s}$ as the set of possible arguments for $\psi_1(\vec{\mathbf{x}}_1)\cdots\psi_s(\vec{\mathbf{x}}_s)$, while $X^{n_1+\cdots+n_s}$ is the set of possible arguments for $\psi_1(\vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_s)$, where $\circ$ is the concatenation operator of part (iii).]

The support of $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}$ is

$$\text{supp}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) = \bigcup_{j=1}^{s} \text{supp}(\vec{\mathbf{x}}_j).$$

If $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1}) \in \mathbf{X}^{(s-1)}$, then $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},-)$ denotes the element of $\mathbf{X}^{(s)}$ having $n(\vec{\mathbf{x}}_s)=0$. In particular, $X^0=\{-\}$ and $\phi(-)=1$.

(iii) We define the concatenation of $\vec{\mathbf{x}}=(\mathbf{x}_1,\ldots,\mathbf{x}_n) \in X^n$ and $\vec{\mathbf{y}}=(\mathbf{y}_1,\ldots,\mathbf{y}_m) \in X^m$ to be

$$\vec{\mathbf{x}} \circ \vec{\mathbf{y}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_m) \in X^{n+m}.$$

For $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)$, $(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_s) \in \mathbf{X}^{(s)}$

$$(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) \circ (\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_s) = (\vec{\mathbf{x}}_1 \circ \vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{x}}_s \circ \vec{\mathbf{y}}_s)$$

Definition 2.2: (Coefficient systems)

(i) A coefficient system of length s is a function $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)$ that assigns a complex number to each $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}$. It is called symmetric if, for each $1 \leqslant j \leqslant s$, $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)$ is invariant under permutations of the components of $\vec{\mathbf{x}}_j$.

(ii) Let $f(\alpha_1,\ldots,\alpha_s)$ be a function, which is defined and analytic on a neighborhood of the origin in $\mathbb{C}^{s|X|}$. Then, f has a unique expansion of the form

$$f(\alpha_1, \ldots, \alpha_s) = \sum_{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}} a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s)\alpha_1(\vec{\mathbf{x}}_1) \cdots \alpha_s(\vec{\mathbf{x}}_s)$$

with $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)$ as a symmetric coefficient system. This coefficient system is called the symmetric coefficient system of f.

Definition 2.3: (Weight systems) A weight system of length s is a function that assigns a positive extended number $w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s) \in (0,\infty]$ to each $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}$ and satisfies the following conditions.

(a) For each $1 \leqslant j \leqslant s$, $w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)$ is invariant under permutations of the components of $\vec{\mathbf{x}}_j$.

(b)

$$w((\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) \circ (\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_s)) \leqslant w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s)w(\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_s)$$

for all $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)$, $(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_s) \in \mathbf{X}^{(s)}$ with $\text{supp}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s) \cap \text{supp}(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_s) \neq \emptyset$.

Example 2.4: (Weight systems)

(i) If $\kappa_1,\ldots,\kappa_s$ are functions from X to $(0, \infty]$ (called weight factors), then

$$w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) = \prod_{j=1}^{s} \prod_{\ell=1}^{n(\vec{\mathbf{x}}_j)} \kappa_j(\mathbf{x}_{j,\ell})$$

is a weight system of length s.

(ii) Let d be a metric on X. The length of a tree T with vertices in X is simply the sum of the lengths of all edges of T (where the length of an edge is the distance between its vertices). For a subset $S \subset X$, denote by $\tau_d(S)$ the length of the shortest tree in X whose set of vertices contains S. Then,

$$w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) = e^{\tau_d(\mathrm{supp}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s))}$$

is a weight system of length s.

(iii) Assume again that d is a metric on X. Let $\Omega \subset X$. The "maximum distance" of any subset S of X to Ω^c is $D(S,\Omega^c) = \sup_{\mathbf{x} \in S} d(\mathbf{x},\Omega^c)$. Then,

$$w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) = e^{D(\mathrm{supp}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s),\Omega^c)}$$

is a weight system of length s.

(iv) Let $\kappa \geqslant 1$, $N \in \mathbb{N}$, $1 \leqslant s_0 \leqslant s$, and $Y \subset X$. Denote by $\nu_Y(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s_0})$ the number of components of $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s_0}$ that are in Y. Then,

$$w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) = \kappa^{\max\{N - \nu_Y(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s_0}),0\}}$$

is a weight system of length s. The verification of condition (b) of Definition 2.3 follows from

$$\kappa^{\max\{N-m-n,0\}} \leqslant \kappa^{\max\{N-m,0\}} \kappa^{\max\{N-n,0\}} \quad \text{for all } m,n \geqslant 0.$$

(v) If $w_1(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s)$ and $w_2(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s)$ are two weight systems of length s, then

$$w_3(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) = w_1(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) w_2(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s)$$

is also a weight systems of length s.

Definition 2.5: Let d be a metric on X. Given weight factors $\kappa_j : X \to (0,\infty]$ for $j = 1, \ldots, s$, we call

$$w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s) = e^{\tau_d(\mathrm{supp}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s))} \prod_{j=1}^{s} \prod_{\ell=1}^{n(\vec{\mathbf{x}}_j)} \kappa_j(\mathbf{x}_{j,\ell})$$

the weight system with metric d that associates the weight factor κ_j with the field α_j. It follows from parts (i), (ii), and (v) of Example 2.4 that this is indeed a weight system.

Using weight systems as defined above, we define norms for functions that depend analytically on the complex fields $\alpha_1, \ldots, \alpha_s$ and the additional "history" field $\mathfrak{h}$.

Definition 2.6: (Norms) Let w be a weight system of length $s+1$. Let $0 \leqslant s' \leqslant s$. We think of the fields α_j with $1 \leqslant j \leqslant s'$ as being sources (that is, we differentiate with respect to these fields to generate correlation functions), the fields α_j with $s' < j \leqslant s$ as being internal fields (that is, they will be integrated out) and the field α_{s+1} as the history field.

(i) For any $n_1, \ldots, n_{s+1} \geqslant 0$ and any function $b(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s, \vec{\mathbf{x}}_{s+1})$ on $X^{n_1} \times \cdots \times X^{n_{s+1}}$, we define the norm $\|b\|_{n_1,\ldots,n_{s+1}}$ as follows:

- If there are external fields, that is, if $\Sigma_{j=1}^{s'} n_j \neq 0$, then

$$\|b\|_{n_1,\ldots,n_{s+1}} = \max_{\substack{\vec{\mathbf{x}}_\ell \in X^{n_\ell} \\ 1 \leqslant \ell \leqslant s'}} \sum_{\substack{\vec{\mathbf{x}}_\ell \in X^{n_\ell} \\ s' < \ell \leqslant s+1}} |b(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s, \vec{\mathbf{x}}_{s+1})|.$$

- If there are no external fields but there are internal fields, that is, if $\Sigma_{j=1}^{s'} n_j = 0$, but $\Sigma_{j=1}^{s} n_j \neq 0$, then

$$\|b\|_{n_1,\ldots,n_{s+1}} = \max_{\mathbf{x} \in X} \max_{\substack{s' < j \le s \\ n_j \neq 0}} \max_{1 \le i \le n_j} \sum_{\substack{\vec{\mathbf{x}}_\ell \in X^{n_\ell} \\ s' < \ell \le s+1 \\ (\vec{\mathbf{x}}_j)_i = \mathbf{x}}} |b(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s, \vec{\mathbf{x}}_{s+1})|.$$

  Here $(\vec{\mathbf{x}}_j)_i$ is the ith component of the n_j-tuple $\vec{\mathbf{x}}_j$.
- If there are no external or internal fields but there are history fields, that is, if $\Sigma_{j=1}^{s} n_j = 0$, but $n_{s+1} \neq 0$, then we take the pure L^1 norm

$$\|b\|_{n_1,\ldots,n_{s+1}} = \sum_{\vec{\mathbf{x}}_{s+1} \in X^{n_{s+1}}} |b(-, \ldots, -, \vec{\mathbf{x}}_{s+1})|.$$

- Finally, for the constant term, that is, if $\Sigma_{j=1}^{s+1} n_j = 0$,

$$\|b\|_{n_1,\ldots,n_{s+1}} = |b(-, \ldots, -)|.$$

(ii) We define the norm, with weight w, of a coefficient system a of length $s+1$ to be

$$|a|_w = \sum_{n_1,\ldots,n_{s+1} \ge 0} \|w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s, \vec{\mathbf{x}}_{s+1}) a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s, \vec{\mathbf{x}}_{s+1})\|_{n_1,\ldots,n_{s+1}}.$$

In some applications, it will be more convenient to turn this norm into a seminorm by ignoring the constant term $a(-)$. The results of this paper apply equally well to such seminorms.

(iii) Let $f(\alpha_1, \ldots, \alpha_s, \alpha_{s+1})$ be a function that is defined and analytic on a neighborhood of the origin in $\mathbb{C}^{(s+1)|X|}$. The norm $\|f\|_w$ of f with weight w is defined to be $|a|_w$, where a is the symmetric coefficient system of f. (This definition also applies when f depends only on a subset of the variables $\alpha_1, \ldots, \alpha_{s+1}$.)

Remark 2.7:

(i) Let a be a (not necessarily symmetric) coefficient system of length $s+1$ and

$$f(\alpha_1, \ldots, \alpha_{s+1}) = \sum_{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1}) \in \mathbf{X}^{(s+1)}} a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s+1}) \alpha_1(\vec{\mathbf{x}}_1) \cdots \alpha_s(\vec{\mathbf{x}}_{s+1}).$$

Then $\|f\|_w \le |a|_w$ for any weight system w. We call a a not necessarily symmetric coefficient system for f.

(ii) In Lemma B.1 we show how to convert a norm bound on f into a supremum bound.

III. THE MAIN THEOREM

In (1.3), we integrate out the last two internal fields z_*, z using the measure $d\mu(z^*, z)$ of (1.1). Recall that this measure is supported in $\|z\|_\infty \le r$. The weight systems that are adapted to this situation fulfil the Definition 3.1, below, with $\rho = 4r$.

Definition 3.1: A weight system of length $s+3$ "gives weight at least ρ to the last two internal fields" if

$$w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s; \vec{\mathbf{y}}_*, \vec{\mathbf{y}}; \vec{\mathbf{x}}) \ge \rho^{n(\vec{\mathbf{y}}_*) + n(\vec{\mathbf{y}})} w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s; -, -; \vec{\mathbf{x}})$$

for all $(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s, \vec{\mathbf{x}}) \in X^{(s+1)}$ and $(\vec{\mathbf{y}}_*, \vec{\mathbf{y}}) \in X^{(2)}$.

Example 3.2: Assume that d is a metric on X and $\kappa_j : X \to (0, \infty]$, $j = 1, \ldots, s$ are weight functions. The weight system with metric d that associates the weight factor κ_j with the field α_j, j

$=1,\ldots,s$, the constant weight factor ρ to the fields α_{s+1} and α_{s+2} and constant weight factor 1 to the history field $\mathfrak{h}$, fulfills Definition 3.1.

We fix, for the rest of this section, a weight system w of length $s+3$ that gives weight at least $4r$ to the last two internal fields. Furthermore, we fix the number $0\leqslant s'\leqslant s$ of source fields for Definition 2.6 of $\|\cdot\|_w$.

Remark 3.3:

(i) If h is an analytic function for which $h(0,\ldots,0;z_*,z;\mathfrak{h})$ is independent of z_* and z, then

$$\left\|\int h(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z)\right\|_w \leqslant \|h(\alpha_1,\ldots,\alpha_s;z_*,z;\mathfrak{h})\|_w.$$

(ii) Assume that the measure $d\mu(z^*,z)$ on $\mathbb{C}$ is rotation invariant. For each $j=1,2$, let $h_j(\alpha_1,\ldots,\alpha_s;z_*,z;\mathfrak{h})$ be an analytic function with $h_j(0,\ldots,0;z_*,z;\mathfrak{h})=0$. Further, assume that the symmetric coefficient system $a_j(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})$ of h_j obeys $a_j(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})=0$ whenever $\vec{\mathbf{y}}=\vec{\mathbf{y}}_*$. Then,

$$\int h_j(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z)=0 \quad \text{for } j=1,2$$

and

$$\left\|\int h_1(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})h_2(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z)\right\|_w \leqslant \|h_1\|_w\|h_2\|_w.$$

Proof:

(i) This follows from the observation that

$$\left|\int z(\vec{\mathbf{y}}_*)^* z(\vec{\mathbf{y}})d\mu(z^*,z)\right| \leqslant r^{n(\vec{\mathbf{y}}_*)+n(\vec{\mathbf{y}})}$$

for all $\vec{\mathbf{y}}_*,\vec{\mathbf{y}}\in X^{(2)}$.

(ii) Write

$$\begin{aligned}\tilde{h}(\alpha_1,\ldots,\alpha_s;\mathfrak{h}) &= \int h_1(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})h_2(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z)\\ &= \sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)\in\mathbf{X}^{(s)}\\ \vec{\mathbf{x}}\in\mathbf{X}^{(1)}}} \tilde{a}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}})\alpha_1(\vec{\mathbf{x}}_1)\cdots\alpha_s(\vec{\mathbf{x}}_s)\mathfrak{h}(\vec{\mathbf{x}})\end{aligned}$$

with, for each $\zeta\in X^{(s)}$ and $\vec{\mathbf{x}}\in\mathbf{X}^{(1)}$,

$$\tilde{a}(\zeta;\vec{\mathbf{x}})=\sum_{\substack{\xi,\xi'\in X^{(s)}\\ \xi\circ\xi'=\zeta\\ \vec{\mathbf{x}}'',\vec{\mathbf{x}}'\in X^{(1)}\\ \vec{\mathbf{x}}''\circ\vec{\mathbf{x}}'=\vec{\mathbf{x}}}}\ \sum_{\substack{(\vec{\mathbf{y}}_*,\mathbf{y})\in\mathbf{X}^{(2)}\\ (\vec{\mathbf{y}}'_*,\mathbf{y}')\in\mathbf{X}^{(2)}}} a_1(\xi;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}}'')a_2(\xi';\vec{\mathbf{y}}'_*,\vec{\mathbf{y}}';\vec{\mathbf{x}}')\int z(\vec{\mathbf{y}}_*\circ\vec{\mathbf{y}}'_*)^* z(\vec{\mathbf{y}}\circ\vec{\mathbf{y}}')d\mu(z^*,z).$$

We claim that only terms with $\mathrm{supp}(\vec{\mathbf{y}}_*,\vec{\mathbf{y}})\cap\mathrm{supp}(\vec{\mathbf{y}}'_*,\vec{\mathbf{y}}')\neq\emptyset$ can be nonzero. By hypothesis, $a_1(\xi;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}}'')=0$ if $\vec{\mathbf{y}}_*=\vec{\mathbf{y}}$. So we may assume that there is a $\mathbf{y}\in\mathrm{supp}(\vec{\mathbf{y}}_*,\vec{\mathbf{y}})$ with the multiplicities of $\vec{\mathbf{y}}_*$ and $\vec{\mathbf{y}}$ at $\mathbf{y}$ being different. Since $\int z(\vec{\mathbf{y}}_*\circ\vec{\mathbf{y}}'_*)^* z(\vec{\mathbf{y}}\circ\vec{\mathbf{y}}')d\mu(z^*,z)$ vanishes unless $\vec{\mathbf{y}}_*\circ\vec{\mathbf{y}}'_*=\vec{\mathbf{y}}\circ\vec{\mathbf{y}}'$, the multiplicities of $\vec{\mathbf{y}}'_*$ and $\vec{\mathbf{y}}'$ at $\mathbf{y}$ must also be different. [To see this, let $\mathbf{y}\in X$ and suppose that the multiplicity, say p_*, of $\vec{\mathbf{y}}_*\circ\vec{\mathbf{y}}'_*$ at $\mathbf{y}$ is different from the multiplicity, say p, of $\vec{\mathbf{y}}\circ\vec{\mathbf{y}}'$ at $\mathbf{y}$.

Because $d\mu$ is invariant under $z(\mathbf{y})\mapsto z_\theta(\mathbf{y})=e^{i\theta}z(\mathbf{y})$, while $[z_\theta(\mathbf{y})^*]^{p_*}z_\theta(\mathbf{y})^p = e^{i(p-p_*)\theta}[z(\mathbf{y})^*]^{p_*}z(\mathbf{y})^p$, the integral $\int[z(\mathbf{y})^*]^{p_*}z(\mathbf{y})^p d\mu(z^*,z)$ must vanish.]

In particular, $\mathbf{y}\in\mathrm{supp}(\vec{\mathbf{y}}'_*,\vec{\mathbf{y}}')$.

Consequently, by (1.2),

$$|\tilde{a}(\zeta;\vec{\mathbf{x}})| \leqslant \sum_{\substack{\xi,\xi'\in X^{(s)}\\ \xi\circ\xi'=\zeta\\ \vec{\mathbf{x}}'',\vec{\mathbf{x}}'\in X^{(1)}\\ \vec{\mathbf{x}}''\circ\vec{\mathbf{x}}'=\vec{\mathbf{x}}}} \sum_{\substack{\eta,\eta'\in\mathbf{X}^{(2)}\\ \mathrm{supp}(\eta)\cap\mathrm{supp}(\eta')\neq\emptyset}} |a_1(\xi;\eta;\vec{\mathbf{x}}'')||a_2(\xi';\eta';\vec{\mathbf{x}}')|r^{n(\eta)+n(\eta')}.$$

Since w gives weight at least $2r$ to the last two fields,

$$w(\zeta;\vec{\mathbf{x}})|\tilde{a}(\zeta;\vec{\mathbf{x}})| \leqslant \sum_{\substack{\xi,\xi'\in X^{(s)}\\ \xi\circ\xi'=\zeta\\ \vec{\mathbf{x}}'',\vec{\mathbf{x}}'\in X^{(1)}\\ \vec{\mathbf{x}}''\circ\vec{\mathbf{x}}'=\vec{\mathbf{x}}}} \sum_{\substack{\eta,\eta'\in\mathbf{X}^{(2)}\\ \mathrm{supp}(\eta)\cap\mathrm{supp}(\eta')\neq\emptyset}} w(\zeta;\eta\circ\eta';\vec{\mathbf{x}})|a_1(\xi;\eta;\vec{\mathbf{x}}'')||a_2(\xi';\eta';\vec{\mathbf{x}}')|2^{-n(\eta)-n(\eta')}$$

$$\leqslant \sum_{\substack{\xi,\xi'\in X^{(s)}\\ \xi\circ\xi'=\zeta\\ \vec{\mathbf{x}}'',\vec{\mathbf{x}}'\in X^{(1)}\\ \vec{\mathbf{x}}''\circ\vec{\mathbf{x}}'=\vec{\mathbf{x}}}} \sum_{\substack{\eta,\eta'\in\mathbf{X}^{(2)}\\ \mathrm{supp}(\eta)\cap\mathrm{supp}(\eta')\neq\emptyset}} w(\xi;\eta;\vec{\mathbf{x}}'')|a_1(\zeta;\eta;\vec{\mathbf{x}}'')|w(\xi';\eta';\vec{\mathbf{x}}')|a_2(\xi';\eta';\vec{\mathbf{x}}')|2^{-n(\eta)-n(\eta')}$$

If $\xi=(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)\in X^{(s)}$, we write $\xi_i\in X$ for $(\vec{\mathbf{x}}_1\circ\cdots\circ\vec{\mathbf{x}}_s)_i$. By hypothesis, $\tilde{a}(\zeta;\vec{\mathbf{x}})=0$ if $\zeta=-$. So, if there are no source fields,

$$\|\tilde{h}\|_w = \sum_{\substack{n_1,\ldots,n_{s+1}\geqslant 0\\ n_1+\cdots+n_s>0}} \max_{\mathbf{u}\in X} \max_{1\leqslant i\leqslant n_1+\cdots+n_s} \sum_{\substack{\zeta\in X^{n_1}\times\cdots\times X^{n_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}}\\ \xi_i=\mathbf{u}}} w(\zeta;\vec{\mathbf{x}})|\tilde{a}(\zeta;\vec{\mathbf{x}})|$$

$$\leqslant \sum_{\substack{n_1,\ldots,n_{s+1}\geqslant 0\\ n'_1,\ldots,n'_{s+1}\geqslant 0\\ m_*,m\geqslant 0\\ m'_*,m'\geqslant 0\\ \Sigma_{j\leqslant s}(n_j+n'_j)\geqslant 1}} \max_{\mathbf{u}\in X} \max_{1\leqslant i\leqslant \Sigma_{j\leqslant s}(n_j+n'_j)} \sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}},\ \vec{\mathbf{x}}\in X^{n'_{s+1}}\\ (\xi\circ\xi')_i=\mathbf{u}}} \sum_{\substack{\eta\in X^{m_*}\times X^m\\ n'\in X^{m'_*}\times X^{m'}\\ \mathrm{supp}(\eta)\cap\mathrm{supp}(\eta')\neq\emptyset}} w(\xi;\eta;\vec{\mathbf{x}})$$

$$\times|a_1(\xi;\eta;\vec{\mathbf{x}})|w(\xi';\eta';\vec{\mathbf{x}}')|a_2(\xi';\eta';\vec{\mathbf{x}}')|2^{-(m_*+m'_*+m+m')}$$

Fix, temporarily, $n_1,\ldots,n_{s+1},n'_1,\ldots,n'_{s+1},m_*,m,m'_*,m'\geqslant 0$. If there are no source fields, also fix, temporarily, $1\leqslant i\leqslant\Sigma_{j\leqslant s}(n_j+n'_j)$. There are source fields if $s'>0$ (s' was specified in Definition 2.6) and $\Sigma_{j=1}^{s'}(n_j+n'_j)\geqslant 1$. In this case,

$$\max_{\mathbf{u}\in X} \sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}},\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}\\ (\xi\circ\xi')_i=\mathbf{u}}}$$

is replaced by

$$\max_{\substack{\mathbf{u}_{j,\ell}\in X\\ 1\leqslant j<s'\\ 1\leqslant \ell\leqslant n_j}} \max_{\substack{\mathbf{u}'_{j,\ell}\in X\\ 1\leqslant j<s'\\ 1\leqslant \ell\leqslant n'_j}} \sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}},\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}\\ (\vec{\mathbf{x}}_j)_\ell=\mathbf{u}_{j,\ell},\ (\vec{\mathbf{x}}'_j)_{\ell'}=\mathbf{u}'_{j,\ell'}\\ 1\leqslant j\leqslant s',\ 1\leqslant \ell\leqslant n_j,\ 1\leqslant \ell'\leqslant n'_j}}.$$

If we translate the $(\vec{\mathbf{x}}_j)_\ell$, $(\vec{\mathbf{x}}'_j)_{\ell'}$ notation into the corresponding components of ξ,ξ', we may write both of these max/sums in the form

$$\max_{\substack{\mathbf{u}_p\in X\\ p\in A}} \max_{\substack{\mathbf{u}'_p\in X\\ p\in A'}} \sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}},\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}\\ \xi_p=\mathbf{u}_p,\ p\in A\\ \xi'_p=\mathbf{u}'_p,\ p\in A'}}$$

for some subsets $A\subset\{p\in\mathbb{N}\,|\,1\leqslant p\leqslant \Sigma_{j\leqslant s}n_j\}$ and $A'\subset\{p\in\mathbb{N}\,|\,1\leqslant p\leqslant \Sigma_{j\leqslant s}n'_j\}$ with $A\cup A'$ nonempty.

Fix $\mathbf{u}_p\in X$ for each $p\in A$ and $\mathbf{u}'_p\in X$ for each $p\in A'$. For notational simplicity, suppose that $p=1\in A$. Then,

$$\begin{aligned}
&\sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}},\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}\\ \xi_p=\mathbf{u}_p,\ p\in A\\ \xi'_p=\mathbf{u}'_p,\ p\in A'}} \sum_{\substack{\eta\in X^{m_*}\times X^{m}\\ \eta'\in X^{m'_*}\times X^{m'}\\ \mathrm{supp}(\eta)\cap\mathrm{supp}(\eta')\neq\emptyset}} w(\xi;\eta;\vec{\mathbf{x}})|a_1(\xi;\eta;\mathbf{x})|w(\xi';\eta';\vec{\mathbf{x}}')|a_2(\xi';\eta';\vec{\mathbf{x}}')|2^{-(m_*+m'_*+m+m')}\\
&\quad\leqslant \sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}},\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}\\ \xi_p=\mathbf{u}_p,\ p\in A\\ \xi'_p=\mathbf{u}'_p,\ p\in A'}} \sum_{\substack{1\leqslant k\leqslant m_*+m\\ 1\leqslant \ell\leqslant m'_*+m'}} \sum_{\substack{\eta\in X^{m_*}\times X^{m}\\ \eta'\in X^{m'_*}\times X^{m'}\\ \eta_k=\eta'_\ell}} w(\xi;\eta;\vec{\mathbf{x}})|a_1(\xi;\eta);\vec{\mathbf{x}}|w(\xi';\eta';\vec{\mathbf{x}}')\\
&\qquad\times|a_2(\xi';\eta';\vec{\mathbf{x}}')|2^{-(m_*+m'_*+m+m')}\leqslant \max_{\substack{1\leqslant k\leqslant m_*+m\\ 1\leqslant \ell\leqslant m'_*+m'}} \sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \vec{\mathbf{x}}\in X^{n_{s+1}},\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}\\ \xi_p=\mathbf{u}_p,\ p\in A\\ \xi'_p=\mathbf{u}'_p,\ p\in A'}} \sum_{\substack{\eta\in X^{m_*}\times X^{m}\\ \eta'\in X^{m'_*}\times X^{m'}\\ \eta_k=\eta'_\ell}} w(\xi;\eta;\vec{\mathbf{x}})
\end{aligned}$$

$$\times a_1(\xi;\eta;\vec{\mathbf{x}})|w(\xi';\eta';\vec{\mathbf{x}}')|a_2(\xi';\eta';\vec{\mathbf{x}}')| \leq \left(\sum_{\substack{\xi\in X^{n_1}\times\cdots\times X^{n_s}\\ \eta\in X^{m_*}\times X^{m}\\ \vec{\mathbf{x}}\in X^{n_{s+1}}\\ \xi_p=\mathbf{u}_p,\ p\in A}} w(\xi;\eta;\vec{\mathbf{x}})|a_1(\xi;\eta;\vec{\mathbf{x}})| \right)$$

$$\times \left(\max_{1\leq\ell\leq m'_*+m'} \max_{\mathbf{y}\in X} \sum_{\substack{\xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \xi'_p=\mathbf{u}'_p,\ p\in A'\\ \eta'\in X^{m'_*}\times X^{m'}\\ \eta'_\ell=\mathbf{y}\\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}}} w(\xi';\eta';\vec{\mathbf{x}}')|a_2(\xi';\eta';\vec{\mathbf{x}}')| \right).$$

If A' is empty (i.e., there are no source fields in ξ'), the second large bracket is

$$\left(\max_{1\leq\ell\leq m'_*+m'} \max_{\mathbf{y}\in X} \sum_{\substack{\xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \eta'\in X^{m'_*}\times X^{m'}\\ \eta'_\ell=\mathbf{y}\\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}}} w(\xi';\eta';\vec{\mathbf{x}}')|a_2(\xi';\eta';\vec{\mathbf{x}}')| \right).$$

If A' is not empty (i.e., there are source fields in ξ'), we bound the second large bracket by

$$\left(\sum_{\substack{\xi'\in X^{n'_1}\times\cdots\times X^{n'_s}\\ \xi'_p=\mathbf{u}'_p,\ p\in A'\\ \eta'\in X^{m'_*}\times X^{m'}\\ \vec{\mathbf{x}}'\in X^{n'_{s+1}}}} w(\xi';\eta';\vec{\mathbf{x}}')|a_2(\xi';\eta';\vec{\mathbf{x}}')| \right).$$

Taking the maximum over $\mathbf{u}_p$'s and $\mathbf{u}'_p$'s, and possibly over i, and the remaining sums gives the desired bound. Recall that $a_1(\xi;\eta;\vec{\mathbf{x}})=0$ if $\xi=-$ and $a_2(\xi';\eta';\vec{\mathbf{x}}')=0$ if $\xi'=-$. ■

Recall that we have fixed a weight system w of length $s+3$ that gives weight at least $4r$ to the last two internal fields and that we have fixed the number $0\leq s'\leq s$ of source fields.

Theorem 3.4: If $f(\alpha_1,\ldots,\alpha_s;z_*,z;\mathfrak{h})$ obeys $\|f\|_w<\frac{1}{16}$, then there is an analytic function $g(\alpha_1,\ldots,\alpha_s;\mathfrak{h})$ such that

$$\frac{\int e^{f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{f(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)} = e^{g(\alpha_1,\ldots,\alpha_s;\mathfrak{h})} \tag{3.1}$$

and

$$\|g\|_w \leq \frac{\|f\|_w}{1-16\|f\|_w}.$$

Proof: The proof of this theorem is virtually identical to the proof of Ref. 2 (Theorem 3.4). Let $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})$ be the symmetric coefficient system for f. We first introduce some shorthand notation.

- For $\eta=(\vec{\mathbf{y}}_*,\vec{\mathbf{y}})\in X^{(2)}$, we write $z(\eta)=z(\vec{\mathbf{y}}_*)^*z(\vec{\mathbf{y}})$ and

$$a(\eta)=\sum_{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s,\vec{\mathbf{x}})\in X^{(s+1)}} a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})\alpha_1(\vec{\mathbf{x}}_1)\cdots\alpha_s(\vec{\mathbf{x}}_s)\mathfrak{h}(\vec{\mathbf{x}}).$$

With this notation

$$f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})=\sum_{\eta\in X^{(2)}} a(\eta)z(\eta).$$

By factoring $e^{f(\alpha_1,\ldots,\alpha_s;0,0;\mathfrak{h})}$ out of the integral in the numerator of (3.1), we may assume that $a(-,-)=0$.

- Let $X_1,\ldots,X_\ell$ be subsets of X. The incidence graph $G(X_1,\ldots,X_\ell)$ of $X_1,\ldots,X_\ell$ is the labeled graph with the set of vertices $(1,\ldots,\ell)$ and edges between $i\neq j$ whenever $X_i\cap X_j\neq\emptyset$.
- For a subset of $Z\subset X$, we denote by $\mathcal{C}(Z)$ the set of all ordered tuples $(\eta_1,\ldots,\eta_n)$ for which the incidence graph $G(\text{supp }\eta_1,\ldots,\text{supp }\eta_n)$ is connected and for which $Z=\text{supp }\eta_1\cup\cdots\cup\text{supp }\eta_n$.

Expanding the exponential

$$e^{f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}=1+\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\sum_{\substack{Z\subset X\\ Z\neq\emptyset}}\sum_{\substack{\eta_1,\ldots,\eta_\ell\in X^{(2)}\\ Z=\text{supp }\eta_1\cup\cdots\cup\text{supp }\eta_\ell}} a(\eta_1)\cdots a(\eta_\ell)z(\eta_1)\cdots z(\eta_\ell).$$

As in Ref. 2 [(3.5)],

$$\int e^{f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)=1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{Z_1,\ldots,Z_n\subset X\\ \text{pairwise disjoint}}}\prod_{j=1}^{n}\Phi(Z_j),$$

where for $\emptyset\neq Z\subset X$, the function $\Phi(Z)(\alpha_1,\ldots,\alpha_s)$ is defined by

$$\Phi(Z)=\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{(\eta_1,\ldots,\eta_k)\in\mathcal{C}(Z)} a(\eta_1)\cdots a(\eta_k)\int z(\eta_1)\cdots z(\eta_k)d\mu(z^*,z)$$

and $\Phi(\emptyset)=0$. Again as in Ref. 2 [(3.7)],

$$\ln\int e^f d\mu=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{Z_1,\ldots,Z_n\subset X}\rho^T(Z_1,\ldots,Z_n)\prod_{j=1}^{n}\Phi(Z_j) \tag{3.2}$$

where

$$\rho^T(Z_1,\ldots,Z_n)=\sum_{\substack{g\in\mathcal{C}_n\\ g\subset G(Z_1,\ldots,Z_n)}}(-1)^{|g|}$$

and $\mathcal{C}_n$ is the set of all connected graphs on the set of vertices $(1,\ldots,n)$ that have at most one edge joining each pair of distinct vertices and no edges joining a vertex to itself. [In particular, $\rho^T(Z_1,\ldots,Z_n)=0$ if $G(Z_1,\ldots,Z_n)$ is not connected.]

From (3.2) one determines, as in Ref. 2 [(3.10)], a not necessarily symmetric coefficient system for g. Namely,

$$g(\alpha_1,\ldots\alpha_s;\mathfrak{h})=\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s,\vec{\mathbf{x}})\in X^{(s+1)}\\ (\mathbf{x}_1,\ldots,\vec{\mathbf{x}}_s)\neq(-,\ldots,-)}} a'(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}})\alpha_1(\vec{\mathbf{x}}_1)\cdots\alpha_s(\vec{\mathbf{x}}_s)\mathfrak{h}(\vec{\mathbf{x}}),$$

where for $\xi\in X^{(s)}$ and $\vec{\mathbf{x}}\in X^{(1)}$,

$$a'(\xi;\vec{\mathbf{x}}) = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{\xi_1,\dots,\xi_n\in X^{(s)}\\ \xi_1\circ\cdots\circ\xi_n=\xi\\ \vec{\mathbf{u}}_1,\dots,\vec{\mathbf{u}}_n\in X^{(1)}\\ \vec{\mathbf{u}}_1\circ\cdots\circ\vec{\mathbf{u}}_n=\vec{\mathbf{x}}}}\ \sum_{\eta_1,\dots,\eta_n\in X^{(2)}}\rho^T(\text{supp }\eta_1,\dots,\text{supp }\eta_n)\prod_{j=1}^{n}\tilde{a}(\xi_j;\eta_j;\vec{\mathbf{u}}_j) \tag{3.3}$$

with for each $\xi\in X^{(s)}$ and $\eta\in X^{(2)}$,

$$\tilde{a}(\xi;\eta;\vec{\mathbf{x}}) = \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{\substack{(\eta_1,\dots,\eta_k)\in\mathcal{C}(\text{supp }\eta)\\ \eta_1\circ\cdots\circ\eta_k=\eta}}\ \sum_{\substack{\xi_1,\dots,\xi_k\\ \xi_1\circ\cdots\circ\xi_k=\xi\\ \vec{\mathbf{u}}_1,\dots,\vec{\mathbf{u}}_k\\ \vec{\mathbf{u}}_1\circ\cdots\circ\vec{\mathbf{u}}_n=\vec{\mathbf{x}}}} a(\xi_1;\eta_1;\vec{\mathbf{u}}_1)\cdots a(\xi_k;\eta_k;\vec{\mathbf{u}}_k)\int z(\eta)d\mu(z^*,z) \tag{3.4}$$

By Remark 2.7, part (i), $\|g\|_w \le |a'|_w$. The bounds on $|a'|_w$ that are necessary for the proof of the theorem are derived as in Ref. 2 [(3.12)–(3.16)]. ■

Corollary 3.5: Let $f(\alpha_1,\dots,\alpha_s;z_*,z;\mathfrak{h})$ *obey* $\|f\|_w < \frac{1}{32}$ *and define, for each complex number* ζ *with* $|\zeta|\|f\|_w < \frac{1}{16}$, *the function* $G(\zeta)=G(\zeta;\alpha_1,\dots,\alpha_s;\mathfrak{h})$ *by the condition*

$$\frac{\int e^{\zeta f(\alpha_1,\dots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{\zeta f(0,\dots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)} = e^{G(\zeta;\alpha_1,\dots,\alpha_s;\mathfrak{h})} \tag{3.5}$$

as in Theorem 3.4. Then, $G(\zeta)$ *is a (Banach space valued) analytic function of* ζ *and, for each* $n\in\mathbb{N}$, *the* $g(\alpha_1,\dots,\alpha_s;\mathfrak{h})=G(1)$ *of Theorem 3.4 obeys*

$$\left\| g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!}\frac{d^nG}{d\zeta^n}(0)\right\|_w \le \left(\frac{\|f\|_w}{\frac{1}{20}-\|f\|_w}\right)^{n+1}.$$

We have $G(0)=0$,

$$\frac{dG}{d\zeta}(0) = \int [f(\alpha_1,\dots,\alpha_s;z^*,z;\mathfrak{h}) - f(0,\dots,0;z^*,z;\mathfrak{h})]d\mu(z^*,z)$$

and

$$\frac{d^2G}{d^2\zeta^2}(0) = \int f(\alpha_1,\dots,\alpha_s;z^*,z;\mathfrak{h})^2 d\mu(z^*,z) - \int f(0,\dots,0;z^*,z;\mathfrak{h})^2 d\mu(z^*,z)$$
$$- \left[\int f(\alpha_1,\dots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z)\right]^2 + \left[\int f(0,\dots,0;z^*,z;\mathfrak{h})d\mu(z^*,z)\right]^2.$$

If, in addition, the measure $d\mu(z^*,z)$ *on* $\mathbb{C}$ *is rotation invariant,* $f(0,\dots,0;z^*,z;\mathfrak{h})=0$ *and the symmetric coefficient system* $a(\vec{\mathbf{x}}_1,\dots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})$ *of* f *obeys* $a(\vec{\mathbf{x}}_1,\dots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})=0$ *whenever* $\vec{\mathbf{y}}=\vec{\mathbf{y}}_*$, *then* $(dG/d\zeta)(0)=0$ *and*

$$\frac{d^2G}{d\zeta^2}(0) = \int f(\alpha_1,\dots,\alpha_s;z^*,z;\mathfrak{h})^2 d\mu(z^*,z).$$

Proof: The analyticity of $G(\zeta)$ is obvious since the series (3.3) and (3.4), with a replaced by ζa converge in the norm $\|\cdot\|_w$ uniformly in ζ for $|\zeta|\|f\|_w$ bounded by any constant strictly less than $\frac{1}{16}$. That $G(0)=0$ and $G(1)=g$ is also obvious. So by Taylor's formula with remainder,

$$g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!}\frac{d^nG}{d\zeta^n}(0) = \frac{1}{(n+1)!}\frac{d^{n+1}G}{d\zeta^{n+1}}(u)$$

for some $0<u<1$. By the Cauchy integral formula,

$$\frac{1}{(n+1)!}\frac{d^{n+1}G}{d\zeta^{n+1}}(u) = \frac{1}{2\pi i}\int_{|\zeta|=U}\frac{G(\zeta)}{(\zeta-u)^{n+2}}d\zeta$$

for any $u<U<1/16\|f\|_w$. Hence,

$$\left\|g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!}\frac{d^nG}{d\zeta^n}(0)\right\|_w \leq \frac{1}{2\pi}2\pi U \sup_{|\zeta|=U}\frac{\|G(\zeta)\|_w}{(U-1)^{n+2}} \leq U\frac{U\|f\|_w}{1-16U\|f\|_w}\frac{1}{(U-1)^{n+2}}$$
$$= \frac{\|f\|_w^{n+1}}{(U\|f\|_w-\|f\|_w)^{n+2}}\frac{(U\|f\|_w)^2}{1-16U\|f\|_w}.$$

Choosing U so that $U\|f\|_w=\frac{1}{20}$ gives

$$\left\|g - \frac{dG}{d\zeta}(0) - \cdots - \frac{1}{n!}\frac{d^nG}{d\zeta^n}(0)\right\|_w \leq \frac{1}{80}\frac{\|f\|_w^{n+1}}{\left(\frac{1}{20}-\|f\|_w\right)^{n+2}} \leq \left(\frac{\|f\|_w}{\frac{1}{20}-\|f\|_w}\right)^{n+1}$$

since $\|f\|_w<\frac{1}{32}$.

To get the formulas for $(dG/d\zeta)(0)$ and $(d^2G/d\zeta^2)(0)$, differentiate (3.5) to give

$$\frac{dG}{d\zeta}(\zeta)e^{G(\zeta;\alpha_1,\ldots,\alpha_s;\mathfrak{h})} = \frac{\int f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})e^{\zeta f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{\zeta f(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)}$$
$$- \frac{\int f(0,\ldots,0;z^*,z;\mathfrak{h})e^{\zeta f(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)\int e^{\zeta f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{[\int e^{\zeta f(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)]^2}$$

and hence

$$\frac{dG}{d\zeta}(\zeta) = \frac{\int f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})e^{\zeta f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{\zeta f(\alpha_1,\ldots,\alpha_2;z^*,z;\mathfrak{h})}d\mu(z^*,z)}$$
$$- \frac{\int f(0,\ldots,0,z^*,z;\mathfrak{h})e^{\zeta f(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{\zeta f(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)}.$$

Setting $\zeta=0$ gives the formula for $(dG/d\zeta)(0)$. Differentiating again with respect to ζ and then setting $\zeta=0$ gives the formula for $(d^2G/d\zeta^2)(0)$. When the measure $d\mu(z^*,z)$ is rotation invariant and the symmetric coefficient system $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})$ of f obeys $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{y}}_*,\vec{\mathbf{y}};\vec{\mathbf{x}})=0$, whenever $\vec{\mathbf{y}}=\vec{\mathbf{y}}_*$, we have

$$\int f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z) = 0,$$

as in Remark 3.3, part (ii), and the remaining formulas follow. ■

Corollary 3.6: Denote by $\mathcal{F}$ the Banach space of functions $f(\alpha_1,\ldots,\alpha_s;z_*,z;\mathfrak{h})$ with $\|f\|_w<\infty$. Let $f,f'\in\mathcal{F}$ with

$$\int f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z) - \int f'(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})d\mu(z^*,z) = 0$$

and $\|f\|_w+\|f'-f\|_w<\frac{1}{17}$. Define $g,g'\in\mathcal{F}$ by the conditions

$$\frac{\int e^{f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{f(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)} = e^{g(\alpha_1,\ldots,\alpha_s;\mathfrak{h})} \quad \frac{\int e^{f'(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{f'(0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)} = e^{g'(\alpha_1,\ldots,\alpha_s;\mathfrak{h})},$$

as in Theorem 3.4. Then,

$$\|g'-g\|_w \leqslant 4\frac{\|f'-f\|_w(\|f\|_w+\|f'-f\|_w)}{\left(\frac{1}{17}-\|f\|_w-\|f'-f\|_w\right)^2}$$

Proof: Define for all $\zeta,\zeta'\in\mathbb{C}$ with $|\zeta|\leqslant U^{-1}\equiv\left(\frac{1}{17}-\|f\|_w-\|f'-f\|_w\right)/(2\|f'-f\|_w$ and $|\zeta|\leqslant U'^{-1}\equiv\left(\frac{1}{17}-\|f\|_w-\|f'-f\|_w\right)/(2\|f\|_w)$,

$$F(\zeta,\zeta';\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h}) = \zeta' f(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h}) + \zeta(f'-f)(\alpha_1,\ldots\alpha_s;z^*,z;\mathfrak{h})$$

and define $G(\zeta)$ by

$$\frac{\int e^{F(\zeta,\zeta';\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{F(\zeta,\zeta';0,\ldots,0;z^*,z;\mathfrak{h})}d\mu(z^*,z)} = e^{G(\zeta,\zeta';\alpha_1,\ldots\alpha_s;\mathfrak{h})},$$

as in Theorem 3.4. Then, $G(\zeta,\zeta')$ is an $\mathcal{F}$-valued analytic function of ζ,ζ' since the series (3.4) and (3.3) with a replaced by the appropriate $a(\zeta,\zeta')$ converge in the norm $\|\cdot\|_w$ uniformly in ζ,ζ' for $|\zeta'|\|f\|_w+|\zeta|\|f'-f\|_w$ bounded by any constant strictly less than $\frac{1}{16}$. Furthermore, $G(0,1)=g$ and $G(1,1)=g'$ so that

$$g'-g-\int_0^1\frac{\partial G}{\partial\zeta}(s,1)ds = \int_0^1\frac{\partial G}{\partial\zeta}(s,0)ds + \int_0^1\int_0^1\frac{\partial^2 G}{\partial\zeta\partial\zeta'}(s,s')dsds'.$$

By hypothesis,

$$\frac{\partial G}{\partial\zeta}(s,0) = \frac{\int(f-f')(\alpha_1,\ldots,\alpha_s;z^*,z;\mathfrak{h})e^{s(f-f')(\alpha_1,\ldots\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{s(f-f')(\alpha_1,\ldots\alpha_s;z^*,z;\mathfrak{h})}d\mu(z^*,z)} - \frac{\int(f-f')(0,\ldots,0;z^*,z;\mathfrak{h})e^{s(f-f')(0,\ldots0;z^*,z;\mathfrak{h})}d\mu(z^*,z)}{\int e^{s(f-f')(0,\ldots0;z^*,z;\mathfrak{h})}d\mu(z^*,z)}$$

vanishes when $s=0$ so that

$$g'-g = \int_0^1 ds\int_0^s ds''\frac{\partial^2 G}{\partial\zeta^2}(s'',0) + \int_0^1\int_0^1\frac{\partial^2 G}{\partial\zeta\partial\zeta'}(s,s')dsds'$$

and

$$\|g'-g\|_w \leqslant \frac{1}{2}\sup_{s\in[0,1]}\left\|\frac{\partial^2 G}{\partial\zeta^2}(s,0)\right\|_w + \sup_{s,s'\in[0,1]}\left\|\frac{\partial^2 G}{\partial\zeta\partial\zeta'}(s,s')\right\|_w$$

By the Cauchy integral formula,

$$\frac{1}{2}\frac{\partial^2 G}{\partial\zeta^2}(s,0) = \frac{1}{2\pi i}\int_{|\zeta|=U^{-1}} d\zeta\frac{G(s+\zeta,0)}{\zeta^3},$$

$$\frac{\partial^2 G}{\partial\zeta\partial\zeta'}(s,s') = \frac{1}{2\pi i}\int_{|\zeta|=U^{-1}} d\zeta\frac{1}{2\pi i}\int_{|\zeta'|=U'^{-1}} d\zeta'\frac{G(s+\zeta,s'+\zeta')}{\zeta^2\zeta'^2}.$$

By hypothesis, for all $0\leqslant s,s'\leqslant 1,\zeta,\zeta'$ with $|\zeta|\leqslant 1/U$, $|\zeta'|\leqslant 1/U'$,

$$\|F(s+\zeta,s'+\zeta')\|_w = \|(s'+\zeta')f+(s+\zeta)(f'-f)\|_w \leqslant (1+|\zeta'|)\|f\|_w + (1+|\zeta|)\|f'-f\|_w \leqslant \frac{1}{17},$$

and hence

$$\|G(s+\zeta,s'+\zeta')\|_w \leqslant \frac{\|F(s+\zeta,x'+\zeta')\|_w}{1-16\|F(s+\zeta,s'+\zeta')\|_w} \leqslant \frac{\frac{1}{17}}{1-\frac{16}{17}} = 1.$$

So

$$\|g'-g\|_w \leqslant \frac{1}{2\pi}2\pi U^{-1}\frac{1}{U^{-3}} + \frac{1}{2\pi}2\pi U^{-1}\frac{1}{2\pi}2\pi U'^{-1}\frac{1}{U^{-2}}\frac{1}{U'^{-2}} = U^2 + UU'.$$

■

IV. THE HISTORY FIELD AND LINEAR TRANSFORMATIONS

In this section, we assume that we are given a metric d and weight factors $\kappa_1,\ldots,\kappa_s$ on X. We consider analytic functions $f(\alpha_1,\ldots,\alpha_s;\mathfrak{h})$ of the complex fields $\alpha_1,\ldots,\alpha_s$ and the additional history field $\mathfrak{h}$, which takes values in $\{0,1\}$. Denote by w the weight system with metric d that associates the weight factor κ_j to the field α_j, and the constant weight factor 1 to the history field $\mathfrak{h}$ (see Definition 2.6). Also fix the number $0 \leqslant s' \leqslant s$ of source fields.

As pointed out in Sec. I, the purpose of the history field is to keep track of all the points that have ever been used in the construction of a particular function or monomial. We will deal with linear changes in the α-fields which may be compositions of several such changes of variables. In each composition it may be relevant which points where involved. This is the motivation for the following

Definition 4.1:

(i) An $\mathfrak{h}$-operator or $\mathfrak{h}$-linear map A on $\mathbb{C}^X$ is a linear operator on $\mathbb{C}^X$ whose kernel is of the form

$$A(\mathbf{x},\mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_\ell)\in X^\ell} A(\mathbf{x};\mathbf{x}_1,\ldots,\mathbf{x}_\ell;\mathbf{y})\mathfrak{h}(\mathbf{x})\mathfrak{h}(\mathbf{x}_1)\cdots\mathfrak{h}(\mathbf{x}_\ell)\mathfrak{h}(\mathbf{y}).$$

(ii) The composition $A\circ B$ of two $\mathfrak{h}$-operators A, B on $\mathbb{C}^X$ is by definition the $\mathfrak{h}$-operator with kernel

$$(A\circ B)(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{z}\in X} A(\mathbf{x},\mathbf{z})B(\mathbf{z},\mathbf{y}) = \sum_{\mathbf{z}\in X}\sum_{\ell,\ell'\geqslant 0}\sum_{\substack{\mathbf{x}_1,\ldots,\mathbf{x}_\ell\\ \mathbf{y}_1,\ldots,\mathbf{y}_{\ell'}}} A(\mathbf{x};\mathbf{x}_1,\ldots,\mathbf{x}_\ell;\mathbf{z})B(\mathbf{z};\mathbf{y}_1,\ldots,\mathbf{y}_{\ell'};\mathbf{y})$$
$$\mathfrak{h}(\mathbf{x})\mathfrak{h}(\mathbf{x}_1)\cdots\mathfrak{h}(\mathbf{x}_\ell)\mathfrak{h}(\mathbf{z})\mathfrak{h}(\mathbf{y}_1)\cdots\mathfrak{h}(\mathbf{y}_{\ell'})\mathfrak{h}(\mathbf{y}).$$

Here, we used that $\mathfrak{h}^2=\mathfrak{h}$.

(iii) For an "ordinary" linear operator J on $\mathbb{C}^X$ with kernel $J(\mathbf{x},\mathbf{y})$, we define the associated $\mathfrak{h}$-operators by

$$\bar{J}(\mathbf{x},\mathbf{y}) = \mathfrak{h}(\mathbf{x})J(\mathbf{x},\mathbf{y})\mathfrak{h}(\mathbf{y})$$

and the associated $\mathfrak{h}$-exponential as

$$\mathrm{exph}(J) = \mathfrak{h} + \sum_{\ell=1}^{\infty}\frac{1}{\ell!}\bar{J}^{\ell}.$$

(iv) If ϕ is any field on X and A an $\mathfrak{h}$-operator, we set

$$(A\phi)(\mathbf{x}) = \sum_{\mathbf{y}\in X} A(\mathbf{x},\mathbf{y})\phi(\mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_\ell,\mathbf{y}} A(\mathbf{x};\mathbf{x}_1,\ldots,\mathbf{x}_\ell;\mathbf{y})\mathfrak{h}(\mathbf{x})\mathfrak{h}(\mathbf{x}_1)\cdots\mathfrak{h}(\mathbf{x}_\ell)\mathfrak{h}(\mathbf{y})\phi(\mathbf{y}).$$

Definition 4.2: (Weighted L^1-L^∞ operator norms) Let $\kappa,\kappa':X\to(0,\infty]$ be weight factors and δ an arbitrary metric and let A be an $\mathfrak{h}$-linear map on $\mathbb{C}^X$. We define the operator norm

$$N_\delta(A;\kappa,\kappa') = \left\| \sum_{\substack{\mathbf{x},\mathbf{y}\in X\\ \vec{\mathbf{x}}\in X^{(1)}}} A(\mathbf{x};\vec{\mathbf{x}};\mathbf{y})\mathfrak{h}(\mathbf{x})\beta_l(\mathbf{x})\mathfrak{h}(\vec{\mathbf{x}})\mathfrak{h}(\mathbf{y})\beta_r(\mathbf{y}) \right\|_\omega ,$$

where ω is the weight system with metric δ that associates the weight $1/\kappa$ to β_l, the weight κ' to β_r and the weight 1 to $\mathfrak{h}$, and where β_l and β_r are thought of as internal fields. For an ordinary operator J on $\mathbb{C}^X$, we set $N_\delta(J;\kappa,\kappa')=N_\delta(\bar{J};\kappa,\kappa')$.

Remark 4.3:

(i) Definition 4.2 may be equivalently formulated as follows. For each integer $\ell\geq 0$, set

$$L_\ell(A;\kappa,\kappa') = \sup_{\mathbf{x}\in X}\sum_{\mathbf{y}\in X}\sum_{\vec{\mathbf{x}}\in X^\ell}\frac{1}{\kappa(\mathbf{x})}|A(\mathbf{x};\vec{\mathbf{x}};\mathbf{y})|\kappa'(\mathbf{y})e^{\tau_\delta(\mathrm{supp}(\mathbf{x},\vec{\mathbf{x}},\mathbf{y}))},$$

$$R_\ell(A;\kappa,\kappa') = \sup_{\mathbf{y}\in X}\sum_{\mathbf{x}\in X}\sum_{\vec{\mathbf{x}}\in X^\ell}\frac{1}{\kappa(\mathbf{x})}|A(\mathbf{x};\vec{\mathbf{x}};\mathbf{y})|\kappa'(\mathbf{y})e^{\tau_\delta(\mathrm{supp}(\mathbf{x},\vec{\mathbf{x}},\mathbf{y}))}.$$

Then,

$$N_\delta(A;\kappa,\kappa') = \sum_{\ell\geq 0}\max\{L_\ell(A;\kappa,\kappa'),R_\ell(A;\kappa,\kappa')\}.$$

(ii) Clearly,

$$N_\delta(A\circ B;\kappa,\kappa'') \leq N_\delta(A;\kappa,\kappa')N_\delta(B;\kappa,\kappa'') \tag{4.1}$$

for any two $\mathfrak{h}$-operators A, B and any three weight factors κ, κ', and κ''.

Proposition 4.4: Let A_j, $1\leq j\leq s$, be $\mathfrak{h}$-operators on $\mathbb{C}^X$, and let $f(\alpha_1,\ldots,\alpha_s;\mathfrak{h})$ be an analytic function on a neighborhood of the origin in $\mathbb{C}^{(s+1)|X|}$. Define $\tilde{f}$ by

$$\tilde{f}(\alpha_1,\ldots,\alpha_s;\mathfrak{h}) = f(A_1\alpha_1,\ldots,A_s\alpha_s;\mathfrak{h}).$$

Let $\kappa_1,\ldots,\kappa_s$, $\tilde{\kappa}_1,\ldots,\tilde{\kappa}_s$ be weight factors. Denote by w and $\tilde{w}$ the weight systems with metric d that associate to the field α_j the weight factors κ_j and $\tilde{\kappa}_j$, respectively, and the constant weight factor 1 to the history field $\mathfrak{h}$ (see Definition 2.5).

If $N_d(A_j;\kappa_j,\tilde{\kappa}_j)\leq 1$ for $1\leq j\leq s$, then

$$\|\tilde{f}\|_{\tilde{w}} \leq \|f\|_w.$$

Proof: Let $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})$ be a symmetric coefficient system for f. Define, for each $n(\vec{\mathbf{x}}_j)=n_j\geq 0$, $1\leq j\leq s+1$,

$$\tilde{a}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1}) = \sum_{\vec{\mathbf{y}}_1\in X^{n_1}}\cdots\sum_{\vec{\mathbf{y}}_s\in X^{n_s}}\sum_{\vec{\mathbf{y}}_{s+1}\circ\vec{\mathbf{z}}_{1,1}\circ\cdots\circ\vec{\mathbf{z}}_{s,n_s}=\vec{\mathbf{x}}_{s+1}} a(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_s;\vec{\mathbf{y}}_{s+1})\prod_{j=1}^{s}\left[\prod_{\ell=1}^{n_j}A_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})\right],$$

where $\vec{\mathbf{x}}_j=(\mathbf{x}_{j,1},\ldots,\mathbf{x}_{j,n_j})$. Then, $\tilde{a}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})$ is a coefficient system for $\tilde{f}$. Since

$$\tau_d(\mathrm{supp}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s,\vec{\mathbf{y}}_{s+1}\circ\vec{\mathbf{z}}_{1,1}\circ\cdots\circ\vec{\mathbf{z}}_{s,n_s}))\leq\tau_d(\mathrm{supp}(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_s,\vec{\mathbf{y}}_{s+1}))+\sum_{\substack{1\leq j\leq s\\1\leq\ell\leq n_j}}\tau_d(\mathrm{supp}(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})),$$

we have

$$\begin{aligned}\tilde{\omega}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|\tilde{a}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|&\\ \leq\sum_{\substack{\vec{\mathbf{y}}_j\in X^{n_j}\\1\leq j\leq s}}\ \sum_{\vec{\mathbf{y}}_{s+1}\circ\vec{\mathbf{z}}_{1,1}\circ\cdots\circ\vec{\mathbf{z}}_{s,n_s}=\vec{\mathbf{x}}_{s+1}}&\omega(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_s,\vec{\mathbf{y}}_{s+1})|a(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_s,\vec{\mathbf{y}}_{s+1})|\\ \times\prod_{j=1}^{s}\left[\prod_{\ell=1}^{n_j}e^{\tau_d(\mathrm{supp}(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell}))}|A_j(\mathrm{supp}(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell}))|\frac{\tilde{\kappa}_j(\mathbf{x}_{j,\ell})}{\kappa_j(\mathbf{y}_{j,\ell})}\right].&\end{aligned}$$

We first observe that when $\vec{\mathbf{x}}_1=\cdots=\vec{\mathbf{x}}_s=-$, we have

$$\tilde{\omega}(-,\ldots,-;\vec{\mathbf{x}}_{s+1})|\tilde{a}(-,\ldots,-;\vec{\mathbf{x}}_{s+1})|=w(-,\ldots,-;\vec{\mathbf{x}}_{s+1})|a(-,\ldots,-;\vec{\mathbf{x}}_{s+1})|$$

so that the corresponding contributions to $\|\tilde{f}\|_{\tilde{w}}$ and $\|f\|_w$ are identical. Therefore, we may assume, without loss of generality, that $f(0,\ldots,0;\mathfrak{h})=0$.

If there are no source fields, we are to bound

$$\|\tilde{f}\|_{\tilde{w}}=\sum_{\substack{n_1,\ldots,n_{s+1}\geq 0\\n_1+\cdots+n_s\geq 1}}\max_{\mathbf{x}\in X}\max_{\substack{1\leq\bar{j}\leq s\\n_{\bar{j}}\neq 0}}\max_{1\leq\bar{\iota}\leq n_{\bar{j}}}\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\in X^{n_1}\times\cdots\times X^{n_{s+1}}\\(\vec{\mathbf{x}}_{\bar{j}})_{\bar{\iota}}=\mathbf{x}}}\tilde{w}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|\tilde{a}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|.$$

If there are source fields, that is, if $s'>0$ (s' was specified in Definition 2.6) and $\Sigma_{j=1}^{s'}n_j\geq 1$

$$\max_{x\in X}\max_{\substack{1\leq\bar{j}\leq s\\n_{\bar{j}}\neq 0}}\max_{1\leq\bar{\iota}\leq n_{\bar{j}}}\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\in X^{n_1}\times\cdots\times X^{n_{s+1}}\\(\vec{\mathbf{x}}_{\bar{j}})_{\bar{\iota}}=\mathbf{x}}}$$

is replaced by

$$\max_{\substack{\tilde{\mathbf{x}}_{j,\ell}\in X\\1\leq j\leq s'\\1\leq\ell\leq n_j}}\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\in X^{n_1}\times\cdots\times X^{n_{s+1}}\\(\vec{\mathbf{x}}_j)_\ell=\tilde{\mathbf{x}}_{j,\ell}\\1\leq j\leq s',\ 1\leq\ell\leq n_j}}.$$

Choose any $n_1,\ldots,n_{s+1}\geq 0$. If there are no source fields also choose a $1\leq\bar{j}\leq s$ with $n_{\bar{j}}\neq 0$ and an $1\leq\bar{\iota}\leq n_{\bar{j}}$. This is equivalent to choosing the singleton subset $I=\{(\bar{j},\bar{\iota})\}$ of $\{(j,\ell)|1\leq j\leq s,1\leq\ell\leq n_j\}$. If there are source fields, set

$$I=\{(j,\ell)|1\leq j\leq s',\ 1\leq\ell\leq n_j\}.$$

Choose an $\tilde{\mathbf{x}}_{j,\ell}\in X$ for each $(j,\ell)\in I$.

To get $\|\tilde{f}\|_{\tilde{w}}$, we are to bound the sum, over the choices of $n_1,\ldots,n_{s+1}$, of the max, over the choices of I and the $\tilde{\mathbf{x}}_{j,\ell}$'s, of

$$\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\mathbf{x}_{s+1})\in X^{n_1}\times\cdots\times X^{n_{s+1}}\\(\vec{\mathbf{x}}_j)_\ell=\vec{\mathbf{x}}_{j,\ell}\text{ for }(j,\ell)\in I}}\ \sum_{\substack{\vec{\mathbf{y}}_j\in X^{n_j},\ 1\le j\le s\\ \vec{\mathbf{y}}_{s+1}\in X^{(1)}}}\ \sum_{\substack{\vec{\mathbf{z}}_{j,\ell}\in X^{(1)},\ 1\le j\le s,\ 1\le\ell\le n_j\\ \vec{\mathbf{y}}_{s+1}\circ\vec{\mathbf{z}}_{1,1}\circ\cdots\circ\vec{\mathbf{z}}_{s,n_s}=\vec{\mathbf{x}}_{s+1}}} w(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})$$
$$\times|a(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|\prod_{j=1}^{s}\left[\prod_{\ell=1}^{n_j}\mathcal{A}_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})|\right], \tag{4.2}$$

where, for each $1\le j\le s$,

$$\mathcal{A}_j(\mathbf{y};\vec{\mathbf{z}};\mathbf{x})=e^{\tau_d(\operatorname{supp}(\mathbf{y};\vec{\mathbf{z}};\mathbf{x}))}|A_j(\mathbf{y};\vec{\mathbf{z}};\mathbf{x})|\frac{\tilde{\kappa}_j(\mathbf{x})}{\kappa_j(\mathbf{y})}.$$

Observe that

$$(4.2)=\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\in X^{n_1}\times\cdots\times X^{n_{s+1}}\\(\vec{\mathbf{x}}_j)_\ell=\vec{\mathbf{x}}_{j,\ell}\text{ for }(j,\ell)\in I}}\ \sum_{\substack{m_{s+1}\ge 0\\ m_{j,\ell}\ge 0\text{ for }1\le j\le s,\ 1\le\ell\le n_j\\ m_{s+1}+\Sigma_{j,\ell}m_{j,\ell}=n_{s+1}}}\ \sum_{\substack{\vec{\mathbf{y}}_j\in X^{n_j},\ 1\le j\le s\\ \vec{\mathbf{y}}_{s+1}\in X^{m_{s+1}}}}\ \sum_{\substack{\vec{\mathbf{z}}_{j,\ell}\in X^{m_{j,\ell}},\ 1\le j\le s,\ 1\le\ell\le n_j\\ \vec{\mathbf{y}}_{s+1}\circ\vec{\mathbf{z}}_{1,1}\circ\cdots\circ\vec{\mathbf{z}}_{s,n_s}=\vec{\mathbf{x}}_{s+1}}}$$
$$w(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|a(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|\prod_{j=1}^{s}\left[\prod_{\ell=1}^{n_j}\mathcal{A}_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})|\right].$$

In taking the sum, over the choices of $n_1,\ldots,n_{s+1}$, of the max, over the choices of I and the $\tilde{\mathbf{x}}_{j,\ell}$'s, of (4.2), we apply "max $\Sigma\le\Sigma$ max" to give

$$\|\tilde{f}\|_{\tilde{w}}\le\sum_{n_1,\ldots,n_s\ge 0}\ \sum_{\substack{m_{s+1}\ge 0\\ m_{j,\ell}\ge 0\text{ for }1\le j\le s,\ 1\le\ell\le n_j}}\ \max_I\ \max_{\substack{\tilde{\mathbf{x}}_{j,\ell}\in\mathbf{x}\\(j,\ell)\in I}}\ \sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)\in X^{n_1}\times\cdots\times X^{n_s}\\(\vec{\mathbf{x}}_j)_\ell=\vec{\mathbf{x}}_{j,\ell}\text{ for }(j,\ell)\in I}}\ \sum_{\substack{\vec{\mathbf{y}}_j\in X^{n_j},\ 1\le j\le s\\ \vec{\mathbf{y}}_{s+1}\in X^{m_{s+1}}}}\ \sum_{\substack{\vec{\mathbf{z}}_{j,\ell}\in X^{m_{j,\ell}}\\ 1\le j\le s,\ 1\le\ell\le n_j}}$$
$$\times w(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|a(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|\prod_{j=1}^{s}\left[\prod_{\ell=1}^{n_j}\mathcal{A}_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})|\right],$$

where I is chosen as above.

For each $(j,\ell)\notin I$ and $\mathbf{y}_{j,\ell}\in X$, we have, in the notation of Remark 4.3, part (i),

$$\sum_{\substack{\mathbf{x}_{j,\ell}\in X\\ \vec{\mathbf{z}}_{j,\ell}\in X^{m_{j,\ell}}}}\mathcal{A}_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})\le L_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j).$$

Thus,

$$\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)\in X^{n_1}\times\cdots\times X^{n_s}\\(\vec{\mathbf{x}}_j)_\ell=\vec{\mathbf{x}}_{j,\ell}\text{ for }(j,\ell)\in I}}\ \sum_{\substack{\vec{\mathbf{y}}_j\in X^{n_j},\ 1\le j\le s\\ \vec{\mathbf{y}}_{s+1}\in X^{m_{s+1}}}}\ \sum_{\substack{\vec{\mathbf{z}}_{j,\ell}\in X^{m_{j,\ell}}\\ 1\le j\le s,\ 1\le\ell\le n_j}}$$
$$w(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|a(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|\prod_{j=1}^{s}\left[\prod_{\ell=1}^{n_j}\mathcal{A}_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})|\right]$$
$$\le\sum_{\substack{\mathbf{y}_{j,\ell}\\(j,\ell)\in I}}\left(\sum_{\substack{\mathbf{y}_{j,\ell}\\(j,\ell)\notin I}}\ \sum_{\vec{\mathbf{y}}_{s+1}\in X^{m_{s+1}}}w(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|a(\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{s+1})|\right)$$
$$\times\sum_{\substack{\vec{\mathbf{z}}_{j,\ell}\in X^{m_{j,\ell}}\\(j,\ell)\in I}}\ \prod_{(j,\ell)\in I}\mathcal{A}_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})|\prod_{(j,\ell)\notin I}L_{m,\ell}(A_j;\kappa_j,\tilde{\kappa}_j)$$

$$\leqslant \left(\max_{\substack{\mathbf{y}_{j,\ell} \\ (j,\ell)\in I}} \sum_{\substack{\mathbf{y}_{j,\ell} \\ (j,\ell)\notin I}} \sum_{\vec{\mathbf{y}}_{s+1}\in X^{m_{s+1}}} w(\vec{\mathbf{y}}_1, \ldots ,\vec{\mathbf{y}}_{s+1})|a(\vec{\mathbf{y}}_1, \ldots ,\vec{\mathbf{y}}_{s+1})| \right)$$

$$\times \sum_{\substack{\mathbf{y}_{j,\ell} \\ (j,\ell)\in I}} \sum_{\substack{\vec{\mathbf{z}}_{j,\ell}\in X^{m_{j,\ell}} \\ (j,\ell)\in I}} \prod_{(j,\ell)\in I} \mathcal{A}_j\,(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell})| \prod_{(j,\ell)\notin I} L_{m,\ell}(A_j;\kappa_j,\tilde{\kappa}_j)$$

$$\leqslant \|wa\|_{n_1,\ldots,n_s,m_{s+1}} \prod_{(j,\ell)\in I} R_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j) \prod_{(j,\ell)\notin I} L_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j)$$

since, for each $(j,\ell)\in I$,

$$\sum_{\substack{\mathbf{y}_{j,\ell}\in X \\ \vec{\mathbf{z}}_{j,\ell}\in X^{m_{j,\ell}}}} \mathcal{A}_j(\mathbf{y}_{j,\ell};\vec{\mathbf{z}}_{j,\ell};\mathbf{x}_{j,\ell}) \leqslant R_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j).$$

Hence,

$$\|\tilde{f}\|_{\tilde{w}} \leqslant \sum_{n_1,\ldots,n_s\geqslant 0} \sum_{\substack{m_{s+1}\geqslant 0 \\ m_{j,\ell}\geqslant 0}} \max_I \|wa\|_{n_1,\ldots,n_s m_{s+1}} \prod_{(j,\ell)\in I} R_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j) \prod_{(j,\ell)\notin I} L_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j)$$

$$\leqslant \sum_{n_1,\ldots,n_s\geqslant 0} \sum_{\substack{m_{s+1}\geqslant 0 \\ m_{j,\ell}\geqslant 0}} \|wa\|_{n_1,\ldots,n_s m_{s+1}} \prod_{(j,\ell)} \max\{R_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j),L_{m_{j,\ell}}(A_j;\kappa_j,\tilde{\kappa}_j)\}$$

$$\leqslant \sum_{n_1,\ldots,n_s\geqslant 0} \sum_{m_{s+1}\geqslant 0} \|wa\|_{n_1,\ldots,n_s m_{s+1}} \prod_{j=1}^{s} N_d(A_j;\kappa_k,\tilde{\kappa}_j)^{n_j} \leqslant \|f\|_w.$$

■

Proposition 4.4 treats substitutions "field by field." Corollary 4.6, below generalizes Proposition 4.4 to allow substitutions that mix fields. In preparation for the corollary, we have the following.

Lemma 4.5: Let $f(\alpha_1, \ldots ,\alpha_s;\mathfrak{h})$ be an analytic function on a neighborhood of the origin in $\mathbb{C}^{(s+1)|X|}$ with $0\leqslant s' < s$ source fields (see Definition 2.6).

(i) *Let $r\in\mathbb{N}$ and define $\tilde{f}$ by*

$$\tilde{f}(\alpha_1, \ldots ,\alpha_{s-1},\beta_1, \ldots ,\beta_r;\mathfrak{h}) = f(\alpha_1, \ldots ,\alpha_{s-1},\beta_1 + \cdots + \beta_r;\mathfrak{h})$$

Let $\tilde{\kappa}_1, \ldots ,\tilde{\kappa}_r$ be weight factors and assume that

$$\sum_{j=1}^{r} \sup_{x\in X} \frac{\tilde{\kappa}_j(\mathbf{x})}{\kappa_s(\mathbf{x})} \leqslant 1.$$

Denote by $\tilde{w}$ the weight system with metric d that associates the weight factor κ_i to the field α_i, for $1\leqslant i\leqslant s-1$, the weight factor $\tilde{\kappa}_j$ to the field β_j, for $1\leqslant j\leqslant r$, and the constant weight factor 1 to the history field $\mathfrak{h}$. Then,

$$\|\tilde{f}\|_{\tilde{w}} \leqslant \|f\|_w.$$

(ii) *Let $\{1, \ldots ,s\}=I_1\cup\cdots\cup I_r$ be a partition of $\{1, \ldots ,s\}$ into disjoint nonempty subsets. If the number of source fields $s'\geqslant 1$, assume that $I_j=\{j\}$ for all $1\leqslant j\leqslant s'$. Define $\tilde{f}$ by*

$$\tilde{f}(\beta_1, \ldots ,\beta_r;\mathfrak{h}) = f(\alpha_1, \ldots ,\alpha_s;\mathfrak{h})\Big|_{\substack{\alpha_i=\beta_j \text{ if } i\in I_j \\ 1\leqslant i\leqslant s,\ 1\leqslant j\leqslant r}}.$$

Let $\tilde{\kappa}_1, \ldots ,\tilde{\kappa}_r$ be weight factors and assume that $\tilde{\kappa}_j\leqslant\kappa_i$ for all $1\leqslant i\leqslant s$ and $1\leqslant j\leqslant r$ with

$i \in I_j$. Denote by $\tilde{w}$ the weight system with metric d that associates the weight factor $\tilde{\kappa}_j$ to the field β_j, for $1 \leqslant j \leqslant r$, and the constant weight factor 1 to the history field $\mathfrak{h}$. Then,

$$\|\tilde{f}\|_{\tilde{w}} \leqslant \|f\|_w .$$

(iii) *Let $r \in \mathbb{N}$ and let $X = \Lambda_1 \cup \cdots \cup \Lambda_r$ be a partition of X into disjoint subsets. Define f by*

$$\tilde{f}(\alpha_1, \ldots, \alpha_{s-1}, \beta_1, \ldots, \beta_r; \mathfrak{h}) = f(\alpha_1, \ldots, \alpha_{s-1}, \Lambda_1\beta_1 + \cdots + \Lambda_r\beta_r; \mathfrak{h}).$$

Let $\tilde{\kappa}_1, \ldots, \tilde{\kappa}_r$ be weight factors and assume that there is a $\nu > 1$ such that

$$\max_{1 \leqslant j \leqslant r} \sup_{\mathbf{x} \in \Lambda_j} \frac{\tilde{\kappa}_j(\mathbf{x})}{\kappa_s(\mathbf{x})} \leqslant \frac{1}{\nu} < 1 .$$

Denote by $\tilde{w}$ the weight system with metric d that associates the weight factor κ_i to the field α_i, for $1 \leqslant i \leqslant s-1$, the weight factor $\tilde{\kappa}_j$ to the field β_j, for $1 \leqslant j \leqslant r$, and the constant weight factor 1 to the history field $\mathfrak{h}$. Then,

$$\|\tilde{f}\|_{\tilde{w}} \leqslant C_{r,\nu} \|f\|_w \quad \text{with } C_{r,\nu} = \begin{cases} \nu\left(\dfrac{r}{e \ln \nu}\right) & \text{if } \nu \leqslant e^{r/2} \\ \dfrac{2^r}{\nu} & \text{if } e^{r/2} \leqslant \nu < 2^r \\ 1 & \text{if } \nu \geqslant 2^r . \end{cases}$$

Proof:

(i) Let $a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s; \vec{\mathbf{z}})$ be a symmetric coefficient system for f. Since a is invariant under permutation of its $\mathbf{x}_s$ components,

$$\begin{aligned}
&\tilde{f}(\alpha_1, \ldots, \alpha_{s-1}, \beta_1, \ldots, \beta_r; \mathfrak{h}) \\
&\quad = f(\alpha_1, \ldots, \alpha_{s-1}, \beta_1 + \cdots + \beta_r; \mathfrak{h}) \\
&\quad = \sum_{\substack{\vec{\mathbf{x}}_i \in \mathbf{x}^{(1)},\ 1 \leqslant i \leqslant s \\ \vec{\mathbf{z}} \in X^{(1)}}} a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s; \vec{\mathbf{z}}) \left(\prod_{\ell=1}^{s-1} \alpha_\ell(\vec{\mathbf{x}}_\ell)\right) \left(\sum_{j=1}^{r} \beta_j\right)(\vec{\mathbf{x}}_s) \mathfrak{h}(\vec{\mathbf{z}}) \\
&\quad = \sum_{\substack{\vec{\mathbf{x}}_i \in \mathbf{x}^{(1)},\ 1 \leqslant i \leqslant s \\ \vec{\mathbf{z}} \in X^{(1)}}} \sum_{\substack{\vec{\mathbf{y}}_i \in \mathbf{x}^{(1)},\ 1 \leqslant i \leqslant r \\ \vec{\mathbf{x}}_s = \vec{\mathbf{y}}_1 \circ \vec{\mathbf{y}}_1 \circ \cdots \circ \vec{\mathbf{y}}_r}} a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s-1}, \vec{\mathbf{y}}_1 \circ \cdots \circ \vec{\mathbf{y}}_r; \vec{\mathbf{z}}) \\
&\qquad \times \left(\prod_{\ell=1}^{s-1} \alpha_\ell(\vec{\mathbf{x}}_\ell)\right) \binom{n(\vec{\mathbf{y}}_1) + \cdots + n(\vec{\mathbf{y}}_r)}{n(\vec{\mathbf{y}}_1), \ldots, n(\vec{\mathbf{y}}_r)} \left(\prod_{j=1}^{r} \beta_j(\vec{\mathbf{y}}_j)\right) \mathfrak{h}(\vec{\mathbf{z}}),
\end{aligned}$$

we have that

$$\tilde{a}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s-1}, \vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_r; \vec{\mathbf{z}}) = a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s-1}, \vec{\mathbf{y}}_1 \circ \cdots \circ \vec{\mathbf{y}}_r; \vec{\mathbf{z}}) \binom{n(\vec{\mathbf{y}}_1) + \cdots + n(\vec{\mathbf{y}}_r)}{n(\vec{\mathbf{y}}_1), \ldots, n(\vec{\mathbf{y}}_r)}$$

is a symmetric coefficient system for $\tilde{f}$.

When $\vec{\mathbf{x}}_1 = \cdots = \vec{\mathbf{x}}_{s-1} = \vec{\mathbf{y}}_1 = \cdots = \vec{\mathbf{y}}_r = -$, we have

$$\tilde{w}(-,\ldots,-;\vec{\mathbf{z}})|\tilde{a}(-,\ldots,-;\vec{\mathbf{z}})| = w(-,\ldots,-;\vec{\mathbf{z}})|a(-,\ldots,-;\vec{\mathbf{z}})|$$

so that the corresponding contributions to $\|\tilde{f}\|_{\tilde{w}}$ and $\|f\|_w$ are identical. Therefore, we may assume, without loss of generality, that $\tilde{f}(0,\ldots,0;\mathfrak{h})=f(0,\ldots,0;\mathfrak{h})=0$.

Set, for each $1\le j\le r$, $t_j=\sup_{\mathbf{x}\in X}[\tilde{\kappa}_j(\mathbf{x})/\kappa_s(\mathbf{x})]$. Thus, $t_1+\cdots+t_r\le 1$ and $\tilde{\kappa}_j(\mathbf{x})\le t_j\kappa_s(\mathbf{x})$ for all $1\le j\le r$ and all $\mathbf{x}\in X$. If there are no source fields, we have

$$\begin{aligned}
\|\tilde{f}\|_{\tilde{w}} = & \sum_{\substack{n_i\ge 0,\ 1\le i\le s-1\\ m_j\ge 0,\ 1\le j\le r\\ \ell\ge 0}} \max_{\substack{\mathbf{x}\in X\\ 1\le i\le \Sigma n_i+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell\\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\le i\le s-1\\ \vec{\mathbf{y}}_j\in X^{m_j},\ 1\le j\le r\\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r)_i=\mathbf{x}}} \tilde{w}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r;\vec{\mathbf{z}})\\
&\times|\tilde{a}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r;\vec{\mathbf{z}})|\\
\le & \sum_{\substack{n_i\ge 0,\ 1\le i\le s-1\\ m_j\ge 0,\ 1\le j\le r\\ \ell\ge 0}} \max_{\substack{\mathbf{x}\in X\\ 1\le i\le \Sigma n_i+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell\\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\le i\le s-1\\ \vec{\mathbf{y}}_j\in X^{m_j},\ 1\le j\le r\\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r)_i=\mathbf{x}}} t_1^{m_1}\cdots t_r^{m_r} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1\circ\ldots\circ\vec{\mathbf{y}}_r;\vec{\mathbf{z}})\\
&\times\binom{m_1+\cdots+m_r}{m_1,\ldots,m_r}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1\circ\ldots\circ\vec{\mathbf{y}}_r;\vec{\mathbf{z}})|\\
\le & \sum_{\substack{n_i\ge 0,\ 1\le i\le s-1\\ m_j\ge 0,\ 1\le j\le r\\ \ell\ge 0}} \binom{m_1+\cdots+m_r}{m_1,\ldots,m_r} t_1^{m_1}\cdots t_r^{m_r} \max_{\substack{\mathbf{x}\in X\\ 1\le i\le \Sigma n_i+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell\\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\le i\le s-1\\ \vec{\mathbf{x}}_s\in X^{\Sigma m_j}\\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_i=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})\\
&\times|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|\\
\le & \sum_{\substack{n_i\ge 0,\ 1\le i\le s\\ \ell\ge 0}} (t_1+\cdots+t_r)^{n_s} \max_{\substack{\mathbf{x}\in X\\ 1\le i\le \Sigma n_i}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell\\ \vec{\mathbf{x}}\in X^{n_i},\ 1\le i\le s\\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_i=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|\\
\le & \|f\|_w.
\end{aligned}$$

The case when there are source fields is similar.

(ii) It suffices to consider the case that $r=s-1$, $I_r=\{s-1,s\}$, and $I_j=\{j\}$ for all $1\le j\le s-2$. The remaining cases follow by repeatedly reordering the field indices and applying the special case. Let $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{r-1},\vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2;\vec{\mathbf{x}}_{r+1})$ be a symmetric coefficient system for f. As

$$\begin{aligned}
\tilde{f}(\beta_1,\ldots,\beta_r;\mathfrak{h}) = f(\beta_1,\ldots,\beta_{r-1},\beta_r,\beta_r;\mathfrak{h}) &= \sum_{\substack{\vec{\mathbf{x}}_i\in\mathbf{x}^{(1)},\ 1\le i\le r-1\\ \vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2,\vec{\mathbf{z}}\in X^{(1)}}} a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{r-1},\vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2;\vec{\mathbf{z}})\\
&\quad\times\left(\prod_{\ell-1}^{r-1}\beta_\ell(\vec{\mathbf{x}}_\ell)\right)\beta_r(\vec{\mathbf{y}}_1)\beta_r(\vec{\mathbf{y}}_2)\mathfrak{h}(\vec{\mathbf{z}})\\
&= \sum_{\substack{\vec{\mathbf{x}}_i\in\mathbf{x}^{(1)},\ 1\le i\le r\\ \vec{\mathbf{z}}\in X^{(1)}}} \sum_{\substack{\vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2\in\mathbf{x}^{(1)}\\ \vec{\mathbf{x}}_r=\vec{\mathbf{y}}_1\circ\vec{\mathbf{y}}_2}} a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{r-1},\vec{\mathbf{y}}_1,\vec{\mathbf{y}}_2;\vec{\mathbf{z}})\left(\prod_{\ell-1}^{r}\beta_\ell(\vec{\mathbf{x}}_\ell)\right)\mathfrak{h}(\vec{\mathbf{z}}),
\end{aligned}$$

we have that

$$\tilde{a}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_r; \vec{\mathbf{z}}) = \sum_{\substack{\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2 \in x^{(1)} \\ \vec{\mathbf{x}}_r = \vec{\mathbf{y}}_1 \circ \vec{\mathbf{y}}_2}} a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2; \vec{\mathbf{z}})$$

is a, not necessarily symmetric, coefficient system for $\tilde{f}$. Again we may assume, without loss of generality, that $\tilde{f}(0, \ldots, 0; \mathfrak{h}) = f(0, \ldots, 0, 0; \mathfrak{h}) = 0$.

If there are no source fields, we have, by Remark 2.7, part (i),

$$\begin{aligned}
\|\tilde{f}\|_{\tilde{w}} &\leqslant \sum_{\substack{n_1 + \cdots + n_r \geqslant 1 \\ \ell \geqslant 0}} \max_{\mathbf{x} \in X} \max_{\substack{1 \leqslant \bar{J} \leqslant r \\ n_{\bar{J}} \neq 0}} \max_{1 \leqslant \bar{\imath} \leqslant n_{\bar{J}}} \sum_{\substack{\vec{\mathbf{z}} \in X^\ell \\ (\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_r) \in X^{n_1} \times \cdots \times X^{n_r} \\ (\vec{\mathbf{x}}_{\bar{J}})_{\bar{J}} = \mathbf{x}}} \tilde{w}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_r; \vec{\mathbf{z}}) |\tilde{a}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_r; \vec{\mathbf{z}})| \\
&\leqslant \sum_{\substack{n_1 + \cdots n_r \geqslant 1 \\ \ell \geqslant 0}} \max_{\mathbf{x} \in X} \max_{\substack{1 \leqslant \bar{J} \leqslant r \\ n_{\bar{J}} \neq 0}} \max_{1 \leqslant \bar{\imath} \leqslant n_{\bar{J}}} \sum_{\substack{\vec{\mathbf{z}} \in X^\ell \\ \vec{\mathbf{x}}_j \in X^{n_j},\ 1 \leqslant j \leqslant r \\ (\vec{\mathbf{x}}_{\bar{J}})_{\bar{\imath}} = \mathbf{x}}} \sum_{\substack{\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2 \in X^{(1)} \\ \vec{\mathbf{x}}_r = \vec{\mathbf{y}}_1 \circ \vec{\mathbf{y}}_2}} w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2; \vec{\mathbf{z}}) \\
&\quad \times |a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2; \vec{\mathbf{z}})| \\
&= \sum_{\substack{n_1 + \cdots + n_r \geqslant 1 \\ \ell \geqslant 0}} \max_{\substack{\mathbf{x} \in X \\ 1 \leqslant i \leqslant n_1 + \cdots + n_r}} \sum_{\substack{m_1, m_2 \geqslant 0 \\ m_1 + m_2 = n_r}} \sum_{\substack{\vec{\mathbf{z}} \in X^\ell \\ \vec{\mathbf{y}}_1 \in X^{m_1},\ \vec{y}_2 \in X^{m_2} \\ \vec{\mathbf{x}}_j \in x^{n_j},\ 1 \leqslant j \leqslant r-1 \\ (\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2)_i = \mathbf{x}}} w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2; \vec{\mathbf{z}}) \\
&\quad \times |a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2; \vec{\mathbf{z}})| \\
&\leqslant \sum_{\substack{n_j \geqslant 0,\ 1 \leqslant j \leqslant r-1 \\ m_1, m_2 \geqslant 0 \\ \ell \geqslant 0}} \max_{\substack{\mathbf{x} \in X \\ 1 \leqslant i \leqslant n_1 + \cdots + n_{r-1} + m_1 + m_2}} \sum_{\substack{\vec{\mathbf{z}} \in X^\ell \\ \vec{\mathbf{y}}_1 \in X^{m_1},\ \vec{y}_2 \in X^{m_2} \\ \vec{\mathbf{x}}_j \in x^{n_j},\ 1 \leqslant j \leqslant r-1 \\ (\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2)_i = \mathbf{x}}} w(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2; \vec{\mathbf{z}}) \\
&\quad \times |a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{r-1}, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2; \vec{\mathbf{z}})| = \|f\|_w .
\end{aligned}$$

The case when there are source fields is similar.

(iii) Let $a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_s; \vec{\mathbf{z}})$ be a symmetric coefficient system for f. As in part (i), we have that

$$\tilde{a}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s-1}, \vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_r; \vec{\mathbf{z}}) = a(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s-1}, \vec{\mathbf{y}}_1 \circ \cdots \circ \vec{\mathbf{y}}_r; \vec{\mathbf{z}}) \left(\prod_{j=1}^r \Lambda_j(\vec{\mathbf{y}}_j) \right) \times \binom{n(\vec{\mathbf{y}}_1) + \cdots + n(\vec{\mathbf{y}}_r)}{n(\vec{\mathbf{y}}_1), \cdots, n(\vec{\mathbf{y}}_r)}$$

is a symmetric coefficient system for $\tilde{f}$. Again, we may assume, without loss of generality, that $\tilde{f}(0, \ldots, 0; \mathfrak{h}) = f(0, \ldots, 0; \mathfrak{h}) = 0$.

By hypothesis, $\tilde{\kappa}_j(\mathbf{x}) \leqslant (1/v)\kappa_s(\mathbf{x})$ for all $1 \leqslant j \leqslant r$ and all $\mathbf{x} \in \Lambda_j$. If there are no source fields, we have

$$\|\tilde{f}\|_{\tilde{w}} = \sum_{\substack{n_i \geq 0,\ 1\leq i\leq s-1 \\ m_j\geq 0,\ 1\leq j\leq r \\ \ell\geq 0}} \max_{\substack{\mathbf{x}\in X \\ 1\leq p\leq \Sigma n_i+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{y}}_j\in X^{m_j},\ 1\leq j\leq r \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r)_p=\mathbf{x}}} \tilde{w}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r;\vec{\mathbf{z}})$$

$$\times|\tilde{a}(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r;\vec{\mathbf{z}})|$$

$$\leq \sum_{\substack{n_i \geq 0,\ 1\leq i\leq s-1 \\ m_j\geq 0,\ 1\leq j\leq r \\ \ell\geq 0}} \max_{\substack{\mathbf{x}\in X \\ 1\leq p\leq \Sigma n_j+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{y}}_j\in \Lambda_j^{m_j},\ 1\leq j\leq r \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r)_p=\mathbf{x}}} \left(\frac{1}{v}\right)^{m_1+\cdots+m_r} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r;\vec{\mathbf{z}})$$

$$\times\binom{m_1+\cdots+m_r}{m_1,\cdots,m_r}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r;\vec{\mathbf{z}})|. \tag{4.3}$$

We claim that

$$\max_{\substack{\mathbf{x}\in X \\ 1\leq p\leq \Sigma n_j+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{y}}_j\in \Lambda_j^{m_j},\ 1\leq j\leq r \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r)_p=\mathbf{x}}} \binom{m_1+\cdots+m_r}{m_1,\cdots,m_r} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r;\vec{\mathbf{z}})|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1$$

$$\circ\cdots\circ\vec{\mathbf{y}}_r;\vec{\mathbf{z}})| \leq (m_1+\cdots+m_r) \max_{\substack{\mathbf{x}\in X \\ 1\leq p\leq \Sigma n_j+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{x}}_s\in X^{\Sigma m_j} \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_p=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})$$

$$\times|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|. \tag{4.4}$$

To see this, fix $\mathbf{x}$ to be element of X and p to be the integer between 1 and $\Sigma n_i+\Sigma m_j$ that give the maximum for the left hand side. In the event that $p>\Sigma_{i=1}^{s-1}n_i$, set $\bar{p}=p-\Sigma_{i=1}^{s-1}n_i$ and let $1\leq\bar{J}\leq r$ obey $\mathbf{x}\in\Lambda_{\bar{J}}$. Denote by $n_\Lambda(\vec{\mathbf{x}})$ the number of components of $\vec{\mathbf{x}}$ that are in Λ. Then,

$$\sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{x}}_s\in X^{\Sigma m_j} \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_p=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})| \geq \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i\in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{x}}_s\in X^{\Sigma m_j} \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_p=\mathbf{x} \\ n_{\Lambda_j}(\vec{\mathbf{x}})=m_j,\ 1\leq j\leq r}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})$$

$$\times|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|.$$

To each $\vec{\mathbf{x}}_s\in X^{\Sigma m_j}$ with $n_{\Lambda_j}(\vec{\mathbf{x}}_s)=m_j$ for each $1\leq j\leq r$ and with $(\vec{\mathbf{x}}_s)_{\bar{p}}=\mathbf{x}$ (if $\bar{p}$ is defined) we assign the unique permutation $\pi\in S_{\Sigma m_j}$ with

- $\pi(\bar{p})=\bar{p}$ (if $\bar{p}$ is defined) and
- $\pi(1)$ being the index of the first component of $\vec{\mathbf{x}}_s$ [other than $(\vec{\mathbf{x}}_s)_{\bar{p}}$ if $\bar{p}$ is defined] with $(\vec{\mathbf{x}}_s)_{\pi(1)}\in\Lambda_1$,
- $\pi(2)$ being the index of the second component of $\vec{\mathbf{x}}_s$ [other than $(\vec{\mathbf{x}}_s)_{\bar{p}}$ if $\bar{p}$ is defined] with $(\vec{\mathbf{x}}_s)_{\pi(2)}\in\Lambda_1$,

- $\pi(\Sigma m_j)$ being the index of the last component of $\vec{\mathbf{x}}_s$ [other than $(\vec{\mathbf{x}}_s)_{\bar{p}}$ if $\bar{p}$ is defined] with $(\vec{\mathbf{x}}_s)_{\pi(\Sigma m_j)} \in \Lambda_r$.

The set Π of such permutations contains exactly $\binom{m_1+\cdots+m_r}{m_1,\ldots,m_r}$ elements if $\bar{p}$ is not defined and

$$\binom{m_1+\cdots+m_r-1}{m_1,\ldots,m_{\bar{j}}-1,\ldots,m_r} = \frac{m_{\bar{j}}}{m_1+\cdots+m_r}\binom{m_1+\cdots+m_r}{m_1,\ldots,m_r} \geq \frac{1}{m_1+\cdots+m_r}\binom{m_1+\cdots+m_r}{m_1,\ldots,m_r}$$

elements if $\bar{p}$ is defined. Hence, if we rename the components of $\vec{\mathbf{x}}_s$ (in order) that are in Λ_j to be $\vec{\mathbf{y}}_j$ so that $\vec{\mathbf{x}}_s = \pi^{-1}(\vec{\mathbf{y}}_1 \circ \cdots \circ \vec{\mathbf{y}}_r)$,

$$\begin{aligned}
&\sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i \in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{x}}_s \in X^{\Sigma m_j} \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_{\bar{p}}=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{x}}_s;\vec{\mathbf{z}})| \\
&\geq \sum_{\pi\in\Pi} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i \in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{y}}_j \in \Lambda_j^{m_j},\ 1\leq j\leq r \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\pi^{-1}(\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r))_p=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})\, |a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|\big|_{\vec{\mathbf{x}}_s=\pi^{-1}(\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r)} \\
&= \sum_{\pi\in\Pi} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i \in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{y}}_j \in \Lambda_j^{m_j},\ 1\leq j\leq r \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r)_p=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})\, |a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|\big|_{\vec{\mathbf{x}}_s=\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r} \\
&= \#(\Pi) \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i \in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{y}}_j \in \Lambda_j^{m_j},\ 1\leq j\leq r \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r)_p=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})\, |a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})|\big|_{\vec{\mathbf{x}}_s=\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r}
\end{aligned}$$

since $w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})$ and $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})$ are invariant under permutations of the components of $\vec{\mathbf{x}}_s$ and the condition $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\pi^{-1}(\vec{\mathbf{y}}_1\circ\cdots\circ\vec{\mathbf{y}}_r))_p=\mathbf{x}$ is equivalent to the condition $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s-1},\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_r)_p=\mathbf{x}$. This yields (4.4).

Substituting (4.4) into (4.3), we have

$$\begin{aligned}
\|\tilde{f}\|_{\tilde{w}} &\leq \sum_{\substack{n_i\geq 0,\ 1\leq i\leq s-1 \\ m_j\geq 0,\ 1\leq j\leq r \\ \ell\geq 0}} \left(\frac{1}{\nu}\right)^{m_1+\cdots+m_r}(m_1+\cdots+m_r) \max_{\substack{\mathbf{x}\in X \\ 1\leq p\leq \Sigma n_i+\Sigma m_j}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i \in X^{n_i},\ 1\leq i\leq s-1 \\ \vec{\mathbf{x}}_s\in X^{\Sigma m_j} \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_p=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}}) \\
&\quad\times|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})| \leq \sum_{\substack{n_i\geq 0,\ 1\leq i\leq s \\ \ell\geq 0}} (n_s+1)^{r-1} n_s\left(\frac{1}{\nu}\right)^{n_s} \max_{\substack{\mathbf{x}\in X \\ 1\leq p\leq \Sigma n_i}} \sum_{\substack{\vec{\mathbf{z}}\in X^\ell \\ \vec{\mathbf{x}}_i \in X^{n_i},\ 1\leq i\leq s \\ (\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)_p=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}}) \\
&\quad\times|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{z}})| \leq \|f\|_w \sup_{n_s\geq 0}(n_s+1)^r\left(\frac{1}{\nu}\right)^{n_s} \leq C_{r,\nu}\|f\|_w,
\end{aligned}$$

where

$$C_{r,\nu} = \begin{cases} \nu\left(\dfrac{r}{e \ln \nu}\right)^r & \text{if } \nu \leqslant e^{1/2} \\ \dfrac{2^r}{\nu} & \text{if } e^{1/2} \leqslant \nu < 2^r \\ 1 & \text{if } \nu \geqslant 2^r. \end{cases}$$

The case when there are source fields is similar, but easier since there is no condition $(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{s-1}, \vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_r)_p = \mathbf{x}$ with $p > \sum_{i=1}^{s-1} n_i$. ■

Corollary 4.6: Let $h(\gamma_1, \ldots, \gamma_r; \mathfrak{h})$ *be an analytic function on a neighborhood of the origin in* $\mathbb{C}^{(r+1)|X|}$*, and let* Γ^i_j *be* $\mathfrak{h}$*-operators on* $\mathbb{C}^X$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$. *Set*

$$\widetilde{h}(\alpha_1, \ldots, \alpha_s; \mathfrak{h}) = h\left(\sum_{i=1}^{s} \Gamma^i_1 \alpha_i, \ldots, \sum_{i=1}^{s} \Gamma^i_r \alpha_i; \mathfrak{h}\right).$$

Furthermore, let $\lambda_j : X \to (0, \infty]$ *for* $1 \leqslant j \leqslant r$ *be weight factors. Let* w_λ *be the weight system with metric d that associates the weight factor* λ_j *and the constant weight factor* 1 *to the history field* $\mathfrak{h}$. *Assume that*

$$\sum_{i=1,\ldots,s} N_d(\Gamma^i_j; \lambda_j, \kappa_i) \leqslant 1$$

for each $1 \leqslant j \leqslant r$. *Then*,

$$\|\widetilde{h}\|_w \leqslant \|h\|_{w_\lambda}.$$

Proof: We introduce auxiliary fields $\{\alpha_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}$ and $\{\beta_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}$ and define

$$h'(\{\beta_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}; \mathfrak{h}) = h\left(\sum_{i=1}^{s} \beta_{i1}, \ldots, \sum_{i=1}^{s} \beta_{ir}; \mathfrak{h}\right),$$

$$h''(\{\alpha_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}; \mathfrak{h}) = h'(\{\Gamma^i_j \alpha_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}; \mathfrak{h}).$$

Then,

$$\widetilde{h}(\alpha_1, \ldots, \alpha_s; \mathfrak{h}) = h''(\{\alpha_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}; \mathfrak{h})\Big|_{1\leqslant i\leqslant s,\ 1\leqslant j\leqslant r}^{\alpha_{i,j}=\alpha_i}.$$

Set $t_{i,j} = N_d(\Gamma^i_j; \lambda_j, \kappa_i)$ and introduce the auxiliary weight system w' with metric d that associates the weight factor one to the history field,

- the weight factor κ_i to the field $\alpha_{i,j}$ and
- the weight factor $t_{i,j}\lambda_j$ to the field $\beta_{i,j}$.

By part (ii) of Lemma 4.5,

$$\|\widetilde{h}(\alpha_1, \ldots, \alpha_s; \mathfrak{h})\|_w \leqslant \|h''(\{\alpha_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}; \mathfrak{h})\|_{w'}.$$

By Proposition 4.4,

$$\|h''(\{\alpha_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}; \mathfrak{h})\|_{w'} \leqslant \|h'(\{\beta_{i,j}\}_{1\leqslant j\leqslant r}^{1\leqslant i\leqslant s}; \mathfrak{h})\|_{w'}$$

since

$$N_d(\Gamma^i_j; t_{i,j}\lambda_j, \kappa_i) = \frac{1}{t_{i,j}} N_d(\Gamma^i_j; \lambda_j, \kappa_i) = 1.$$

By part (i) of Lemma 4.5,

$$\|h'(\{\beta_{i,j}\}_{1\le j\le r}^{1\le i\le s};\mathfrak{h})\|_{w'} \le \|h(\gamma_1,\ldots,\gamma_r;\mathfrak{h})\|_{w_\lambda}$$

since, for each $1 \le j \le r$,

$$\sum_{i=1}^{s} \sup_{\mathbf{x}} \frac{t_{i,j}\lambda_j(\mathbf{x})}{\lambda_j(\mathbf{x})} = \sum_{i=1}^{s} t_{i,j} = \sum_{i=1}^{s} N_d(\Gamma_j^i;\lambda_j,\kappa_i) \le 1 .$$

■

APPENDIX A: CHANGE OF VARIABLES FORMULAS IN A SIMPLE SETTING

The results of Secs. II–IV will be applied in their full generality to the construction of the temporal ultraviolet limit of a many boson system in Ref. 4. In the second part of the paper,[3] we discuss the small field part of this construction. For it we need neither history nor source fields and can do with constant weight factors as in (1.4). For this reason, we specialize our main change of variables formulas, Corollary 4.6 to this setting (see Corollary A.2, below). We also provide another change of variables formula, Proposition A.3, which is special for multilinear functions.

For an abstract framework, we consider analytic functions $f(\alpha_1,\ldots,\alpha_s)$ of the complex fields $\alpha_1,\ldots,\alpha_s$, none of which are history or source fields. We assume that we are given a metric d and *constant* weight factors $\kappa_1,\ldots,\kappa_s$. Denote by w the weight system with metric d that associates the weight factor κ_j to the field α_j.

In this environment, Definition 2.6, for the norm of the function

$$f(\alpha_1,\ldots,\alpha_s) = \sum_{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)\in\mathbf{x}(s)} a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)\alpha_1(\vec{\mathbf{x}}_1)\cdots\alpha_s(\vec{\mathbf{x}}_s)$$

with $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)$ a symmetric coefficient system, simplifies to

$$\|f\|_w = |a(-)| + \sum_{\substack{n_1,\ldots,n_s\ge 0\\ n_1,\ldots,n_s\ge 1}} \max_{\mathbf{x}\in X} \max_{\substack{1\le j\le s\\ n_j\ne 0}} \max_{1\le i\le n_j} \sum_{\substack{\vec{\mathbf{x}}_\ell\in X^{n_\ell}\\ 1\le \ell\le s\\ (\vec{\mathbf{x}}_j)_i=\mathbf{x}}} |a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)|\kappa_1^{n_1}\cdots\kappa_s^{n_s}e^{\tau_d(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s)}. \quad \text{(A1)}$$

As well, for the change in variable formulas of Sec. IV, one only needs a special case of the operator norm in Definition 4.2.

Definition A.1: Let A be a linear map on $\mathbb{C}^X$. We define the operator norm

$$|||A||| = N_d(A;\,1,\,1) = \max\left\{\sup_{\mathbf{x}\in X}\sum_{\mathbf{y}\in X} e^{d(\mathbf{x},\mathbf{y})}|A(\mathbf{x},\mathbf{y})|, \sup_{\mathbf{y}\in X}\sum_{\mathbf{x}\in X} e^{e(\mathbf{x},\mathbf{y})}|A(\mathbf{x},\mathbf{y})|\right\}.$$

Observe that for constant weight factors λ and κ, $N_d(A;\lambda,\kappa) = (\kappa/\lambda)|||A|||$. Hence, Corollary 4.6 is simplified as follows.

Corollary A.2: Let $h(\gamma_1,\ldots,\gamma_r)$ be an analytic function on a neighborhood of the origin in $\mathbb{C}^{r|X|}$, and let Γ_j^i be linear operators on $\mathbb{C}^X$, $1\le i\le s$, $1\le j\le r$. Set

$$\tilde{h}(\alpha_1,\ldots,\alpha_s) = h\left(\sum_{i=1}^{s}\Gamma_1^i\alpha_i,\ldots,\sum_{i=1}^{s}\Gamma_r^i\alpha_i\right).$$

Furthermore, let $\lambda_1,\ldots,\lambda_r$ be constant weight factors and let w_λ be the weight system with metric d that associates the weight factor λ_j to the field γ_j. Assume that

$$\sum_{i=1}^{s}\frac{\kappa_i}{\lambda_j}|||\Gamma_j^i||| \le 1$$

for each $1 \leqslant j \leqslant r$. *Then,*

$$\|\tilde{h}\|_w \leqslant \|h\|_{w_\lambda}.$$

Proposition A.3: Let $h(\gamma_1, \ldots, \gamma_r)$ *be a multilinear form in the fields* $\gamma_1, \ldots, \gamma_r$. *Furthermore, let*

$$\Gamma_i^j, \quad \tilde{\Gamma}_i^j, \quad A_j, \quad \text{and} \quad \tilde{A}_j$$

$(i=1, \ldots, r;\ j=1, \ldots, s)$ *be linear operators on* $\mathbb{C}^X$. *Set*

$$f_1(\alpha_1, \ldots, \alpha_s) = h\left(\sum_{j=1}^{s} \Gamma_1^j \alpha_j, \ldots, \sum_{j=1}^{s} \Gamma_r^j \alpha_j\right),$$

$$f_2(\alpha_1, \ldots, \alpha_s) = h\left(\sum_{j=1}^{s} \Gamma_1^j \alpha_j, \ldots, \sum_{j=1}^{s} \Gamma_r^j \alpha_j\right) - h\left(\sum_{j=1}^{s} \tilde{\Gamma}_1^j \alpha_j, \ldots, \sum_{j=1}^{s} \tilde{\Gamma}_r^j \alpha_j\right),$$

$$f_3(\alpha_1, \ldots, \alpha_s) = h\left(\sum_{j=1}^{s} A_1 \Gamma_1^j \alpha_j, \ldots, \sum_{j=1}^{s} A_r \Gamma_r^j \alpha_j\right) - h\left(\sum_{j=1}^{s} \tilde{A}_1 \Gamma_1^j \alpha_j, \ldots, \sum_{j=1}^{s} \tilde{A}_r \Gamma_r^j \alpha_j\right)$$
$$- h\left(\sum_{j=1}^{s} A_1 \tilde{\Gamma}_1^j \alpha_j, \ldots, \sum_{j=1}^{s} A_r \tilde{\Gamma}_r^j \alpha_j\right) + h\left(\sum_{j=1}^{s} \tilde{A}_1 \tilde{\Gamma}_1^j \alpha_j, \ldots, \sum_{j=1}^{s} \tilde{A} \tilde{\Gamma}_r^j \alpha_j\right).$$

Furthermore, let $\lambda_1, \ldots, \lambda_r$ *be constant weight factors and let* w_λ *be the weight system with metric* d *that associates the weight factor* λ_j *to the field* γ_j. *Then,*

(i)

$$\|f_1\|_w \leqslant \|h\|_{w_\lambda} \prod_{i=1}^{r} \left(\sum_{j=1}^{s} \frac{\kappa_j}{\lambda_i} |||\Gamma_i^j|||\right).$$

(ii)

$$\|f_2\|_w \leqslant r \|h\|_{w_\lambda} \sigma_\delta \, \sigma^{r-1},$$

where

$$\sigma = \max_{i=1,\ldots,r} \max\left\{\sum_{j=1}^{s} \frac{\kappa_j}{\lambda_i} |||\Gamma_i^j|||, \sum_{j=1}^{s} \frac{\kappa_j}{\lambda_i} |||\tilde{\Gamma}_i^j|||\right\},$$

$$\sigma_\delta = \max_{i=1,\ldots,r} \sum_{j=1}^{s} \frac{\kappa_j}{\lambda_i} |||\Gamma_i^j - \tilde{\Gamma}_i^j|||.$$

(iii)

$$\|f_3\|_w \leqslant r^2 \|h\|_{\omega_\lambda} \sigma_\delta \, a_\delta \, (\sigma a)^{r-1},$$

where

$$a = \max_{i=1,\ldots,r} \max\{|||A_i|||, |||\tilde{A}_i|||\}, \quad a_\delta = \max_{i=1,\ldots,r} |||A_i - \tilde{A}_i|||.$$

Proof:

(i) Define, for each $1 \leq i \leq r$, $t_i = \Sigma_{j=1}^{s} (\kappa_j/\lambda_i) |||\Gamma_i^j|||$. Set

$$\tilde{h}(\alpha_1, \ldots, \alpha_s) = h\left(\sum_{j=1}^{s} \frac{1}{t_1}\Gamma_1^j \alpha_j, \cdots, \sum_{j=1}^{s} \frac{1}{t_r}\Gamma_r^j \alpha_j\right).$$

By multilinearity, $f_1 = (t_1 \cdots t_r)\tilde{h}$, and, by Corollary A.2, $\|\tilde{h}\|_w \leq \|h\|_{w_\lambda}$.

(ii) Write the telescoping sum

$$f_2(\alpha_1, \ldots, \alpha_s) = h\left(\sum_{j=1}^{s} (\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \sum_{j=1}^{s} \Gamma_2^j \alpha_j, \ldots, \sum_{j=1}^{s} \Gamma_r^j \alpha_j\right) + h\left(\sum_{j=1}^{s} \tilde{\Gamma}_1^j \alpha_j, \sum_{j=1}^{s} (\Gamma_2^j - \tilde{\Gamma}_2^j)\alpha_j, \ldots, \sum_{j=1}^{s} \Gamma_r^j \alpha_j\right) + \cdots + h\left(\sum_{j=1}^{s} \tilde{\Gamma}_1^j \alpha_j, \sum_{j=1}^{s} \tilde{\Gamma}_2^j \alpha_j, \ldots, \sum_{j=1}^{s} (\tilde{\Gamma}_r^j - \tilde{\Gamma}_r^j)\alpha_j\right)$$

and apply part (i) to each term.

(iii) Write the telescoping sum

$$\begin{aligned} f_3(\alpha_1, \ldots, \alpha_s) = {} & h\left(\sum_{j=1}^{s} A_1(\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \ldots, \sum_{j=1}^{s} A_r \Gamma_r^j \alpha_j\right) - h\left(\sum_{j=1}^{s} \tilde{A}_1(\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \ldots, \sum_{j=1}^{s} \tilde{A}_r \Gamma_r^j \alpha_j\right) \\ & + \cdots + h\left(\sum_{j=1}^{s} A_1 \tilde{\Gamma}_1^j \alpha_j, \ldots, \sum_{j=1}^{s} A_r(\tilde{\Gamma}_r^j - \tilde{\Gamma}_r^j)\alpha_j\right) \\ & - h\left(\sum_{j=1}^{s} \tilde{A}_1 \tilde{\Gamma}_1^j \alpha_j, \ldots, \sum_{j=1}^{s} \tilde{A}_r(\tilde{\Gamma}_r^j - \tilde{\Gamma}_r^j)\alpha_j\right). \end{aligned}$$

We claim that the $\|\cdot\|_w$ norm of each of the r lines is bounded by $r\|h\|_{w_\lambda} \sigma_\delta a_\delta (\sigma a)^{r-1}$. We prove this for the first line. The proof for the other lines is similar. We again write a telescoping sum

$$\begin{aligned} & h\left(\sum_{j=1}^{s} A_1(\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \ldots, \sum_{j=1}^{s} A_r \Gamma_r^j \alpha_j\right) - h\left(\sum_{j=1}^{s} \tilde{A}_1(\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \ldots, \sum_{j=1}^{s} \tilde{A}_r \Gamma_r^j \alpha_j\right) \\ & = h\left(\sum_{j=1}^{s} (A_1 - \tilde{A}_1)(\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \sum_{j=1}^{s} A_2 \Gamma_2^j \alpha_j, \ldots, \sum_{j=1}^{s} A_r \Gamma_r^j \alpha_j\right) \\ & \quad + h\left(\sum_{j=1}^{s} \tilde{A}_1(\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \sum_{j=1}^{s} (A_2 - \tilde{A}_2)\Gamma_2^j \alpha_j, \ldots, \sum_{j=1}^{s} A_r \Gamma_r^j \alpha_j\right) \\ & \quad + \cdots + h\left(\sum_{j=1}^{s} \tilde{A}_1(\Gamma_1^j - \tilde{\Gamma}_1^j)\alpha_j, \sum_{j=1}^{s} \tilde{A}_2 \Gamma_2^j \alpha_j, \ldots, \sum_{j=1}^{s} (A_r - \tilde{A}_r)\Gamma_r^j \alpha_j\right). \end{aligned}$$

By the first bound, the $\|\cdot\|_w$ norm of the first term is bounded by

$$\begin{aligned} & \|h\|_{w_\lambda} \left(\sum_{j=1}^{s} \frac{\kappa_j}{\lambda_1} |||(A_1 - \tilde{A}_1)(\Gamma_1^j - \tilde{\Gamma}_1^j)|||\right) \prod_{i=2}^{r} \left(\sum_{j=1}^{s} \frac{\kappa_j}{\lambda_i} |||A_i \Gamma_i^j|||\right) \\ & \quad \leq \|h\|_{w_\lambda} |||A_1 - \tilde{A}_1||| \left(\sum_{j=1}^{s} \frac{\kappa_j}{\lambda_1} |||\Gamma_1^j - \tilde{\Gamma}_1^j|||\right) \prod_{i=2}^{r} \left(|||A_i||| \sum_{j=1}^{s} \frac{\kappa_2}{\lambda_i} |||\Gamma_i^j|||\right) \\ & \quad \leq \|h\|_{w_\lambda} \sigma_\delta a_\delta (\sigma a)^{r-1} \end{aligned}$$

by Remark 4.3, part (ii). The norm of the second term is bounded by

$$\|h\|_{w_\lambda}\left(\sum_{j=1}^{s}\frac{\kappa_j}{\lambda_1}|||\tilde{A}_1(\Gamma_1^j-\tilde{\Gamma}_1^j)|||\right)\left(\sum_{j=1}^{s}\frac{\kappa_j}{\lambda_2}|||(A_2-\tilde{A}_2)\Gamma_2^j|||\right)\prod_{i=3}^{r}\left(\sum_{j=1}^{s}\frac{\kappa_j}{\lambda_i}|||A_i\Gamma_i^j|||\right)$$
$$\leqslant \|h\|_{w_\lambda}(\sigma_\delta a)(\sigma a_\delta)(\sigma a)^{r-2}.$$

Similarly, one bounds the norms of each of the other $r-2$ terms by $\|h\|_{w_\lambda}\sigma_\delta a_\delta(\sigma a)^{r-1}$. ■

APPENDIX B: A SUPREMUM BOUND

Lemma B.1: Let $f(\alpha_1,\ldots,\alpha_s;\mathfrak{h})$ be a function that is defined and analytic on a neighborhood of the origin in $\mathbb{C}^{(s+1)|X|}$. Let w be the weight system with metric d that associates the weight factor κ_j to the field α_j. Let $S\subset X$ be nonempty and assume that $f|_{\substack{\mathfrak{h}(\mathbf{x})=0\\ \text{for all }\mathbf{x}\in S}}=0$. Then,

$$\sup\{|f(\alpha_1,\ldots,\alpha_s;\mathfrak{h})|\,|\,|\alpha_j(\mathbf{x})|\leqslant\kappa_j(\mathbf{x}),\ |\mathfrak{h}(\mathbf{x})|\leqslant 1 \textit{ for all } \mathbf{x}\in X,\ 1\leqslant j\leqslant s\}\leqslant K_d\|f\|_w\#S,$$

where $K_d=\sup_{\mathbf{y}\in X}\Sigma_{\mathbf{x}\in X}e^{-d(\mathbf{x},\mathbf{y})}$.

Proof: Denote by $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})$ the symmetric coefficient system for f. Then, if $|\alpha_j(\mathbf{x})|\leqslant\kappa_j(\mathbf{x})$ and $|\mathfrak{h}(\mathbf{x})|\leqslant 1$ for all $\mathbf{x}\in X$ and $1\leqslant j\leqslant s$, we have

$$|f(\alpha_1,\ldots,\alpha_s;\mathfrak{h})|\leqslant\sum_{n_1,\ldots,n_{s+1}}\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\\ \in X^{n_1}\times\cdots\times X^{n_{s+1}}}}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})\alpha_1(\vec{\mathbf{x}}_1)\cdots\alpha_s(\vec{\mathbf{x}}_s)\mathfrak{h}(\vec{\mathbf{x}}_{s+1})|$$
$$\leqslant\sum_{n_1,\ldots,n_{s+1}}\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\\ \in X^{n_1}\times\cdots\times X^{n_{s+1}}}}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|\kappa_1(\vec{\mathbf{x}}_1)\cdots\kappa_s(\vec{\mathbf{x}}_s).$$

It suffices to prove that, for each $n_1,\ldots,n_{s+1}\geqslant 0$,

$$\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\\ \in X^{n_1}\times\cdots\times X^{n_{s+1}}}}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|\kappa_1(\vec{\mathbf{x}}_1)\cdots\kappa_s(\vec{\mathbf{x}}_s)$$
$$\leqslant K_d\#S\|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})\|_{n_1,\ldots,n_{s+1}}.$$

The cases with $n_1+\cdots+n_s=0$ are trivial, so fix any $n_1,\ldots,n_{s+1}\geqslant 0$ with $n_1+\cdots+n_s>0$. For notational convenience, suppose that $n_1>0$. By hypothesis $a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})=0$ unless at least one component of $\vec{\mathbf{x}}_{s+1}$ is in S, in which case $\tau_d(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})$ is greater than the distance from the first component of $\vec{\mathbf{x}}_1$ to S. So, suppressing from the notation that $(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\in X^{n_1}\times\cdots\times X^{n_{s+1}}$,

$$\sum_{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|\kappa_1(\vec{\mathbf{x}}_1)\cdots\kappa_s(\vec{\mathbf{x}}_s)$$
$$\leqslant\sum_{\mathbf{x}\in X}\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\\ (\vec{x}_1)_1=\mathbf{x}}}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|\kappa_1(\vec{\mathbf{x}}_1)\cdots\kappa_s(\vec{\mathbf{x}}_s)$$
$$\leqslant\sum_{\mathbf{x}\in X}e^{-d(\mathbf{x},S)}\sum_{\substack{(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_{s+1})\\ (\vec{x}_1)_1=\mathbf{x}}}|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})|w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})$$
$$\leqslant\|a(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})w(\vec{\mathbf{x}}_1,\ldots,\vec{\mathbf{x}}_s;\vec{\mathbf{x}}_{s+1})\|_{n_1,\ldots,n_{s+1}}\sum_{\mathbf{x}\in X}e^{-d(\mathbf{x},S)}.$$

The lemma now follows from

$$\sum_{\mathbf{x}\in X} e^{-d(\mathbf{x},S)} \leqslant \sum_{\mathbf{x}\in X}\sum_{\mathbf{y}\in S} e^{-d(\mathbf{x},\mathbf{y})} \leqslant K_d \# S.$$

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