Power series representations for complex bosonic effective actions. II. A small field renormalization group flow

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In a previous paper, we developed a power series representation and estimates for an effective action of the form $\ln[\int e^{f(\alpha_1,\ldots,\alpha_s;z^*,z)}d\mu(z^*,z)/\int e^{f(0,\ldots,0;z^*,z)}d\mu(z^*,z)]$. Here, $f(\alpha_1,\ldots,\alpha_s;z_*,z)$ is an analytic function of the complex fields $\alpha_1(\mathbf{x}),\ldots,\alpha_s(\mathbf{x}),z_*(\mathbf{x}),z(\mathbf{x})$ indexed by $\mathbf{x}$ in a finite set X and $d\mu(z^*,z)$ is a compactly supported product measure. Such effective actions occur in the small field region for a renormalization group analysis. We illustrate the technique by a model renormalization group flow motivated by the ultraviolet regime in many boson systems. © *2010 American Institute of Physics.* [doi:10.1063/1.3329938]

I. INTRODUCTION

Consider the grand canonical partition function, at temperature T and chemical potential μ, for a many boson system moving in a metric space X with a finite number of points and metric d. Suppose that the Hamiltonian H is the sum of a single particle operator (for example, the discrete Laplacian) with kernel $\mathrm{h}(\mathbf{x},\mathbf{y})$ and a two body operator given by a real, symmetric, repulsive pair potential $2v(\mathbf{x},\mathbf{y})$.

In Ref. 2 (Theorem 2.2), we proved the functional integral representation

$$\mathrm{Tr}\, e^{-(1/kT)(H-\mu N)} = \lim_{\varepsilon\to 0}\int \prod_{\tau\in\varepsilon\mathbb{Z}\cap(0,1/kT]} [d\tilde{\mu}_{R_\varepsilon}(\alpha_\tau^*,\alpha_\tau)\zeta_\varepsilon(\alpha_{\tau-\varepsilon},\alpha_\tau)e^{\langle\alpha_{\tau-\varepsilon}^*,j(\varepsilon)\alpha_\tau\rangle-\varepsilon\langle\alpha_{\tau-\varepsilon}^*\alpha_\tau,v\alpha_{\tau-\varepsilon}^*\alpha_\tau\rangle}] \tag{1.1}$$

for the partition function, under the convention that $\alpha_0=\alpha_{1/kT}$ and the limit $\varepsilon\to 0$ is restricted to ε's dividing $1/kT$ [that is, $1/\varepsilon\in(kT)\mathbb{N}$]. Here, N is the number operator and, for any $r>0$,

$$d\tilde{\mu}_r(\alpha^*,\alpha) = \prod_{\mathbf{x}\in X}\frac{d\alpha^*(\mathbf{x})\wedge d\alpha(\mathbf{x})}{2\pi\imath}e^{-\alpha^*(\mathbf{x})\alpha(\mathbf{x})}\chi(|\alpha(\mathbf{x})|<r)$$

denotes the unnormalized Gaussian measure, cutoff at radius r, and $\zeta_\varepsilon(\alpha,\beta)$ is the characteristic function of

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$$\{\alpha,\beta : X \to \mathbb{C} \mid \|\alpha-\beta\|_\infty < p_0(\varepsilon)\}.$$

The cutoffs R_ε and $p_0(\varepsilon)$ grow at an appropriate rate as $\varepsilon \to 0$. [One can think of R_ε as growing a bit faster than $1/\sqrt[4]{\varepsilon}$ and of $p_0(\varepsilon)$ as a power of $\ln(1/\varepsilon)$ or a very small power of $1/\varepsilon$.] Furthermore, for any $\varepsilon>0$, the operator $j(\varepsilon)=e^{-\varepsilon(\mathrm{h}-\mu)}$. We write the ($\mathbb{R}$-style) scalar product, $\langle f,g\rangle = \Sigma_{\mathbf{x}\in X} f(\mathbf{x})g(\mathbf{x})$ for any two fields $f,g:X\to\mathbb{C}$. (Thus, the usual scalar product over $\mathbb{C}^{|X|}$ is $\langle f^*,g\rangle$.)

The representation (1.1) is the first part of a program to resolve the mathematical difficulties inherent in the time-ultraviolet limit of the formal, coherent state functional integral representation for the partition function and correlation functions of many boson systems [see Ref. 5, (2.66) and the discussion in the introduction to Ref. 1]. In Ref. 4 this program is completed using techniques of renormalization group analysis. This paper is a description of the "small field part" of that construction.

In Ref. 4 we obtain a representation of the functional integral of (1.1) which can be used to analyze infrared problems. We do so by applying a simple version of a renormalization group procedure, namely, "decimation." In each decimation step we integrate out every "second" variable. In the first step, we integrate out $\alpha_{\tau'}$ with $\tau'=\varepsilon,3\varepsilon,5\varepsilon,\ldots$. The integral with respect to these variables factorizes into the product, over $\tau=2\varepsilon,4\varepsilon,6\varepsilon,\ldots$, of the independent integrals

$$\int d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\alpha^*_{\tau-\varepsilon},\alpha_{\tau-\varepsilon})\zeta_\varepsilon(\alpha_{\tau-2\varepsilon},\alpha_{\tau-\varepsilon})e^{\langle\alpha^*_{\tau-2\varepsilon},j(\varepsilon)\alpha_{\tau-\varepsilon}\rangle-\varepsilon\langle\alpha^*_{\tau-2\varepsilon}\alpha_{\tau-\varepsilon}v\alpha^*_{\tau-2\varepsilon}\alpha_{\tau-\varepsilon}\rangle}$$
$$\times e^{\langle\alpha^*_{\tau-\varepsilon},j(\varepsilon)\alpha_\tau\rangle-\varepsilon\langle\alpha^*_{\tau-\varepsilon}\alpha_\tau v\alpha^*_{\tau-\varepsilon}\alpha_\tau\rangle}\zeta_\varepsilon(\alpha_{\tau-\varepsilon},\alpha_\tau).$$

That is, assuming that $1/kT \in 2\varepsilon\mathbb{N}$,

$$\int \prod_{\tau\in\varepsilon\mathbb{Z}\cap(0,1/kT]}[d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\alpha^*_\tau,\alpha_\tau)\zeta_\varepsilon(\alpha_{\tau-\varepsilon},\alpha_\tau)e^{\langle\alpha^*_{\tau-\varepsilon},j(\varepsilon)\alpha_\tau\rangle-\varepsilon\langle\alpha^*_{\tau-\varepsilon}\alpha_\tau v\alpha^*_{\tau-\varepsilon}\alpha_\tau\rangle}]$$
$$=\int \prod_{\tau\in 2\varepsilon\mathbb{Z}\cap(0,1/kT]} d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\alpha^*_\tau,\alpha_\tau)I_1(\varepsilon;\alpha^*_{\tau-2\varepsilon},\alpha_\tau),$$

where

$$I_1(\varepsilon;\alpha^*,\beta)=\int d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\phi^*,\phi)\zeta_\varepsilon(\alpha,\phi)e^{\langle\alpha^*,j(\varepsilon)\phi\rangle+\langle\phi^*,j(\varepsilon)\beta\rangle}e^{-\varepsilon(\langle\alpha^*\phi,v\alpha^*\phi\rangle+\langle\phi^*\beta,v\phi^*\beta\rangle)}\zeta_\varepsilon(\phi,\beta). \tag{1.2}$$

After $n-1$ additional decimation steps we will have integrated out those α_τ's with $\tau\in(\varepsilon\mathbb{Z}\setminus(2^n\varepsilon)\mathbb{Z})\cap(0,1/kT]$ leaving an integrand which is a function of the α_τ's with $\tau\in(2^n\varepsilon)\mathbb{Z}\cap[0,\frac{1}{kT}]$. Thus, for $\frac{1}{kT}\in 2^n\varepsilon\mathbb{N}$, we write

$$\int \prod_{\tau\in\varepsilon\mathbb{Z}\cap(0,1/kT]}[d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\alpha^*_\tau,\alpha_\tau)\zeta_\varepsilon(\alpha_{\tau-\varepsilon},\alpha_\tau)e^{\langle\alpha^*_{\tau-\varepsilon},j(\varepsilon)\alpha_\tau\rangle-\varepsilon\langle\alpha^*_{\tau-\varepsilon}\alpha_\tau v\alpha^*_{\tau-\varepsilon}\alpha_\tau\rangle}]$$
$$=\int \prod_{\tau\in(2^n\varepsilon)\mathbb{Z}\cap(0,1/kT]} d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\alpha^*_\tau,\alpha_\tau)I_n(\varepsilon,\alpha^*_{\tau-2^n\varepsilon},\alpha_\tau), \tag{1.3}$$

where the functions $I_n(\varepsilon;\alpha^*,\beta)$ are recursively defined by (1.2) and

$$I_{n+1}(\varepsilon;\alpha^*,\beta)=\int d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\phi^*,\phi)I_n(\varepsilon;\alpha^*,\phi)I_n(\varepsilon;\phi^*,\beta). \tag{1.4}$$

The main result of Ref. 4 is the construction and description of a functional $I_\theta(\alpha^*,\beta)$, defined for $\theta\in(0,\Theta]$ [where $\Theta=O(1)$, independent of v] such that

$$I_\theta(\alpha^*,\beta) = \lim_{m\to\infty} I_m(2^{-m}\theta;\alpha^*,\beta).$$

Then, for any p such that $1/pkT \in (0,\Theta]$,

$$\mathrm{Tr}\, e^{-(1/kT)(H-\mu N)} = \int \prod_{n=1}^{p}\left[\prod_{\mathbf{x}\in X}\frac{d\phi_n(\mathbf{x})^*\phi_n(\mathbf{x})}{2\pi\imath}e^{-\phi_n(\mathbf{x})^*\phi_n(\mathbf{x})}\right] I_{(1/pkT)}(\phi_{n-1}^*,\phi_n) \tag{1.5}$$

with the convention $\phi_0=\phi_p$. (1.5) can be the starting point for an infrared analysis.

In Ref. 4 we describe the functions $I_n(\varepsilon;\alpha^*,\beta)$ and $I_\theta(\alpha^*,\beta)$ as sums over "large/small field decompositions" of X. The dominant part in the large/small field decomposition is called the "pure small field part" and is obtained by replacing the full integrals (1.4) with integrals over appropriate neighborhoods of stationary points (see Sec. II).

In this note, we discuss a toy model, which we call the "stationary phase (SP) approximation," in which all domains of integration are restricted, simply by fiat, to neighborhoods of stationary points. These neighborhoods will be measured by radii $r(\delta)$, where $\delta\mapsto r(\delta)$ is a positive and monotonically decreasing function. We now give a description of the SP approximation and derive estimates for it. To do this, introduce the notation $\varepsilon_n=2^n\varepsilon$ and

$$\mathcal{V}_\delta(\varepsilon;\alpha^*,\beta) = -\varepsilon\sum_{\tau\in\varepsilon\mathbb{Z}\cap[0,\delta)}\langle[j(\tau)\alpha^*][j(\delta-\tau-\varepsilon)\beta],v[j(\tau)\alpha^*][j(\delta-\tau-\varepsilon)\beta]\rangle. \tag{1.6}$$

It will turn out that there is a function $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$ of the fields α^*,β with the property

$$I_n^{(\mathrm{SP})}(\varepsilon;\alpha^*,\beta) = \mathcal{Z}_{\varepsilon_n}(\varepsilon)^{|X|}e^{\langle\alpha^*,j(\varepsilon_n)\beta\rangle+\mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,\beta)+\mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,\beta)}. \tag{1.7}$$

The function $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$ is defined for real numbers $0<\varepsilon\le\delta\le\Theta$ such that $\delta=2^n\varepsilon$ for some integer $n\ge 0$. The normalization constant $\mathcal{Z}_\delta(\varepsilon)$ is defined in Appendix C. It is chosen so that $\mathcal{E}_\delta(\varepsilon;0,0)=0$. It is extremely close to 1. The "irrelevant" contributions $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$ to the effective action are characterized by the recursion relation

$$\mathcal{E}_\varepsilon(\varepsilon;\alpha^*,\beta)=0,$$

$$\mathcal{E}_{2\delta}(\varepsilon;\alpha^*,\beta) = \mathcal{E}_\delta(\varepsilon;\alpha^*,j(\delta)\beta) + \mathcal{E}_\delta(\varepsilon;j(\delta)\alpha^*,\beta) + \log\frac{\int d\tilde{\mu}_{r(\delta)}(z^*,z)e^{\partial\mathcal{A}_\delta(\varepsilon;\alpha^*,\beta;z^*,z)}}{\int d\tilde{\mu}_{r(\delta)}(z^*,z)}, \tag{1.8}$$

where

$$\begin{aligned}\partial\mathcal{A}_\delta(\varepsilon;\alpha^*,\beta;z_*,z) &= [\mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta+z) - \mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta)]\\ &+ [\mathcal{V}_\delta(\varepsilon;j(\delta)\alpha^*+z_*,\beta) - \mathcal{V}_\delta(\varepsilon;j(\delta)\alpha^*,\beta)]\\ &+ [\mathcal{E}_\delta(\varepsilon;\alpha^*,j(\delta)\beta+z) - \mathcal{E}_\delta(\varepsilon;\alpha^*,j(\delta)\beta)]\\ &+ [\mathcal{E}_\delta(\varepsilon;j(\delta)\alpha^*+z_*,\beta) - \mathcal{E}_\delta(\varepsilon;j(\delta)\alpha^*,\beta)].\end{aligned} \tag{1.9}$$

The motivation for this recursion relation comes from a SP construction and is given in Sec. II.

We estimate $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$ in terms of norms as in Ref. 3 (Definition 2.6 and, more specifically, (A1)]. Assume that X is a metric space. Choose a constant $\mathrm{m}\ge 0$ as a spatial exponential decay rate and a positive monotonically decreasing function $\delta\mapsto\kappa(\delta)$ to measure the radius of convergence of the expansion of $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$ in powers of the fields α^* and β.

We define the norm of the power series

$$f(\alpha^*,\beta)=\sum_{k,\ell\ge 0}\sum_{\substack{\mathbf{x}_1,\ldots,\mathbf{x}_k\in X\\ \mathbf{y}_1,\ldots,\mathbf{y}_\ell\in X}} a(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathbf{y}_1,\ldots,\mathbf{y}_\ell)\alpha(\mathbf{x}_1)^*\cdots\alpha(\mathbf{x}_k)^*\beta(\mathbf{y}_1)\cdots\beta(\mathbf{y}_\ell)$$

[with the coefficients $a(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathbf{y}_1,\ldots,\mathbf{y}_\ell)$ invariant under permutations of $\mathbf{x}_1,\ldots,\mathbf{x}_k$ and of $\mathbf{y}_1,\ldots,\mathbf{y}_\ell$] to be

$$\|f(\alpha^*,\beta)\|_\delta=\sum_{k,\ell\ge 0}\max_{\mathbf{x}\in X}\max_{1\le i\le k+\ell}\sum_{\substack{(\vec{\mathbf{x}},\vec{\mathbf{y}})\in X^k\times X^\ell\\ (\vec{\mathbf{x}},\vec{\mathbf{y}})_i=\mathbf{x}}} w_\delta(\vec{\mathbf{x}};\vec{\mathbf{y}})|a(\vec{\mathbf{x}};\vec{\mathbf{y}})|$$

with the weight system

$$w_\delta(\vec{\mathbf{x}};\vec{\mathbf{y}})=\kappa(\delta)^{k+\ell}e^{\mathrm{m}\tau(\vec{\mathbf{x}},\vec{\mathbf{y}})}\quad\text{for }(\vec{\mathbf{x}},\vec{\mathbf{y}})\in X^k\times X^\ell,$$

where $\tau(\vec{\mathbf{x}},\vec{\mathbf{y}})$ is the minimal length of a tree whose set of vertices contains the points of the set $\{\mathbf{x}_1,\ldots,\mathbf{x}_k,\mathbf{y}_1,\ldots,\mathbf{y}_\ell\}$. In the language of Ref. 3 (Definitions 2.5 and 2.6), w_δ is the weight system with metric md that associates the constant weight factor $\kappa(\delta)$ to the fields α^* and β, and the norm $\|f(\alpha^*,\beta)\|_\delta$ is denoted $\|f(\alpha^*,\beta)\|_{w_\delta}$. For any operator $\mathcal{A}$ on $\mathbb{C}^X$, with kernel $\mathcal{A}(\mathbf{x},\mathbf{y})$, we define the weighted L^1-L^∞ operator norm

$$|||\mathcal{A}|||=\max\left\{\sup_{\mathbf{x}\in X}\sum_{\mathbf{y}\in X}e^{\mathrm{m}d(\mathbf{x},\mathbf{y})}|\mathcal{A}(\mathbf{x},\mathbf{y})|,\sup_{\mathbf{y}\in X}\sum_{\mathbf{x}\in X}e^{\mathrm{m}d(\mathbf{x},\mathbf{y})}|\mathcal{A}(\mathbf{x},\mathbf{y})|\right\},\tag{1.10}$$

as in Ref. 3 (Definition A.1).

The quantities relevant for the estimates of $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$, in addition to the radii $r(\delta)$ and $\kappa(\delta)$, are the norm $|||v|||$ of the interaction, a constant K_j such

$$|||j(\tau)|||\le e^{K_j\tau}\quad\text{and}\quad|||j(\tau)-\mathbb{1}|||\le K_j\tau e^{K_j\tau}\quad\text{for }\tau\ge 0\tag{1.11}$$

(see Corollary B.2), and a constant $0<\Theta\le 1$ that bounds the range for which the constructions work. On these quantities we make the following.

Hypothesis 1.1: We assume that the monotonically decreasing functions $r(t)$ and $\kappa(t)$ do not decrease too quickly. Precisely,

(i) $1\le r(t)\le 2r(2t)$ *and* $1\le\kappa(t)\le 2\kappa(2t)$, *for all* $0\le t\le\Theta/2$.
On the other hand, $r(t)$ and $\kappa(t)$ must decrease quickly enough and $r(t)$ must be sufficiently small compared to $\kappa(t)$ that
(ii) $e^{tK_j}[\kappa(2t)/\kappa(t)]+4[r(t)/\kappa(t)]\le 1$ *for all* $0\le t\le\Theta/2$ *and*
(iii) $r(t)[r(t)-r(2t)]\ge 2$ *for all* $0\le t\le\Theta/2$.
We also assume that there are constants K_E and $q\ge 1$ such that
(iv) $t|||v|||r(t)\kappa(t)^3\le 1/K_E$ *for all* $0\le t\le\Theta$;
(v) $[1/C(\Theta,K_E)](2/q)\le(\kappa(t)/\kappa(2t))^4\le C(\Theta,K_E)4(r(2t)/r(t))^4$ *for all* $0\le t\le\Theta/2$, *where* $C(\Theta,K_E)=e^{-4\Theta K_j}[1-2^{33}e^{14K_j}/K_E]$; and
(vi) $t^2\sum_{k=0}^\infty(q/4)^k r(t/2^k)^2\kappa(t/2^k)^6$ *converges uniformly in* $0\le t\le\Theta$.

Example 1.2: Let $\mathfrak{v}>0$.

(i) Suppose that

$$\kappa(t)=\frac{1}{\sqrt[4]{|||v|||}}\left(\frac{1}{t}\right)^{a_\kappa}\quad\text{and}\quad r(t)=\frac{1}{\sqrt[4]{|||v|||}}\left(\frac{1}{t}\right)^{a_r}$$

for some constants $0<a_r<a_\kappa$ obeying $3a_\kappa+a_r<1$. We prove in Appendix D that there are constants K_E, Θ, and q such that Hypothesis 1.1 is fulfilled for all nonzero v with $|||v|||\le\mathfrak{v}$.

(ii) Suppose that

$$\kappa(t) = \frac{1}{\sqrt[4]{t|||v|||}}\left(\ln\frac{1}{t|||v|||}\right)^b \quad \text{and} \quad r(t) = \left(\ln\frac{1}{t|||v|||}\right)^b$$

for some $b \geq 1$. Again, we prove in Appendix D that there are constants K_E, Θ, and q such that Hypothesis 1.1 is fulfilled for all nonzero v with $|||v||| \leq \mathfrak{v}$.

We choose $p_0(\varepsilon)$ of the functional integral representation (1.1) to be $r(\varepsilon)$.

Theorem 1.3: *Under Hypothesis 1.1*

$$\|\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)\|_\delta \leq K_E \delta^2 |||v|||^2 r(\delta)^2 \kappa(\delta)^6$$

for all $0 \leq \varepsilon \leq \delta \leq \Theta$ for which δ/ε is a power of 2. The function $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$ has degree at least two both in α^* and β. [By this we mean that every monomial appearing in its power series expansion contains a factor of the form $\alpha^*(\mathbf{x}_1)\alpha^*(\mathbf{x}_2)\beta(\mathbf{x}_3)\beta(\mathbf{x}_4)$.]

Theorem 1.3 is proven after Proposition 3.3. The theorem is proven by induction on n, where $\delta = 2^n \varepsilon$. The induction step is prepared by Proposition 3.3, which is proven using the results in Ref. 3. After the proof of Theorem 1.3, we give the following proof.

Theorem 1.4: *The limit*

$$\mathcal{E}_\theta(\alpha^*,\beta) = \lim_{m\to\infty} \mathcal{E}_\theta(2^{-m}\theta;\alpha^*,\beta)$$

exists uniformly in $0 \leq \theta \leq \Theta$. It fulfills the estimate

$$\|\mathcal{E}_\theta(\alpha^*,\beta)\|_\theta \leq K_E \theta^2 |||v|||^2 r(\theta)^2 \kappa(\theta)^6$$

and has degree of at least 2 in both α^* and β.

To take the limit $\varepsilon \to 0$ in (1.7) observe that, for any fixed δ,

$$\lim_{n\to\infty} \mathcal{V}_\delta(2^{-n}\delta;\alpha^*,\beta) = \mathcal{V}_\delta(\alpha^*,\beta),$$

where

$$\mathcal{V}_\delta(\alpha^*,\beta) = -\int_0^\delta \langle [j(t)\alpha^*][j(\delta-t)\beta], v[j(t)\alpha^*][j(\delta-t)\beta]\rangle dt.$$

The SP approximation to the $I_\theta(\alpha^*,\beta)$ constructed in Ref. 4 then is

$$I_\theta^{(\mathrm{SP})}(\alpha^*,\beta) = \lim_{m\to\infty} I_m^{(\mathrm{SP})}(2^{-m}\theta;\alpha^*,\beta) = \mathcal{Z}_\theta^{|X|} e^{\langle\alpha^*, j(\theta)\beta\rangle + \mathcal{V}_\theta(\alpha^*,\beta) + \mathcal{E}_\theta(\alpha^*,\beta)}.$$

The existence of $\mathcal{Z}_\theta = \lim_{m\to\infty} \mathcal{Z}_\theta(\theta/2^m)$ is proven in Lemma C.1.

II. SP AND STOKES' THEOREM

To motivate the recursive definition (1.8) of $\mathcal{E}_\delta(\varepsilon;\alpha^*,\beta)$, we replace I_n by $I_n^{(\mathrm{SP})}$ in the recursion relation (1.4). Inserting (1.7), the resulting integral

$$\int d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\phi^*,\phi) I_n^{(\mathrm{SP})}(\varepsilon;\alpha^*,\phi) I_n^{(\mathrm{SP})}(\varepsilon;\phi^*,\beta)$$

$$= \mathcal{Z}_n^{|2X|} \int d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\phi^*,\phi) e^{\langle \alpha^*, j(\varepsilon_n)\phi\rangle + \langle \phi^*, j(\varepsilon_n)\beta\rangle} e^{\mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi)+\mathcal{V}_{\varepsilon_n}(\varepsilon;\phi^*,\beta)} e^{\mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi)+\mathcal{E}_{\varepsilon_n}(\varepsilon;\phi^*,\beta)}$$

$$= \mathcal{Z}_n^{2|X|} \int \left[\prod_{\mathbf{x}\in X} \frac{d\phi^*(\mathbf{x}) \wedge d\phi(\mathbf{x})}{2\pi\imath} \chi(|\phi(\mathbf{x})| < R_\varepsilon) \right] e^{\mathcal{A}(\alpha^*,\beta;\phi^*,\phi)} \tag{2.1}$$

with $\mathcal{Z}_n = \mathcal{Z}_{\varepsilon_n}(\varepsilon)$ and

$$\mathcal{A}(\alpha^*,\beta;\phi_*,\phi) = -\langle \phi_*,\phi\rangle + \langle \alpha^*, j(\varepsilon_n)\phi\rangle + \langle \phi_*, j(\varepsilon_n)\beta\rangle + \mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi) + \mathcal{V}_{\varepsilon_n}(\varepsilon;\phi_*,\beta)$$
$$+ \mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi) + \mathcal{E}_{\varepsilon_n}(\varepsilon;\phi_*,\beta).$$

Here we have written $\mathcal{A}$ as a function of four independent complex fields α^*, β, ϕ_*, and ϕ. The activity in (2.1) is obtained by evaluating $\mathcal{A}(\alpha^*,\beta;\phi_*,\phi)$ with $\phi_* = \phi^*$, the complex conjugate of ϕ. The reason for introducing independent complex fields ϕ_* and ϕ lies in the fact that the critical point (with respect to the variables ϕ_* and ϕ) of the quadratic part

$$-\langle \phi_*,\phi\rangle + \langle j(\varepsilon_n)\alpha^*,\phi\rangle + \langle \phi_*, j(\varepsilon_n)\beta\rangle = -\langle \phi_* - j(\varepsilon_n)\alpha^*, \phi - j(\varepsilon_n)\beta\rangle + \langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle$$

of $\mathcal{A}$ is "not real." Precisely, the critical point is

$$\phi_*^{\mathrm{crit}} = j(\varepsilon_n)\alpha^*, \quad \phi^{\mathrm{crit}} = j(\varepsilon_n)\beta$$

and, in general $(\phi_*^{\mathrm{crit}})^* \neq \phi^{\mathrm{crit}}$. To do stationary phase, we make the substitution

$$\phi_* = \phi_*^{\mathrm{crit}} + z_*, \quad \phi = \phi^{\mathrm{crit}} + z \tag{2.2}$$

with "fluctuation fields" z_* and z. With this substitution, the quadratic part of $\mathcal{A}$ is equal to $-\langle z_*, z\rangle + \langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle$. So (2.1) becomes

$$\mathcal{Z}_n^{2|X|} e^{\langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle} \left[\prod_{\mathbf{x}\in X} \int_{M(\mathbf{x})} \frac{dz_*(\mathbf{x}) \wedge dz(\mathbf{x})}{2\pi\imath} e^{-z_*(\mathbf{x})z(\mathbf{x})} \right] e^{\tilde{\mathcal{A}}(\alpha^*,\beta;z_*,z)}, \tag{2.3}$$

where

$$\tilde{\mathcal{A}}(\alpha^*,\beta;z_*,z) = \mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi^{\mathrm{crit}}+z) + \mathcal{V}_{\varepsilon_n}(\varepsilon;\phi_*^{\mathrm{crit}}+z_*,\beta) + \mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi^{\mathrm{crit}}+z) + \mathcal{E}_{\varepsilon_n}(\varepsilon;\phi_*^{\mathrm{crit}}+z_*,\beta),$$

$$M(\mathbf{x}) = \{(z_*(\mathbf{x}),z(\mathbf{x})) | (\phi_*^{\mathrm{crit}}(\mathbf{x}) + z_*(\mathbf{x}))^* = \phi^{\mathrm{crit}}(\mathbf{x}) + z(\mathbf{x}) \quad \text{and} \quad |\phi^{\mathrm{crit}}(\mathbf{x}) + z(\mathbf{x})| < R_\varepsilon\}.$$

(2.3) is the integral over a real $2|X|$ dimensional subset in the complex $2|X|$ dimensional space of fields z_* and z.

The first step in the SP approximation is to replace, for each $\mathbf{x}\in X$, the set $M(\mathbf{x})$ in (2.3) by the neighborhood

$$D(\mathbf{x}) = \{(z_*(\mathbf{x}),z(\mathbf{x})) \in \mathbb{C}^2 | |z_*(\mathbf{x})| \le r(\varepsilon_n), \ |z(\mathbf{x})| \le r(\varepsilon_n), \ (z_*(\mathbf{x}) + \phi_*^{\mathrm{crit}}(\mathbf{x}))^* = z(\mathbf{x}) + \phi^{\mathrm{crit}}(\mathbf{x})\}$$

of the critical point. In Ref. 4 (Sec. VI) we show that the error introduced by this approximation is extremely small [even when $D(\mathbf{x})$ is empty]. There, we provide detailed bounds on a "large field–small field" expansion for which the above approximation is the leading term. In Remark 2.1 [part (a), below], we illustrate the sources of the smallness for $n=0$.

By Stokes' theorem [Lemma A.1 with $r=r(\varepsilon_n)$, $\sigma=\sigma_*=0$, and $\rho=(\phi_*^{\mathrm{crit}})^* - \phi^{\mathrm{crit}}$], one can write

$$\mathcal{Z}_n^{2|X|} e^{\langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle} \left[\prod_{\mathbf{x}\in X} \int_{D(\mathbf{x})} \frac{dz_*(\mathbf{x}) \wedge dz(\mathbf{x})}{2\pi \imath} e^{-z_*(\mathbf{x})z(\mathbf{x})} \right] e^{\tilde{\mathcal{A}}(\alpha^*,\beta;z_*,z)} \tag{2.4}$$

as the sum of

$$\mathcal{Z}_n^{2|X|} e^{\langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle} \left[\prod_{\mathbf{x}\in X} \int_{|z(\mathbf{x})|\le r(\varepsilon_n)} \frac{dz^*(\mathbf{x}) \wedge dz(\mathbf{x})}{2\pi \imath} e^{-|z(\mathbf{x})|^2} \right] e^{\tilde{\mathcal{A}}(\alpha^*,\beta;z^*,z)} \tag{2.5}$$

and $\mathcal{Z}_n^{2|X|} e^{\langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle}$ times

$$\sum_{\substack{R\subset X\\ R\neq\emptyset}} \prod_{\mathbf{x}\in R} \left[\int_{C(\mathbf{x})} \frac{dz_*(\mathbf{x}) \wedge dz(\mathbf{x})}{2\pi i} e^{-z_*(\mathbf{x})z(\mathbf{x})} \right]$$
$$\times \prod_{\mathbf{x}\in X\backslash R} \left[\int_{|z(\mathbf{x})|\le r(\varepsilon_n)} \frac{dz(\mathbf{x})^* \wedge dz(\mathbf{x})}{2\pi i} e^{-z_*(\mathbf{x})z(\mathbf{x})} \right] e^{\tilde{\mathcal{A}}(\alpha^*,\beta;z^*,z)}\Big|_{\substack{z_*(\mathbf{x})=z(\mathbf{x})^*\\ \text{for }\mathbf{x}\in X\backslash R}},$$

where, for each $\mathbf{x}\in X$, $C(\mathbf{x})$ is a two real dimensional submanifold of $\mathbb{C}^2$ whose boundary is the union of "circles" $\partial D(\mathbf{x})$ and $\{(z_*(\mathbf{x}),z(\mathbf{x}))\in\mathbb{C}^2 \,|\, z_*^*(\mathbf{x})=z(\mathbf{x}), |z(\mathbf{x})|=r(\varepsilon_n)\}$. In Ref. 4 we argue that $-z_*(\mathbf{x})z(\mathbf{x})$ has an extremely large negative real part whenever $(z_*(\mathbf{x}),z(\mathbf{x}))\in C(\mathbf{x})$. [Also see Remark 2.1, part (b), below.] The second step in the SP approximation is to ignore these terms. That is, to replace (2.4) with (2.5).

Thus, the SP approximation for

$$\int d\tilde{\mu}_{R_\varepsilon}(\phi^*,\phi) I_n^{(\mathrm{SP})}(\varepsilon;\alpha^*,\phi) I_n^{(\mathrm{SP})}(\varepsilon;\phi^*,\beta)$$

is

$$(2.5) = \mathcal{Z}_n^{2|X|} e^{\langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle} \int d\tilde{\mu}_{r(\varepsilon_n)}(z^*,z) e^{\tilde{\mathcal{A}}(\alpha^*,\beta;z^*,z)}.$$

By construction

$$\mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi^{\mathrm{crit}}) + \mathcal{V}_{\varepsilon_n}(\varepsilon;\phi_*^{\mathrm{crit}},\beta) = \mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,j(\varepsilon_n)\beta) + \mathcal{V}_{\varepsilon_n}(\varepsilon;j(\varepsilon_n)\alpha^*,\beta) = \mathcal{V}_{\varepsilon_{n+1}}(\varepsilon;\alpha^*,\beta)$$

so that

$$\begin{aligned}\mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,\phi^{\mathrm{crit}}+z) + \mathcal{V}_{\varepsilon_n}(\varepsilon;\phi_*^{\mathrm{crit}}+z^*,\beta) = \mathcal{V}_{\varepsilon_{n+1}}(\varepsilon;\alpha^*,\beta) &+ [\mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,j(\varepsilon_n)\beta+z) \\ &- \mathcal{V}_{\varepsilon_n}(\varepsilon;\alpha^*,j(\varepsilon_n),\beta)] + [\mathcal{V}_{\varepsilon_n}(\varepsilon;j(\varepsilon_n)\alpha^*+z^*,\beta) \\ &- \mathcal{V}_{\varepsilon_n}(\varepsilon;j(\varepsilon_n)\alpha^*,\beta)].\end{aligned}$$

Consequently, the SP approximation for

$$\int d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\phi^*,\phi) I_n^{(\mathrm{SP})}(\varepsilon;\alpha^*,\phi) I_n^{(\mathrm{SP})}(\varepsilon;\phi^*,\beta)$$

can also be written as

$$\mathcal{Z}_n^{2|X|} e^{\langle \alpha^*, j(\varepsilon_{n+1})\beta\rangle + \mathcal{V}_{\varepsilon_{n+1}}(\varepsilon;\alpha^*,\beta)} e^{\mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,j(\varepsilon_n)\beta)\mathcal{E}_{\varepsilon_n}(\varepsilon;j(\varepsilon_n)\alpha^*,\beta)} \int d\tilde{\mu}_{r(\varepsilon_n)}(z^*,z) e^{\partial\mathcal{A}_{\varepsilon_n}(\varepsilon;\alpha^*,\beta;z_*,z)}.$$

This is compatible with (1.7) and (1.8) since, by the definition of Appendix C,

$$\mathcal{Z}_{2\varepsilon_n}(\varepsilon_n) = \mathcal{Z}_{\varepsilon_n}(\varepsilon)^2 \int_{|z|<r(\varepsilon_n)} \frac{dz^* \wedge dz}{2\pi i} e^{-|z|^2}.$$

Remark 2.1:

(a) We now illustrate why the error introduced by the SP approximation is extremely small by considering the case $n=0$. The initial functional integral representation (1.1) may be written as

$$\mathrm{Tr}\, e^{-(1/kT)(H-\mu N)} = \lim_{\varepsilon\to 0}\int \prod_{\tau\in\varepsilon\mathbb{Z}\cap(0,1/kT]} \left\{ \left[\prod_{\mathbf{x}\in X} \frac{d\alpha_\tau^*(\mathbf{x}) \wedge d\alpha_\tau(\mathbf{x})}{2\pi i} \chi(|\alpha_\tau(\mathbf{x})| < \mathrm{R}_\varepsilon) \right] \zeta_\varepsilon(\alpha_{\tau-\varepsilon},\alpha_\tau) \times e^{-(1/2)\langle\alpha^*_{\tau-\varepsilon},\alpha_{\tau-\varepsilon}\rangle} I_0(\varepsilon;\alpha^*_{\tau-\varepsilon},\alpha_\tau) e^{-(1/2)\langle\alpha^*_\tau,\alpha_\tau\rangle} \right\},$$

where

$$I_0(\varepsilon;\alpha^*,\beta) = e^{\langle\alpha^*, j(\varepsilon)\beta\rangle} e^{-\varepsilon\langle\alpha^*\beta, v\alpha^*\beta\rangle}.$$

Dropping one of the "time derivative small field characteristic functions" $\zeta_\varepsilon(\alpha_{\tau-\varepsilon},\alpha_\tau)$ would introduce only a very small error. This is because writing $\alpha_{\tau-\varepsilon}=\alpha$ and $\alpha_\tau=\beta$, the quadratic part of the exponent of $e^{-(1/2)\langle\alpha^*,\alpha\rangle} I_0(\varepsilon;\alpha^*,\beta) e^{-(1/2)\langle\beta^*,\beta\rangle}$ obeys

$$\mathrm{Re}\left\{-\tfrac{1}{2}\langle\alpha^*,\alpha\rangle + \langle\alpha^*, j(\varepsilon)\beta\rangle - \tfrac{1}{2}\langle\beta^*,\beta\rangle\right\} \approx \mathrm{Re}\left\{-\tfrac{1}{2}\langle\alpha^*,\alpha\rangle + \langle\alpha^*,\beta\rangle - \tfrac{1}{2}\langle\beta^*,\beta\rangle\right\} = -\tfrac{1}{2}\|\alpha-\beta\|_{L^2}^2,$$

which generates a factor on the order of $e^{-(1/2)p_0(\varepsilon)^2} = e^{-(1/2)r(\varepsilon)^2}$ when (α,β) is not in support of $\zeta_\varepsilon(\alpha,\beta)$. The quartic part $-\varepsilon\langle\alpha^*\beta, v\alpha^*\beta\rangle$ of the exponent is roughly $-\varepsilon\langle\alpha^*\alpha, v\alpha^*\alpha\rangle \le 0$ and so cannot generate a large factor. A similar mechanism generates small factors for the right hand side of (1.3) whenever the difference $\alpha_{\tau-\varepsilon_n}-\alpha_\tau$ between the two arguments of I_n is larger than roughly $r(\varepsilon_n)$.

Consequently, we apply the SP approximation to the integral

$$I_1(\varepsilon;\alpha^*,\beta) = \int d\tilde{\mu}_{\mathrm{R}_\varepsilon}(\phi^*,\phi)\zeta_\varepsilon(\alpha,\phi) I_0(\varepsilon;\alpha^*,\phi) I_0(\varepsilon;\phi^*,\beta)\zeta_\varepsilon((\phi^*)^*,\beta)$$

of (1.2) only when the "time derivative small field condition" $\|\alpha-\beta\|_\infty \le r(2\varepsilon)$ is satisfied. The change in variables (2.2) expresses I_1 as

$$I_1(\varepsilon;\alpha^*,\beta) = e^{\langle\alpha^*, j(2\varepsilon)\beta\rangle}\left[\prod_{\mathbf{x}\in X}\int_{M(\mathbf{x})} \frac{dz_*(\mathbf{x}) \wedge dz(\mathbf{x})}{2\pi i} e^{-z_*(\mathbf{x})z(\mathbf{x})}\right] e^{\tilde{\mathcal{A}}(\alpha^*,\beta;z_*,z)} \times \zeta_\varepsilon(\alpha, j(\varepsilon)\beta + z)\zeta_\varepsilon((j(\varepsilon)\alpha^* + z_*)^*,\beta).$$

The characteristic function $\zeta_\varepsilon(\alpha, j(\varepsilon)\beta+z)$ limits the domain of integration to z's, obeying

$$\|z + j(\varepsilon)\beta - \alpha\|_\infty < r(\varepsilon).$$

Since $\|\alpha-\beta\|_\infty \le r(2\varepsilon) \le \frac{1}{2}r(\varepsilon)$ and $\|j(\varepsilon)\beta-\beta\|_\infty \le \mathrm{const}\ \varepsilon \mathrm{R}_\varepsilon \ll r(\varepsilon)$, this condition is roughly equivalent to $\|z\|_\infty < r(\varepsilon)$. On the difference between these two domains of integration, the integrand is extremely small, for reasons like those given above. Similarly, the condition imposed by the second ζ_ε is roughly equivalent to $\|z_*\|_\infty < r(\varepsilon)$. The two conditions $\|z\|_\infty \le r(\varepsilon)$ and $\|z_*\|_\infty \le r(\varepsilon)$ are built into the domains of integration $D(\mathbf{x})$ in (2.4).

(b) The "time derivative small field condition" $\|\alpha-\beta\|_\infty \le r(2\varepsilon) \le \frac{1}{2}r(\varepsilon)$ is also used to ensure that $-z_*(\mathbf{x})z(\mathbf{x})$ has an extremely large negative real part whenever $(z_*(\mathbf{x}), z(\mathbf{x}))$ lies on $C(\mathbf{x})$, the side of the Stokes' "cylinder." This may be seen from Remark A.3 with $r=r(\varepsilon)$, $\sigma=\sigma_*=0$, and $\rho=(\phi_*^{\mathrm{crit}})^* - \phi^{\mathrm{crit}} = j(\varepsilon)[\alpha-\beta]$.

III. THE INDUCTION STEP

In preparation for the proof of Proposition 3.3 below, as well as of the proofs of Theorems 1.3 and 1.4, we collect consequences of Hypotheses 1.1 in the form in which they are actually used in these proofs.

Remark 3.1: Hypothesis 1.1 implies that for all $0 \le \delta \le \Theta/2$,

$$K_E \ge 2^{19} e^{9K_j}, \tag{3.1a}$$

$$\left[\frac{2^{31} e^{14\delta K_j}}{K_E} + e^{2\delta K_j}\left(\frac{k(\delta)}{k(2\delta)}\right)^2\right]\left(\frac{r(\delta)}{r(2\delta)}\right)^2 \le 2, \tag{3.1b}$$

$$2e^{2\delta K_j}\left(\frac{k(2\delta)}{k(\delta)}\right)^4 + 2^{26} e^{9K_j}\delta |||v||| r(\delta)\kappa(2\delta)^3 \le q. \tag{3.1c}$$

Proof: Hypothesis 1.1 [part (v)] forces $C(\Theta, K_E) \ge 0$ and hence $K_E \ge 2^{33} e^{14K_j}$, which implies (3.1a). Hypothesis 1.1 [part (v)] also forces

$$\left(\frac{k(\delta)}{\kappa(2\delta)}\right)^2 \le 2e^{-2\Theta K_j}\sqrt{1 - \frac{2^{33} e^{14K_j}}{K_E}}\left(\frac{r(2\delta)}{r(\delta)}\right)^2 \le 2e^{-2\delta K_j}\left[1 - \frac{2^{32} e^{14K_j}}{K_E}\right]\left(\frac{r(2\delta)}{r(\delta)}\right)^2$$

and hence

$$4\frac{2^{31} e^{14\delta K_j}}{K_E} + e^{2\delta K_j}\left(\frac{\kappa(\delta)}{\kappa(2\delta)}\right)^2\left(\frac{r(\delta)}{r(2\delta)}\right)^2 \le 2.$$

So (3.1b) now follows from Hypothesis 1.1 [part (i)], which ensures that $(r(\delta)/r(2\delta))^2 \le 4$. Finally, by Hypothesis 1.1 [parts (iv) and (v)],

$$\begin{aligned}
2e^{2\delta K_j}\left(\frac{\kappa(2\delta)}{\kappa(\delta)}\right)^4 + 2^{26} e^{9K_j}\delta |||v||| r(\delta)\kappa(2\delta)^3 &\le e^{2\Theta K_j} q C(\Theta, K_E) + \frac{2^{26} e^{9K_j}}{K_E} \\
&= q e^{-2\Theta K_j} - q\frac{2^{33} e^{-2\Theta K_j} e^{14K_j}}{K_E} + \frac{2^{26} e^{9K_j}}{K_E} \\
&\le q - \frac{2^{33} e^{12K_j}}{K_E} + \frac{2^{26} e^{9K_j}}{K_E} \\
&\le q
\end{aligned}$$

since $q \ge 1$ and $0 < \Theta \le 1$. ■

We formulate the recursion relation (1.8) that defines $\mathcal{E}_{\varepsilon_n}(\varepsilon; \alpha^*, \beta)$ more abstractly.

Definition 3.2: Let $0 \le \varepsilon \le \delta$. For an action $\mathcal{E}(\alpha^*, \beta)$, we set

$$\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}](\alpha^*, \beta) = \mathcal{E}(\alpha^*, j(\delta)\beta) + \mathcal{E}(j(\delta)\alpha^*, \beta) + \log\frac{\int d\tilde{\mu}_{r(\delta)}(z^*, z) e^{\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\alpha^*,\beta;z^*,z)}}{\int d\tilde{\mu}_{r(\delta)}(z^*, z)}$$

whenever the logarithm is defined. Here,

$$\begin{aligned}
\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E}; \alpha^*, \beta; z_*, z) = &[\mathcal{V}_\delta(\varepsilon; \alpha^*, j(\delta)\beta + z) - \mathcal{V}_\delta(\varepsilon; \alpha^*, j(\delta)\beta)] \\
&+ [\mathcal{V}_\delta(\varepsilon; j(\delta)\alpha^* + z_*, \beta) - \mathcal{V}_\delta(\varepsilon; j(\delta)\alpha^*, \beta)] \\
&+ [\mathcal{E}(\alpha^*, j(\delta)\beta + z) - \mathcal{E}(\alpha^*, j(\delta)\beta)] \\
&+ [\mathcal{E}(j(\delta)\alpha^* + z_*, \beta) - \mathcal{E}(j(\delta)\alpha^*, \beta)].
\end{aligned}$$

The recursion relation (1.8) is equivalent to

$$\mathcal{E}_{\varepsilon}(\varepsilon;\alpha^*,\beta)=0,$$

$$\mathcal{E}_{\varepsilon_{n+1}}(\varepsilon;\alpha^*,\beta)=\mathfrak{R}_{\varepsilon_n,\varepsilon}[\mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,\beta)]. \tag{3.2}$$

To prove Theorem 1.3, we perform induction on n to successively bound $\mathcal{E}_{\varepsilon_n}(\varepsilon;\cdot)$ for $n=0,\ldots,\log_2(\Theta/\varepsilon)$. For the induction step, we use the following.

Proposition 3.3: Assume that $r(t)$ and $\kappa(t)$ fulfill Hypothesis 1.1. Then, for all $0\le\varepsilon\le\delta\le\Theta/2$, with δ an integer multiple of ε, the following holds.

Let $\mathcal{E}(\alpha^*,\beta)$ be an analytic function which has degree of at least 2 both in α^* and β and which obeys $\|\mathcal{E}(\alpha^*,\beta)\|_\delta\le 2^9 e^{7\delta K_j}\delta|||v|||r(\delta)\kappa(2\delta)^3$. Then $\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}](\alpha^*,\beta)$ is well defined, has degree at least two both in α^* and β, and satisfies the estimate

$$\|\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}]\|_{2\delta}\le 2^{32}e^{14\delta K_j}\delta^2|||v|||^2 r(\delta)^2\kappa(2\delta)^6+2e^{2\delta K_j}\left(\frac{\kappa(2\delta)}{\kappa(\delta)}\right)^4\|\mathcal{E}\|_\delta.$$

Proof: Observe that the functions $\mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta+z)-\mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta)$ and $\mathcal{E}(\alpha^*,j(\delta)\beta+z)-\mathcal{E}(\alpha^*,j(\delta)\beta)$ both have degree of at least 2 in α^*, degree of at least 1 in z, and do not depend on z_*. Similarly, both $\mathcal{V}_\delta(\varepsilon;j(\delta)\alpha^*+z_*,\beta)-\mathcal{V}_\delta(\varepsilon;j(\delta)\alpha^*,\beta)$ and $\mathcal{E}(j(\delta)\alpha^*+z_*,\beta)-\mathcal{E}(j(\delta)\alpha^*,\beta)$ have degree of at least 2 in β, degree of at least 1 in z_*, and do not depend on z. Since the integral of any monomial against $d\tilde{\mu}_{r(\delta)}(z^*,z)$ is zero unless there are the same number of z's and z^*'s,

$$\int d\tilde{\mu}_{r(\delta)}(z^*,z)\ \partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\alpha^*,\beta;z^*,z)=0 \tag{3.3}$$

and

$$\log\frac{\int d\tilde{\mu}_{r(\delta)}(z^*,z)e^{\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\alpha^*,\beta;z^*,z)}}{\int d\tilde{\mu}_{r(\delta)}(z^*,z)}$$

has degree of at least 2 both in α^* and β. This implies that $\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}](\alpha^*,\beta)$ has degree of at least 2 both in α^* and β.

To estimate $\partial A_{\delta,\varepsilon}$ we introduce a second auxiliary weight system w_{fluct}. It has a metric md and associates the constant weight factor $\kappa(2\delta)$ to the fields α^* and β and the constant weight factor $4r(\delta)$ to the fluctuation fields z_* and z. We abbreviate

$\|f(\alpha^*,\beta;z_*,z)\|_{\text{fluct}}=\|f(\alpha^*,\beta;z_*,z)\|_{w_{\text{fluct}}}$. Clearly, $\|f(\alpha^*,\beta)\|_{2\delta}=\|f(\alpha^*,\beta)\|_{\text{fluct}}$ for functions that are independent of the fluctuation fields.

Observe that

$$\mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta+z)-\mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta)=\varepsilon\sum_{\tau\in\varepsilon\mathbb{Z}\cap(0,\delta]}[\langle\gamma_{*\tau-\varepsilon}g_\tau,v\gamma_{*\tau-\varepsilon}g_\tau\rangle-\langle\gamma_{*\tau-\varepsilon}\hat{g}_\tau,v\gamma_{*\tau-\varepsilon}\hat{g}_\tau\rangle]$$

with

$$\gamma_{*\tau}=j(\tau)\alpha^*,\quad g_\tau=j(2\delta-\tau)\beta,\quad \hat{g}_\tau=j(\delta-\tau)(j(\delta)\beta+z)=j(2\delta-\tau)\beta+j(\delta-\tau)z. \tag{3.4}$$

We apply Proposition A.3 [part (ii)] of Ref. 3, with d replaced by md, $r=4$, $s=3$, $h(\gamma_1,\ldots,\gamma_4)=(\gamma_1\gamma_2,v\gamma_3\gamma_4)$, $\alpha_1=\alpha^*$, $\alpha_2=\beta$, $\alpha_3=z$, weights $\lambda_1=\cdots=\lambda_4=1$, and

$$\Gamma_1^1=\Gamma_3^1=j(\tau-\varepsilon),\quad \Gamma_2^2=\Gamma_4^2=j(2\delta-\tau),\quad \Gamma_2^3=\Gamma_4^3=0,$$

$$\tilde{\Gamma}_1^1=\tilde{\Gamma}_3^1=j(\tau-\varepsilon),\quad \tilde{\Gamma}_2^2=\tilde{\Gamma}_4^2=j(2\delta-\tau),\quad \tilde{\Gamma}_2^3=\tilde{\Gamma}_4^3=j(\delta-\tau) \tag{3.5}$$

with all other Γ_i^j's and $\tilde{\Gamma}_i^j$'s being zero. Then

$$\sigma = \kappa(2\delta)\max\left\{ |||j(\tau-\varepsilon)|||,\ |||j(2\delta-\tau)|||,\ |||j(2\delta-\tau)||| + 4\frac{r(\delta)}{\kappa(2\delta)}|||j(\delta-\tau)||| \right\}$$
$$\leq \left(1+4\frac{r(\delta)}{\kappa(2\delta)}\right)e^{2K_j\delta}\kappa(2\delta),$$

$$\sigma_\delta = 4r(\delta)|||j(\delta-\tau)||| \leq 4e^{K_j\delta}r(\delta). \tag{3.6}$$

So, for each $\tau\in\varepsilon\mathbb{Z}\cap(0,\delta]$,

$$\|\langle\gamma_{*\tau-\varepsilon}g_\tau, v\gamma_{*\tau-\varepsilon}g_\tau\rangle - \langle\gamma_{*\tau-\varepsilon}\hat g_\tau, v\gamma_{*\tau-\varepsilon}\hat g_\tau\rangle\|_{\rm fluct} \leq 4\|h\|_{w_\lambda}\sigma_\delta\sigma^3 \leq 2^9e^{7K_j\delta}|||v|||r(\delta)\kappa(2\delta)^3.$$

Here, we used that $r(\delta)\leq\frac14\kappa(\delta)\leq\frac12\kappa(2\delta)$ by Hypothesis 1.1 [parts (i) and (ii)]. Summing over τ gives

$$\|\mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta+z) - \mathcal{V}_\delta(\varepsilon;\alpha^*,j(\delta)\beta)\|_{\rm fluct} \leq 2^9e^{7K_j\delta}\delta|||v|||r(\delta)\kappa(2\delta)^3.$$

Similarly,

$$\|\mathcal{V}_\delta(\varepsilon;j(\delta)\alpha^*+z_*,\beta) - \mathcal{V}_\delta(\varepsilon;j(\delta)\alpha^*+z_*,\beta)\|_{\rm fluct} \leq 2^9e^{7K_j\delta}\delta|||v|||r(\delta)\kappa(2\delta)^3.$$

Next, by Corollary A.2 of Ref. 3 for any analytic function $f(\alpha^*,\beta)$,

$$\|f(\alpha^*,j(\delta)\beta+z) - f(\alpha^*,j(\delta)\beta)\|_{\rm fluct} \leq \|f(\alpha^*,j(\delta)\beta+z)\|_{\rm fluct} \leq \|f(\alpha^*,\beta)\|_\delta \tag{3.7}$$

since

$$\frac{\kappa(2\delta)}{\kappa(\delta)}|||j(\delta)||| + \frac{4(r\delta)}{\kappa(\delta)}|||\mathbb{1}||| \leq e^{\delta K_j}\frac{\kappa(2\delta)}{\kappa(\delta)} + \frac{4r(\delta)}{\kappa(\delta)} \leq 1$$

by Hypothesis 1.1 [part (ii)]. In particular $\|\mathcal{E}(\alpha^*,j(\delta)\beta+z)-\mathcal{E}(\alpha^*,j(\delta)\beta)\|_{\rm fluct}\leq\|\mathcal{E}\|_\delta$. Similarly $\|\mathcal{E}(j(\delta)\alpha^*+z_*,\beta)-\mathcal{E}(j(\delta)\alpha^*,\beta)\|_{\rm fluct}\leq\|\mathcal{E}\|_\delta$.

Combining the bounds of the previous two paragraphs with the assumption on $\|\mathcal{E}\|_\delta$, we get

$$\|\partial A_{\delta,\varepsilon}(\mathcal{E};\cdot)\|_{\rm fluct} \leq 2^{10}e^{7\delta K_j}\delta|||v|||r(\delta)\kappa(2\delta)^3 + 2\|\mathcal{E}\|_\delta \leq 2^{11}e^{7\delta K_j}\delta|||v|||r(\delta)\kappa(2\delta)^3 \leq \tfrac{1}{64} \tag{3.8}$$

by Hypothesis 1.1 [part (iv)] and (3.1a). By (3.3) and Corollary 3.5 (Ref. 3) with $n=1$,

$$\left\|\log\frac{\int d\tilde\mu_{r(\delta)}(z^*,z)e^{\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\alpha^*,\beta;z_*,z)}}{\int d\tilde\mu_{r(\delta)}(z^*,z)}\right\|_{2\delta} \leq \frac{\|\partial A_{\delta,\varepsilon}(\mathcal{E};\cdot)\|^2_{\rm fluct}}{\left(\dfrac{1}{20}-\|\partial A_{\delta,\varepsilon}(\mathcal{E};\cdot)\|_{\rm fluct}\right)^2} \leq 2^{32}e^{14\delta K_j}\delta^2|||v|||^2r(\delta)^2\kappa(2\delta)^6.$$

Combining this estimate and the estimate of Lemma 3.4, below, with $f=\mathcal{E}$, we get the desired bound on $\|\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}]\|_{2\delta}$. ■

Lemma 3.4: Let $f(\alpha^*,\beta)$ be an analytic function that has degree of at least 2 both in α^* and β. Then,

$$\|f(\alpha^*,j(\delta)\beta)\|_{2\delta},\ \|f(j(\delta)\alpha^*,\beta)\|_{2\delta} \leq e^{2\delta K_j}\left(\frac{\kappa(2\delta)}{\kappa(\delta)}\right)^4\|f\|_\delta.$$

Proof: Introduce the auxiliary weight system $w_{\rm aux}$ with, in the language of Ref. 3 [Definitions 2.5 and 2.6 and, more specifically, (A.1)], metric md that associates the constant weight factor to the field α_* and the constant weight factor $e^{-\delta K_j}\kappa(\delta)$ to the field β. Since, by (1.11), $[e^{-\delta K_j}\kappa(\delta)/\kappa(\delta)]|||j(\delta)|||\leq 1$, Corollary A.2 of Ref. 3 gives

$$\|f(\alpha^*,j(\delta)\beta)\|_{w_{\mathrm{aux}}} \le \|f\|_\delta.$$

As $f(\alpha^*,j(\delta)\beta)$ has degree of at least 2 both in α^* and β and $e^{-\delta K_j}\kappa(\delta)\ge\kappa(2\delta)$, by Hypothesis 1.1 [part (ii)],

$$\|f(\alpha^*,j(\delta)\beta)\|_{2\delta} \le \left(\frac{\kappa(2\delta)}{\kappa(\delta)}\right)^2\left(\frac{\kappa(2\delta)}{e^{-\delta K_j}\kappa(\delta)}\right)^2\|f(\alpha^*,j(\delta)\beta)\|_{w_{\mathrm{aux}}} \le e^{2\delta K_j}\left(\frac{\kappa(2\delta)}{\kappa(\delta)}\right)^4\|f\|_\delta.$$

The estimate on $\|f(j(\delta)\alpha^*,\beta)\|_{2\delta}$ is similar. ■

Proof of Theorem 1.3: We write $\delta=\varepsilon_n=2^n\varepsilon$ and prove the statement by induction on n. In the case $n=0$, there is nothing to prove. For the induction step from n to $n+1$, set $\delta=\varepsilon_n$. The hypothesis of Proposition 3.3, with $\mathcal{E}=\mathcal{E}_\delta$, is satisfied since,

$$\|\mathcal{E}_\delta\|_\delta \le K_E\delta^2|||v|||^2 r(\delta)^2\kappa(\delta)^6 \le 8\delta|||v|||r(\delta)\kappa(2\delta)^3 \tag{3.9}$$

by the inductive hypothesis and Hypothesis 1.1 [parts (i) and (iv)]. Using (3.2), Proposition 3.3, and (3.1a) we see that

$$\begin{aligned}
\|\mathcal{E}_{\varepsilon_{n+1}}\|_{\varepsilon_{n+1}} &\le 2^{32}e^{14\delta K_j}\delta^2|||v|||^2 r(\delta)^2\kappa(2\delta)^6 + 2e^{2\delta K_j}\left(\frac{\kappa(2\delta)}{\kappa(\delta)}\right)^4 K_E\delta^2|||v|||^2 r(\delta)^2\kappa(\delta)^6\\
&\le \left[2^{32}e^{14\delta K_j} + 2e^{2\delta K_j}\left(\frac{\kappa(\delta)}{\kappa(2\delta)}\right)^2 K_E\right]\delta^2|||v|||^2 r(\delta)^2\kappa(2\delta)^6\\
&= \frac{1}{2}\left[\frac{2^{31}e^{14\delta K_j}}{K_E} + e^{2\delta K_j}\left(\frac{\kappa(\delta)}{\kappa(2\delta)}\right)^2\right]\left(\frac{r(\delta)}{r(2\delta)}\right)^2 K_E(2\delta)^2|||v|||^2 r(2\delta)^2\kappa(2\delta)^6\\
&\le K_E(2\delta)^2|||v|||^2 r(2\delta)^2\kappa(2\delta)^6\\
&= K_E\varepsilon_{n+1}^2|||v|||^2 r(\varepsilon_{n+1})^2\kappa(\varepsilon_{n+1})^6.
\end{aligned}$$

■

In the proof of Theorem 1.4, we shall compare $\mathcal{E}_\theta(2^{-m-1}\theta;\alpha^*,\beta)$ and $\mathcal{E}_\theta(2^{-m}\theta;\alpha^*,\beta)$ to prove that the sequence $\mathcal{E}_\theta(2^{-m}\theta;\cdot)$ is Cauchy with respect to our norm. To do so, we shall compare $\mathcal{E}_{\varepsilon_n}(\varepsilon/2;\cdot)$ and $\mathcal{E}_{\varepsilon_n}(\varepsilon;\cdot)$ for each $n=0,\ldots,\log_2(\Theta/\varepsilon)$. This is done by induction on n. For the induction step, we use Proposition 3.6, below. To prepare for it, we have as follows.

Lemma 3.5: Set

$$W(\alpha_*,\beta) = \mathcal{V}_\delta(\varepsilon;\alpha_*,\beta) - \mathcal{V}_\delta\left(\frac{\varepsilon}{2};\alpha_*,\beta\right).$$

Then

$$\|W(\alpha_*,z+j(\delta)\beta) - W(\alpha_*,j(\delta)\beta)\|_{\mathrm{fluct}} \le 2^{10}e^{9K_j}\varepsilon\delta|||v|||r(\delta)\kappa(2\delta)^3,$$

$$\|W(z_*+j(\delta)\alpha_*,\beta) - W(j(\delta)\alpha_*,\beta)\|_{\mathrm{fluct}} \le 2^{10}e^{9K_j}\varepsilon\delta|||v|||r(\delta)\kappa(2\delta)^3.$$

Proof: We prove the first inequality. By definition

$$W(\alpha_*,\beta) = W_1(\alpha_*,\beta) + W_2(\alpha_*,\beta)$$

with

$$W_1(\alpha_*,\beta) = -\frac{\varepsilon}{2}\sum_{\tau\in\varepsilon\mathbb{Z}\cap(0,\delta]}[\langle\gamma_{*\tau-\varepsilon}\gamma_\tau, v\,\gamma_{*\tau-\varepsilon}\gamma_\tau\rangle - \langle\gamma_{*\tau-\varepsilon}\gamma_{\tau-\varepsilon/2}, v\,\gamma_{*\tau-\varepsilon}\gamma_{\tau-\varepsilon/2}\rangle],$$

$$W_2(\alpha_*,\beta) = -\frac{\varepsilon}{2}\sum_{\tau\in\varepsilon\mathbb{Z}\cap(0,\delta]}[\langle\gamma_{*\tau-\varepsilon}\gamma_\tau, v\,\gamma_{*\tau-\varepsilon}\gamma_\tau\rangle - \langle\gamma_{*\tau-\varepsilon/2}\gamma_\tau, v\,\gamma_{*\tau-\varepsilon/2}\gamma_\tau\rangle],$$

where

$$\gamma_{*\tau} = j(\tau)\alpha^*, \quad \gamma_\tau = j(\delta-\tau).$$

Using the notation of (3.4),

$$\begin{aligned}
W_1(\alpha_*, j(\delta)\beta) - W_1(\alpha_*, z + j(\delta)\beta) &= -\frac{\varepsilon}{2}\sum_{\tau\in\varepsilon\mathbb{Z}\cap(0,\delta]}[\langle\gamma_{*\tau-\varepsilon}g_\tau, v\,\gamma_{*\tau-\varepsilon}g_\tau\rangle - \langle\gamma_{*\tau-\varepsilon}g_{\tau-\varepsilon/2}, v\,\gamma_{*\tau-\varepsilon}g_{\tau-\varepsilon/2}\rangle \\
&\quad - \langle\gamma_{*\tau-\varepsilon}\hat{g}_\tau, v\,\gamma_{*\tau-\varepsilon}\hat{g}_\tau\rangle + \langle\gamma_{*\tau-\varepsilon}\hat{g}_{\tau-\varepsilon/2}, v\,\gamma_{*\tau-\varepsilon}\hat{g}_{\tau-\varepsilon/2}\rangle] \\
&= -\frac{\varepsilon}{2}\sum_{\tau\in\varepsilon\mathbb{Z}\cap(0,\delta]}\Bigg[\langle\gamma_{*\tau-\varepsilon}g_\tau, v\,\gamma_{*\tau-\varepsilon}g_\tau\rangle \\
&\quad - \left\langle\gamma_{*\tau-\varepsilon}j\left(\frac{\varepsilon}{2}\right)g_\tau, v\,\gamma_{*\tau-\varepsilon}j\left(\frac{\varepsilon}{2}\right)g_\tau\right\rangle \\
&\quad - \langle\gamma_{*\tau-\varepsilon}\hat{g}_\tau, v\,\gamma_{*\tau-\varepsilon}\hat{g}_\tau\rangle + \left\langle\gamma_{*\tau-\varepsilon}j\left(\frac{\varepsilon}{2}\right)\hat{g}_\tau, v\,\gamma_{*\tau-\varepsilon}j\left(\frac{\varepsilon}{2}\right)\hat{g}_\tau\right\rangle\Bigg].
\end{aligned}$$

This time, we apply Proposition A.3 [part (iii)] of Ref. 3 using the Γ_i^j's and $\tilde{\Gamma}_i^j$'s of (3.5) and, in addition,

$$A_1 = \tilde{A}_1 = A_3 = \tilde{A}_3 = \mathbb{1}, \quad A_2 = A_4 = \mathbb{1}, \quad \tilde{A}_2 = \tilde{A}_4 = j\left(\frac{\varepsilon}{2}\right).$$

The corollary bounds the $\|\cdot\|_{\text{fluct}}$ norm of the τ term by $4^2|||v|||\sigma_\delta a_\delta(\sigma a)^3$ with σ and σ_δ of (3.6) and

$$a = \max\left\{|||\mathbb{1}|||, \left|\left|\left|j\left(\frac{\varepsilon}{2}\right)\right|\right|\right|\right\} \le e^{K_j(\varepsilon/2)},$$

$$a_\delta = \left|\left|\left|j\left(\frac{\varepsilon}{2}\right) - \mathbb{1}\right|\right|\right| \le \frac{\varepsilon}{2}K_j e^{K_j(\varepsilon/2)}$$

by (1.11). Inserting and summing over τ, we get

$$\begin{aligned}
&\|W_1(\alpha_*, z + j(\delta)\beta) - W_1(\alpha_*, j(\delta)\beta)\|_{\text{fluct}} \\
&\quad \le \frac{\varepsilon}{2}\frac{\delta}{\varepsilon}4^2|||v|||4e^{K_j\delta}r(\delta)\frac{\varepsilon}{2}K_j e^{K_j(\varepsilon/2)}\left(\left(1 + 4\frac{r(\delta)}{\kappa(2\delta)}\right)e^{2K_j\delta}\kappa(2\delta)e^{K_j(\varepsilon/2)}\right)^3 \\
&\quad \le 2^9 e^{9K_j}\varepsilon\delta|||v|||r(\delta)\kappa(2\delta)^3.
\end{aligned}$$

The same estimate holds for $\|W_2(\alpha_*, z + j(\delta)\beta) - W_2(\alpha_*, j(\delta)\beta)\|_{\text{fluct}}$. ■

Proposition 3.6: Under the hypotheses of Proposition 3.3, assume that there is a second analytic function $\tilde{\mathcal{E}}(\alpha^*,\beta)$, *which has similar properties to* $\mathcal{E}$ *and is close to* $\mathcal{E}$. *Precisely, we assume that* $\tilde{\mathcal{E}}$ *is of degree of at least 2 both in* α^* *and* β *and obeys* $\|\tilde{\mathcal{E}}\|_\delta \le 2^9 e^{7\delta K_j}\delta|||v|||r(\delta)\kappa(2\delta)^3$. *Then*

$$\|\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}] - \mathfrak{R}_{\delta,\varepsilon/2}[\tilde{\mathcal{E}}]\|_{2\delta} \le 2^{36}e^{18K_j}\varepsilon\delta^2|||v|||^2 r(\delta)^2\kappa(2\delta)^6 + q\|\mathcal{E} - \tilde{\mathcal{E}}\|_\delta,$$

where q is the constant in parts (v) and (vi) of Hypothesis 1.1.

Proof: By Definition 3.2,

$$\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}]-\mathfrak{R}_{\delta,\varepsilon/2}[\widetilde{\mathcal{E}}]=\mathcal{B}(\alpha^*,\beta)+\log\frac{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)e^{\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\alpha^*,\beta;z^*,z)}}{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)}-\log\frac{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)e^{\partial\mathcal{A}_{\delta,\varepsilon/2}(\widetilde{\mathcal{E}};\alpha^*,\beta;z^*,z)}}{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)}, \tag{3.10}$$

where $\mathcal{B}(\alpha^*,\beta)=(\mathcal{E}-\widetilde{\mathcal{E}})(\alpha^*,j(\delta)\beta)+(\mathcal{E}-\widetilde{\mathcal{E}})(j(\delta)\alpha^*,\beta)$. By Lemma 3.4,

$$\|\mathcal{B}\|_{2\delta}\le 2e^{2\delta K_j}\left(\frac{\kappa(2\delta)}{\kappa(\delta)}\right)^4\|\mathcal{E}-\widetilde{\mathcal{E}}\|_{\delta} \tag{3.11}$$

With the notation of the previous lemma,

$$\begin{aligned}&\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\alpha^*,\beta;z_*,z)-\partial\mathcal{A}_{\delta,\varepsilon/2}(\widetilde{\mathcal{E}};\alpha^*,\beta;z_*,z)\\&\quad=[W(\alpha^*,z+j(\delta)\beta)-W(\alpha^*,j(\delta)\beta)]+[W(z_*+j(\delta)\alpha^*,\beta)-W(j(\delta)\alpha^*,\beta)]+\mathcal{C}(\alpha^*,\beta;z_*,z),\end{aligned}$$

where

$$\begin{aligned}\mathcal{C}(\alpha^*,\beta;z_*,z)=&[(\mathcal{E}-\widetilde{\mathcal{E}})(\alpha^*,j(\delta)\beta+z)-(\mathcal{E}-\widetilde{\mathcal{E}})(\alpha^*,j(\delta)\beta)]+[(\mathcal{E}-\widetilde{\mathcal{E}})(j(\delta)\alpha^*+z_*,\beta)\\&-(\mathcal{E}-\widetilde{\mathcal{E}})(j(\delta)\alpha^*,\beta)].\end{aligned}$$

By (3.7), $\|\mathcal{C}\|_{\text{fluct}}\le 2\|\mathcal{E}-\widetilde{\mathcal{E}}\|_{\delta}$. Combining this with Lemma 3.5, we get

$$\begin{aligned}\|\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\cdot)-\partial\mathcal{A}_{\delta,\varepsilon/2}(\widetilde{\mathcal{E}};\cdot)\|_{\text{fluct}}&\le 2^{11}e^{9K_j}\varepsilon\delta|||v|||r(\delta)\kappa(2\delta)^3+2\|\mathcal{E}-\widetilde{\mathcal{E}}\|_{\delta}\\&\le\{2^{11}e^{9K_j}\varepsilon+2^{11}e^{7\delta K_j}\}\delta|||v|||r(\delta)\kappa(2\delta)^3.\end{aligned} \tag{3.12}$$

Consequently, by (3.8),

$$\begin{aligned}\|\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\cdot)\|_{\text{fluct}}+\|\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\cdot)-\partial\mathcal{A}_{\delta,\varepsilon/2}(\widetilde{\mathcal{E}};\cdot)\|_{\text{fluct}}&\le\{2^{11}e^{9K_j}\varepsilon+2^{12}e^{7\delta K_j}\}\delta|||v|||r(\delta)\kappa(2\delta)^3\\&\le 2^{13}e^{9K_j}\delta|||v|||r(\delta)\kappa(2\delta)^3\le\frac{1}{17}-\frac{1}{32}\end{aligned}$$

by (3.1a) and Hypothesis 1.1 [part (iv)]. Therefore, the hypotheses of Ref. 3 (Corollary 3.6) are satisfied and we have, using (3.3) and (3.12),

$$\begin{aligned}&\left\|\log\frac{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)e^{\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\alpha^*,\beta;z_*,z)}}{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)}-\log\frac{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)e^{\partial\mathcal{A}_{\delta,\varepsilon/2}(\widetilde{\mathcal{E}};\alpha^*,\beta;z_*,z)}}{\int d\widetilde{\mu}_{r(\delta)}(z^*,z)}\right\|_{2\delta}\\&\quad\le 2^{12}\{\|\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\cdot)\|_{\text{fluct}}+\|\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\cdot)-\partial\mathcal{A}_{\delta,\varepsilon/2}(\widetilde{\mathcal{E}};\cdot)\|_{\text{fluct}}\}\|\partial\mathcal{A}_{\delta,\varepsilon}(\mathcal{E};\cdot)-\partial\mathcal{A}_{\delta,\varepsilon/2}(\widetilde{\mathcal{E}};\cdot)\|_{\text{fluct}}\\&\quad\le 2^{25}e^{9K_j}\delta|||v|||r(\delta)\kappa(2\delta)^3\{2^{11}e^{9K_j}\varepsilon\delta|||v|||r(\delta)\kappa(2\delta)^3+2\|\mathcal{E}-\widetilde{\mathcal{E}}\|_{\delta}\}.\end{aligned}$$

Using this, (3.11) and (3.1c), we bound (3.10) by

$$\|\mathfrak{R}_{\delta,\varepsilon}[\mathcal{E}]-\mathfrak{R}_{\delta,\varepsilon/2}[\widetilde{\mathcal{E}}]\|_{2\delta}\le 2^{36}e^{18K_j}\varepsilon\delta^2|||v|||^2r(\delta)^2\kappa(2\delta)^6+q\|\mathcal{E}-\widetilde{\mathcal{E}}\|_{\delta}.$$

■

Corollary 3.7: For all sufficiently small $\varepsilon>0$ *and integers* $0\le n\le\log_2(\Theta/\varepsilon)$, *we have*

$$\left\|\mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,\beta)-\mathcal{E}_{\varepsilon_n}\left(\frac{\varepsilon}{2};\alpha^*,\beta\right)\right\|_{\varepsilon_n}\le K_Eq^n\varepsilon^2|||v|||^2r(\varepsilon)^2\kappa(\varepsilon)^6+2^{36}e^{18K_j}\varepsilon|||v|||^2\sum_{k=1}^{n}q^{n-k}\varepsilon_k^2r(\varepsilon_k)^2\kappa(\varepsilon_k)^6.$$

Proof: The proof is by induction on n. In the case $n=0$, $\mathcal{E}_{\varepsilon_n}(\varepsilon;\alpha^*,\beta)=0$ and

$$\left\|\mathcal{E}_{\varepsilon_n}\left(\frac{\varepsilon}{2};\alpha^*,\beta\right)\right\|_{\varepsilon_n} \le K_E\varepsilon^2|||v|||^2 r(\varepsilon)^2\kappa(\varepsilon)^6$$

by Theorem 1.3. For the induction step from n to $n+1$, apply Proposition 3.6, with $\delta=\varepsilon_n$, $\mathcal{E}=\mathcal{E}_{\varepsilon_n}(\varepsilon)$, and $\widetilde{\mathcal{E}}=\mathcal{E}_{\varepsilon_n}(\varepsilon/2)$. This gives, using Hypothesis 1.1 [part (i)] and the induction hypothesis on $\|\mathcal{E}_{\varepsilon_n}(\varepsilon)-\mathcal{E}_{\varepsilon_n}(\varepsilon/2)\|_{\varepsilon_n}$,

$$\begin{aligned}\left\|\mathcal{E}_{\varepsilon_{n+1}}(\varepsilon)-\mathcal{E}_{\varepsilon_{n+1}}\left(\frac{\varepsilon}{2}\right)\right\|_{\varepsilon_{n+1}} &\le 2^{36}e^{18K_j}\varepsilon|||v|||^2\varepsilon_n^2 r(\varepsilon_n)^2\kappa(\varepsilon_{n+1})^6 + q\left\|\mathcal{E}_{\varepsilon_n}(\varepsilon)-\mathcal{E}_{\varepsilon_n}\left(\frac{\varepsilon}{2}\right)\right\|_{\varepsilon_n}\\ &\le K_E q^{n+1}\varepsilon^2|||v|||^2 r(\varepsilon)^2\kappa(\varepsilon)^6 + 2^{36}e^{18K_j}\varepsilon|||v|||^2\sum_{k=1}^{n+1} q^{n+1-k}\varepsilon_k^2 r(\varepsilon_k)^2\kappa(\varepsilon_k)^6.\end{aligned}$$

■

Proof of Theorem 1.4: By Corollary 3.7, with $\varepsilon=2^{-m}\theta$ and $n=m$, we have, for sufficiently large m,

$$\begin{aligned}\left\|\mathcal{E}_\theta\left(\frac{\theta}{2^m};\alpha^*,\beta\right)-\mathcal{E}_\theta\left(\frac{\theta}{2^{m+1}};\alpha^*,\beta\right)\right\|_\theta &\le K_E\theta^2|||v|||^2\left(\frac{q}{4}\right)^m r\left(\frac{\theta}{2^m}\right)^2\kappa\left(\frac{\theta}{2^m}\right)^6\\ &\quad + 2^{36}e^{18K_j}|||v|||^2\frac{\theta^3}{2^m}\sum_{\ell=0}^{m-1}\left(\frac{q}{4}\right)^\ell r\left(\frac{\theta}{2^\ell}\right)^2\kappa\left(\frac{\theta}{2^\ell}\right)^6\end{aligned}$$

and, consequently,

$$\begin{aligned}&\sum_{\nu=m}^{\infty}\left\|\mathcal{E}_\theta\left(\frac{\theta}{2^\nu};\alpha^*,\beta\right)-\mathcal{E}_\theta\left(\frac{\theta}{2^{\nu+1}};\alpha^*,\beta\right)\right\|_\theta\\ &\quad\le \text{const }\ \theta^2|||v|||^2\left[\sum_{\nu=m}^{\infty}\left(\frac{q}{4}\right)^\nu r\left(\frac{\theta}{2^\nu}\right)^2\kappa\left(\frac{\theta}{2^\nu}\right)^6 + \sum_{\nu=m}^{\infty}\frac{\theta}{2^\nu}\sum_{\ell=0}^{\nu-1}\left(\frac{q}{4}\right)^\ell r\left(\frac{\theta}{2^\ell}\right)^2\kappa\left(\frac{\theta}{2^\ell}\right)^6\right]\\ &\quad\le \text{const }\ \theta^2|||v|||^2\left[\sum_{k=m}^{\infty}\left(\frac{q}{4}\right)^k r\left(\frac{\theta}{2^k}\right)^2\kappa\left(\frac{\theta}{2^k}\right)^6 + \frac{\theta}{2^m}\sum_{k=0}^{\infty}\left(\frac{q}{4}\right)^k r\left(\frac{\theta}{2^k}\right)^2\kappa\left(\frac{\theta}{2^k}\right)^6\sum_{\nu=0}^{\infty}\frac{1}{2^\nu}\right].\end{aligned}$$

Hence, by Hypothesis 1.1 [part (vi)] and the Cauchy criterion, the sequence $\mathcal{E}_\theta(2^{-m}\theta;\alpha^*,\beta)$ converges uniformly in θ. This gives Theorem 1.4. ■

APPENDIX A: APPENDIX ON STOKES' THEOREM

Lemma A.1: Let $r>0$ and σ , σ_* , $\rho\in\mathbb{C}^X$ obey $|\rho(\mathbf{x})+\sigma_*(\mathbf{x})^*-\sigma(\mathbf{x})|<2r$ for all $\mathbf{x}\in X$. Set

$$D_{\sigma_*,\sigma,\rho}(\mathbf{x})=\{(z_*(\mathbf{x}),z(\mathbf{x}))\in\mathbb{C}^2\,|\,|z_*(\mathbf{x})-\sigma_*(\mathbf{x})|\le r,\ \ |z(\mathbf{x})-\sigma(\mathbf{x})|\le r,\ \ z(\mathbf{x})-z_*(\mathbf{x})^*=\rho(\mathbf{x})\},$$

$$D_{\sigma_*,\sigma,\rho}=\mathop{\mathsf{X}}_{\mathbf{x}\in X}D_{\sigma_*,\sigma,\rho}(\mathbf{x}).$$

Let, for each $\mathbf{x}\in X$, $C_{\sigma_*,\sigma,\rho}(\mathbf{x})$ be any two real dimensional submanifold of $\mathbb{C}^2$ whose boundary is the union of the one real dimensional submanifolds $\partial D_{\sigma_*,\sigma,\rho}(\mathbf{x})$ and the circle $\{(z_*(\mathbf{x}),z(\mathbf{x}))\in\mathbb{C}^2\,|\,z_*^*(\mathbf{x})=z(\mathbf{x}),|z(\mathbf{x})|=r\}$. [By submanifold, we really mean a submanifold with corners. The orientation of $C_{\sigma_*,\sigma,\rho}(\mathbf{x})$ must also be chosen appropriately.] Furthermore, let $f(\alpha_1,\ldots,\alpha_s;z_*,z)$ be a function that is holomorphic in the variables $\alpha_1,\ldots,\alpha_s$ in a neighborhood of the origin in $\mathbb{C}^{sX}$ and in the variables $(z_*,z)\in\mathsf{X}_{\mathbf{x}\in X}\mathcal{P}(\mathbf{x})$, with, for each $\mathbf{x}\in X$, $\mathcal{P}(\mathbf{x})$ being an open poly disk in $\mathbb{C}^2$

that contains $C_{\sigma_*,\sigma,\rho}(\mathbf{x})$. *Then,*

$$\int_{D_{\sigma_*,\sigma,\rho}} \prod_{\mathbf{x}\in X}\left[\frac{dz_*(\mathbf{x})\wedge dz(\mathbf{x})}{2\pi i}e^{-z_*(\mathbf{x})z(\mathbf{x})}\right]e^{f(\alpha_1,\ldots,\alpha_s;z_*,z)}$$

$$= \sum_{R\subset X}\prod_{\mathbf{x}\in R}\left(\int_{C_{\sigma_*,\sigma,\rho}(\mathbf{x})}\frac{dz_*(\mathbf{x})\wedge dz(\mathbf{x})}{2\pi i}e^{-z_*(\mathbf{x})z(\mathbf{x})}\right)$$

$$\times \prod_{\mathbf{x}\in X\setminus R}\left(\int_{|z(\mathbf{x})|\le r}\frac{dz(\mathbf{x})^*\wedge dz(\mathbf{x})}{2\pi i}e^{-z_*(\mathbf{x})z(\mathbf{x})}\right)e^{f(\alpha_1,\ldots,\alpha_s;z_*,z)}\Big|_{\substack{z_*(\mathbf{x})=z(\mathbf{x})^* \\ \text{for } \mathbf{x}\in X\setminus R}}.$$

Proof: In the proof we suppress the subscripts σ_*,σ,ρ. For each $\mathbf{x}\in X$ there is a three real dimensional submanifold $\mathcal{B}(\mathbf{x})\subset\mathcal{P}(\mathbf{x})$ whose boundary is the union of $D(\mathbf{x})$, $C(\mathbf{x})$ and

$$D_{\mathrm{IR}}(\mathbf{x}) = \{(z_*(\mathbf{x}),z(\mathbf{x}))\in\mathbb{C}^2 \mid z_*^*(\mathbf{x})=z(\mathbf{x}),|z(\mathbf{x})|\le r\}.$$

We apply Stokes' theorem once for each point $\mathbf{x}\in X$ to the differential form

$$\omega = \prod_{\mathbf{x}\in X}\frac{dz_*(\mathbf{x})\wedge dz(\mathbf{x})}{2\pi i}\exp\{-\langle z_*,z\rangle + f(\alpha_1,\ldots,\alpha_s;z_*,z)\}.$$

Since ω is a holomorphic $2|X|$ form in $\mathbb{C}^{2|X|}$, $d\omega=0$ and

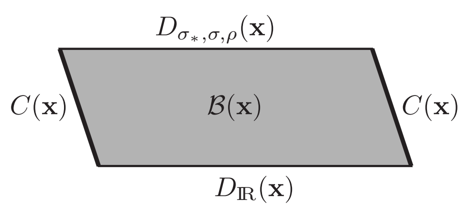
$$\int_D \omega = \sum_{R\subset X}\int_{M_R}\omega, \quad \text{where } M_R=\prod_{\mathbf{x}\notin R}D_{\mathrm{IR}}(\mathbf{x})\times\prod_{\mathbf{x}\in R}C(\mathbf{x}).$$

■

Example A.2: In Lemma A.1, $C(\mathbf{x})=C_{\sigma_*,\sigma,\rho}(\mathbf{x})$ must be a surface whose boundary coincides with the union of the boundaries of $D_{\mathrm{IR}}(\mathbf{x})$ and $D_{\sigma_*,\sigma,\rho}(\mathbf{x})$. A possible choice of such a surface is constructed as follows. Interpolate between $D_{\mathrm{IR}}(\mathbf{x})$ and $D_{\sigma_*,\sigma,\rho}(\mathbf{x})$ by the three dimensional set $\mathcal{B}(\mathbf{x})=\cup_{0\le t\le 1}D_t(\mathbf{x})$, where

$$D_t(\mathbf{x}) = \{(z_*,z)\in\mathbb{C}^2 \mid |z_*-t\sigma_*(\mathbf{x})|\le r,\ \ |z-t\sigma(\mathbf{x})|\le r,\ \ z-z_*^*=t\rho(\mathbf{x})\}.$$

Then $C(\mathbf{x})=\cup_{0<t<1}\partial D_t(\mathbf{x})$ has the required boundary



Remark A.3: In the above example,

$$\mathrm{Re}(z_*z)\ge \tfrac{1}{2}(r^2-|\rho(\mathbf{x})|^2)-r(|\sigma(\mathbf{x})|+|\sigma_*(\mathbf{x})|)$$

for all $(z_*,z)\in C(\mathbf{x})$. Furthermore, the area of $C(\mathbf{x})$ is bounded by $8\pi r[|\sigma|+|\sigma_*|+|\rho|]$.

Proof: Let $(z_*,z)\in C(\mathbf{x})$. We suppress the dependence on $\mathbf{x}$. There is a $0\le t\le 1$ such that $\max\{|z_*-t\sigma_*|,|z-t\sigma|\}=r$ and $z_*=z^*-t\rho^*$. So

$$z_*z = |z-t\sigma|^2 + 2\ \mathrm{Re}(z-t\sigma)t\sigma^* + |t\sigma|^2 - t\rho^*z,$$

$$z_*z = |z_*-t\sigma_*|^2 + 2\ \mathrm{Re}(z_*-t\sigma_*)t\sigma_*^* + |t\sigma_*|^2 + t\rho z^* - |t\rho|^2.$$

Adding and taking the real part,

$$2\,\mathrm{Re}(z_* z) = |z - t\sigma|^2 + |z_* - t\sigma_*|^2 + 2\,\mathrm{Re}(z - t\sigma)t\sigma^* + 2\,\mathrm{Re}(z_* - t\sigma_*)t\sigma_*^* + t^2(|\sigma|^2 + |\sigma_*|^2 - |\rho|^2)$$
$$\geq r^2 - 2r(|\sigma| + |\sigma_*|) - |\rho|^2.$$

By construction, $C(\mathbf{x})$ is contained in the union of the two cylinders

$$\{(r\zeta + t\sigma, r\zeta^* + t(\sigma^* - \rho^*))\,|\,|\zeta| = 1, \quad t \in [0,1]\},$$

$$\{(r\zeta^* + t(\sigma_*^* + \rho), r\zeta + t\sigma_*)\,|\,|\zeta| = 1, \quad t \in [0,1]\}.$$

The area of the first is bounded by $2\pi\sqrt{2r}\sqrt{|\sigma|^2+|\sigma-\rho|^2}$ and the area of the second is bounded by $2\pi\sqrt{2r}\sqrt{|\sigma_*|^2+|\sigma_*^*-\rho|^2}$. ■

APPENDIX B: PROPERTIES OF $j(\tau)$

We discuss the decay properties of the operator $j(\tau)=e^{-\tau(\mathrm{h}-\mu)}$ using the operator norm (1.10).

Lemma B.1:

(a) *For any two operators $\mathcal{A},\mathcal{B}:L^2(X)\to L^2(X)$,*

$$|||\mathcal{AB}||| \leq |||\mathcal{A}|||\,|||\mathcal{B}|||.$$

(b) *For any operator $\mathcal{A}:L^2(X)\to L^2(X)$ and any complex number α,*

$$|||e^{\alpha\mathcal{A}}||| \leq e^{|\alpha||||\mathcal{A}|||} \quad |||e^{\alpha\mathcal{A}} - \mathbb{1}||| \leq |\alpha||||\mathcal{A}|||e^{|\alpha||||\mathcal{A}|||}$$

Proof:

(a) By the triangle inequality, for each $\mathbf{x}\in X$,

$$\sum_{\mathbf{y}\in X} e^{\mathrm{m}d(\mathbf{x},\mathbf{y})}|(\mathcal{AB})(\mathbf{x},\mathbf{y})| \leq \sum_{\mathbf{y},\mathbf{z}\in X} e^{\mathrm{m}d(\mathbf{x},\mathbf{z})}|\mathcal{A}(\mathbf{x},\mathbf{z})|e^{\mathrm{m}d(\mathbf{z},\mathbf{y})}|\mathcal{B}(\mathbf{z},\mathbf{y})|$$
$$\leq \sum_{\mathbf{z}\in X} e^{\mathrm{m}d(\mathbf{x},\mathbf{z})}|\mathcal{A}(\mathbf{x},\mathbf{z})||||\mathcal{B}|||$$
$$\leq |||\mathcal{A}|||\,|||\mathcal{B}|||.$$

The other bound is similar.

(b) By part (a),

$$|||e^{\alpha\mathcal{A}}||| \leq \sum_{n=0}^{\infty}\frac{1}{n!}|||\alpha^n\mathcal{A}^n||| \leq \sum_{n=0}^{\infty}\frac{1}{n!}|\alpha|^n|||\mathcal{A}|||^n = e^{|\alpha|\;|||\mathcal{A}|||}$$

and

$$|||e^{\alpha\mathcal{A}} - \mathbb{1}||| \leq \sum_{n=1}^{\infty}\frac{1}{n!}|||\alpha^n\mathcal{A}^n||| \leq \sum_{n=1}^{\infty}\frac{1}{n!}|\alpha|^n|||\mathcal{A}|||^n \leq |\alpha|\;|||\mathcal{A}|||e^{|\alpha|\;|||\mathcal{A}|||}.$$

■

Corollary B.2: Let $\tau\geq 0$,

$$|||j(\tau)||| \leq e^{\tau(|||\mathrm{h}|||+\mu)}, \quad |||j(\tau) - \mathbb{1}||| \leq \tau(|||\mathrm{h}||| + |\mu|)e^{\tau(|||\mathrm{h}|||+|\mu|)}.$$

Proof: Write $j(\tau)=e^{\tau\mu}e^{-\tau\mathrm{h}}$ and $j(\tau)-\mathbb{1}=e^{\tau\mu}(e^{-\tau\mathrm{h}}-\mathbb{1})+e^{\tau\mu}-\mathbb{1}$. By the previous lemma

$$|||j(\tau)||| = e^{\tau\mu}|||e^{-\tau h}||| \le e^{\tau\mu}e^{\tau|||h|||}$$

and

$$|||j(\tau) - \mathbb{1}||| \le e^{\tau\mu}|||e^{-\tau h} - \mathbb{1}||| + |||e^{\tau\mu} - \mathbb{1}||| \le \tau|||h|||e^{\tau\mu}e^{\tau|||h|||} + |e^{\tau\mu} - 1|.$$

■

APPENDIX C: THE NORMALIZATION CONSTANT

We define the normalization constant $\mathcal{Z}_\delta(\varepsilon)$ by the recursion relations

$$\mathcal{Z}_\varepsilon(\varepsilon) = 1, \quad \mathcal{Z}_{2\delta}(\varepsilon) = \mathcal{Z}_\delta(\varepsilon)^2 \int_{|z|\le r(\delta)} \frac{dz^* \wedge dz}{2\pi i} e^{-|z|^2}.$$

Lemma C.1: For all $0<\varepsilon<\delta$ *with* δ/ε *a positive integer power of 2, we have* $0<\mathcal{Z}_\delta(\varepsilon)<1$ *and*

$$|\ln \mathcal{Z}_\delta(\varepsilon)| \le e^{-r(\delta)^2}.$$

Furthermore, the limit $\mathcal{Z}_\delta = \lim_{n\to\infty} \mathcal{Z}_\delta(\delta/2^n)$ *exists and also obeys* $|\ln \mathcal{Z}_\delta| \le e^{-r(\delta)^2}$.

Proof: Start by fixing any $\varepsilon>0$ and writing $\varepsilon_n = 2^n\varepsilon$. From the inductive definition,

$$\ln \mathcal{Z}_{\varepsilon_{n+1}}(\varepsilon) = 2 \ln \mathcal{Z}_{\varepsilon_n}(\varepsilon) + \ln \mathcal{Z}'_{\varepsilon_n}, \quad \text{where } \mathcal{Z}'_{\varepsilon_n} = \int_{|z|\le r(\varepsilon_n)} \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} < 1$$

so that $0<\mathcal{Z}_{\varepsilon_n}(\varepsilon)<1$ for all $n \in \mathrm{IN}$ and

$$2^{-n-1}|\ln \mathcal{Z}_{\varepsilon_{n+1}}(\varepsilon)| = 2^{-n}|\ln \mathcal{Z}_{\varepsilon_n}(\varepsilon)| + 2^{-n-1}|\ln \mathcal{Z}'_{\varepsilon_n}|$$

which implies that

$$2^{-n}|\ln \mathcal{Z}_{\varepsilon_n}(\varepsilon)| = \sum_{k=0}^{n-1} 2^{-k-1}|\ln \mathcal{Z}'_{\varepsilon_k}|. \tag{C1}$$

Since

$$1 - \mathcal{Z}'_{\varepsilon_k} = \int_{|(x,y)|\ge r(\varepsilon_k)} \frac{dxdy}{\pi} e^{-(x^2+y^2)} = \frac{1}{\pi}\int_{r(\varepsilon_k)}^{\infty} dr \int_0^{2\pi} d\theta r e^{-r^2} = \int_{r(\varepsilon_k)^2}^{\infty} ds e^{-s} = e^{-r(\varepsilon_k)^2}$$

and $|\ln(1-x)| \le |x|/(1-|x|) \le 2|x|$ for all $|x| \le \frac{1}{2}$,

$$e^{r(\varepsilon_n)^2}|\ln \mathcal{Z}_{\varepsilon_n}(\varepsilon)| = \sum_{k=0}^{n-1} 2^{n-k-1} e^{r(\varepsilon_n)^2}|\ln(1 - e^{-r(\varepsilon_k)^2})| \le \sum_{k=0}^{n-1} 2^{n-k} e^{-(r(\varepsilon_k)^2 - r(\varepsilon_n)^2)}.$$

By Hypothesis 1.1 [part (iii)],

$$r(\varepsilon_k)^2 - r(\varepsilon_n)^2 = \sum_{p=k}^{n-1} (r(\varepsilon_p)^2 - r(\varepsilon_{p+1})^2) \ge \sum_{p=k}^{n-1} r(\varepsilon_p)(r(\varepsilon_p) - r(\varepsilon_{p+1})) \ge \sum_{p=k}^{n-1} 2 = 2(n-k)$$

so that

$$e^{r(\varepsilon_n)^2}|\ln \mathcal{Z}_{\varepsilon_n}(\varepsilon)| \le \sum_{k=0}^{n-1} 2^{n-k} e^{-2(n-k)} \le \sum_{\ell=1}^{\infty} \left(\frac{2}{e^2}\right)^{\ell} = \frac{2/e^2}{1-2/e^2} \le 1.$$

For the limit, we rewrite (C1)

$$|\ln \mathcal{Z}_{\varepsilon_n}(\varepsilon)| = \sum_{k=0}^{n-1} 2^{n-k-1}|\ln \mathcal{Z}'_{\varepsilon_k}| = \sum_{\ell=1}^{n} 2^{\ell-1}|\ln \mathcal{Z}'_{\varepsilon_{n-\ell}}|,$$

which implies that

$$\left|\ln \mathcal{Z}_{\delta}\left(\frac{1}{2^n}\delta\right)\right| = \sum_{\ell=1}^{n} 2^{\ell-1}|\ln \mathcal{Z}'_{2^{-\ell}\delta}| \tag{C2}$$

and hence

$$\begin{aligned}\lim_{n\to\infty}\left|\ln \mathcal{Z}_{\delta}\left(\frac{1}{2^n}\delta\right)\right| &= \sum_{\ell=1}^{\infty} 2^{\ell-1}|\ln \mathcal{Z}'_{2^{-\ell}\delta}| \\ &= \sum_{\ell=1}^{\infty} 2^{\ell-1}|\ln(1-e^{-r(2^{-\ell}\delta)^2})| \\ &\le e^{-r(\delta)^2}\sum_{\ell=1}^{\infty} 2^{\ell}e^{-(r(2^{-\ell}\delta)^2-r(\delta)^2)} \le e^{-r(\delta)^2}\sum_{\ell=1}^{\infty} 2^{\ell}e^{-2\ell} \\ &\le e^{-r(\delta)^2}.\end{aligned}$$

APPENDIX D: THE PROOF OF EXAMPLE 1.2

Example 1.2: Let $\mathfrak{v}>0$.

(i) Suppose that

$$\kappa(t) = \frac{1}{\sqrt[4]{|||v|||}}\left(\frac{1}{t}\right)^{a_\kappa} \quad \text{and} \quad r(t) = \frac{1}{\sqrt[4]{|||v|||}}\left(\frac{1}{t}\right)^{a_r}$$

for some constants $0<a_r<a_\kappa$ obeying $3a_\kappa+a_r<1$. Then there are constants K_E, Θ, and q such that Hypothesis 1.1 is fulfilled for all nonzero v with $|||v|||\le\mathfrak{v}$.

(ii) Suppose that

$$\kappa(t) = \frac{1}{\sqrt[4]{t|||v|||}}\left(\ln\frac{1}{t|||v|||}\right)^{b} \quad \text{and} \quad r(t) = \left(\ln\frac{1}{t|||v|||}\right)^{b}$$

for some $b\ge1$. Then there are constants K_E, Θ, and q such that Hypothesis 1.1 is fulfilled for all nonzero v with $|||v|||\le\mathfrak{v}$.

Proof:

(i) We have

$$\frac{\kappa(2t)}{\kappa(t)} = \frac{1}{2^{a_\kappa}}, \quad \frac{r(2t)}{r(t)} = \frac{1}{2^{a_r}}, \quad t|||v|||r(t)\kappa(t)^3 = t^{1-a_r-3a_\kappa}, \quad \frac{r(t)}{\kappa(t)} = t^{a_\kappa-a_r}.$$

So part (i) of the hypothesis is trivially fulfilled if $(1/\sqrt[4]{\mathfrak{v}})\min\{1/\Theta^{a_r},1/\Theta^{a_k}\}\ge1$. Part (ii) of the hypothesis, namely, $e^{tK_j}(1/2^{a_\kappa})+4t^{a_\kappa-a_r}\le1$, is satisfied provided $e^{\Theta K_j}(1/2^{a_\kappa})+4\Theta^{a_\kappa-a_r}\le1$, which is the case if Θ is small enough. Part (iii) of the hypothesis, namely,

$$r(t)[r(t)-r(2t)] = \frac{1}{\sqrt{|||v|||}}\frac{1}{t^{2a_r}}\left[1-\frac{1}{2^{a_r}}\right] \geq 2$$

is satisfied if $(1/\Theta^{2a_r})[1-1/2^{a_r}]\geq 2\sqrt{\mathfrak{v}}$. Part (iv) of the hypothesis, namely, $t^{1-a_r-3a_\kappa} \leq 1/K_E$, is satisfied provided that $\Theta^{1-a_r-3a_\kappa}\leq 1/K_E$. The uniform convergence (for each fixed nonzero v) of

$$t^2\sum_{k=0}^{\infty}\left(\frac{q}{4}\right)^k r\left(\frac{t}{2^k}\right)^2 \kappa\left(\frac{t}{2^k}\right)^6 = t^{2-2a_r-6a_\kappa}|||v|||^{-2}\sum_{k=0}^{\infty}\left(\frac{q}{4}2^{2a_r+6a_\kappa}\right)^k$$

is achieved whenever $q<2^{2(1-a_r-3a_\kappa)}$. Finally, to satisfy part (v), we need

$$\frac{1}{C(\Theta,K_E)}\frac{2}{q} \leq 2^{4a_\kappa} \leq C(\Theta,K_E)\frac{4}{2^{4a_r}}$$

or

$$q \geq \frac{2^{1-4a_\kappa}}{C(\Theta,K_E)} \quad \text{and} \quad 2^{4a_\kappa+4a_r-2} \leq C(\Theta,K_E).$$

Since $a_\kappa+a_r<a_\kappa+\frac{1}{2}a_\kappa+\frac{1}{2}a_r=\frac{1}{2}(3a_\kappa+a_r)<\frac{1}{2}$, we have $1-4a_\kappa<2(1-a_r-3a_\kappa)$ and hence $\max\{1,2^{1-4a_\kappa}\}<2^{2(1-a_r-3a_\kappa)}$. Fix any q obeying $\max\{1,2^{1-4a_\kappa}\}<q<2^{2(1-a_r-3a_\kappa)}$. Then, pick a $C<1$ sufficiently close to 1 that $q\geq 2^{1-4a_\kappa}/C$ and $2^{4a_\kappa+4a_r-2}\leq C$. Then, pick a K_E large enough that $1-(2^{33}e^{14K_j}/K_E)>C$. Finally, choose $0<\Theta<1$ that is small enough that $(1/\sqrt[4]{\mathfrak{v}})\min\{1/\Theta^{a_r},1/\Theta^{a_k}\}\geq 1$, $e^{\Theta K_j}(1/2^{a_\kappa})+4\Theta^{a_\kappa-a_r}\leq 1$, $\Theta^{1-a_r-3a_\kappa}\leq 1/K_E$, $1/\Theta^{2a_r}[1-1/2^{a_r}]\geq 2\sqrt{\mathfrak{v}}$, and $C(\Theta,K_E)\geq C$.

(ii) Set

$$C_\Theta = \left(1-\frac{\ln 2}{\ln\dfrac{2}{\Theta\mathfrak{v}}}\right)^b.$$

Then, for all $0\leq t\leq\Theta/2$ and v with $|||v|||\leq\mathfrak{v}$,

$$\frac{r(2t)}{r(t)} = \left(1-\frac{\ln 2}{\ln\dfrac{1}{t|||v|||}}\right)^b \in [C_\Theta,1)$$

so that

$$\frac{\kappa(2t)}{\kappa(t)} \in \frac{1}{\sqrt[4]{2}}[C_\Theta,1), \quad t|||v|||r(t)\kappa(t)^3 = \sqrt[4]{t|||v|||}\left(\ln\frac{1}{t|||v|||}\right)^{4b}, \quad \frac{r(t)}{\kappa(t)} = \sqrt[4]{t|||v|||}.$$

The proof now continues as in part (i). ∎

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