Ascona Lecture Notes

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Table of Contents

Fermionic Expansions

These lecture notes concern an expansion that can play the role of a single scale cluster expansion in fermionic models. We use this expansion to reduce the problem of constructing two dimensional Fermi liquids to that of controlling the flow of two and four legged vertices.

Notation. Let

- $\mathbf{A}$ be the Grassmann algebra generated by $\{a_1, \cdots, a_n\}$. Think of $\{a_1, \cdots, a_n\}$ as some finite approximation to the set $\{\ \psi_\sigma(x), \bar\psi_\sigma(x)\ \big|\ x \in \mathbb{R}^{d+1}, \sigma \in \{\uparrow, \downarrow\}\ \}$ of fields integrated out in a renormalization group transformation like

$$\mathcal{V}(\psi, \bar\psi) \to \widetilde{\mathcal{V}}(\Psi, \Psi) = -\log \int e^{\mathcal{V}(\psi+\Psi, \bar\psi+\bar\Psi)} d\mu_S(\psi, \bar\psi)$$

- $\mathbf{C}$ be the Grassmann algebra generated by $\{c_1, \cdots, c_n\}$. Think of $\{c_1, \cdots, c_n\}$ as some finite approximation to the set $\{\ \Psi_\sigma(x), \bar\Psi_\sigma(x)\ \big|\ x \in \mathbb{R}^{d+1}, \sigma \in \{\uparrow, \downarrow\}\ \}$ of fields that are arguments of the output of the renormalization group transformation.
- $\mathbf{AC}$ be the Grassmann algebra generated by $\{a_1, \cdots, a_n, c_1, \cdots, c_n\}$.
- $S = (S_{ij})$ be a skew symmetric matrix of order n. Think of S as the "single scale" covariance of the Gaussian measure that is integrated out in the renormalization group step.
- $\int \cdot \ d\mu_S(a)$ be the Grassmann, Gaussian integral with covariance S. It is the unique linear map from $\mathbf{AC}$ to $\mathbf{C}$ satisfying

$$\int e^{\Sigma\, c_i a_i}\ d\mu_S(a)\ =\ e^{-\frac{1}{2}\Sigma\, c_i S_{ij} c_j}$$

  In particular

$$\int a_i a_j\ d\mu_S(a)\ =\ S_{i,j}$$

- $\mathcal{M}_r = \{\ (i_1, \cdots, i_r)\ \big|\ 1 \le i_1, \cdots, i_r \le n\ \}$ be the set of all multi indices of degree $r \ge 0$. For each $\mathrm{I} \in \mathcal{M}_r$ set $a_{\mathrm{I}} = a_{i_1} \cdots a_{i_r}$. By convention, $a_\emptyset = 1$.

- the space $(\mathbf{AC})^0$ of "interactions" is the linear subspace of $\mathbf{AC}$ of even Grassmann polynomials with no constant term. That is, polynomials of the form

  $$W(c,a) \ = \sum_{\substack{l,r\in\mathbb{N}\\ 1\le l+r\in 2\mathbb{Z}}} \ \sum_{\substack{\mathrm{L}\in\mathcal{M}_l\\ \mathrm{J}\in\mathcal{M}_r}} w_{l,r}(\mathrm{L},\mathrm{J})\ c_{\mathrm{L}} a_{\mathrm{J}}$$

  Often, in the renormalization group map, the interaction is of the form $W(c+a)$. We do not require this.

Main Definitions.

i) The **renormalization group map** $\Omega : (\mathbf{AC})^0 \to \mathbf{C}^0$ is

$$\Omega(W)(c) = \log \int e^{W(c,a)}\, d\mu_S(a) - \log \int e^{W(0,a)}\, d\mu_S(a)$$

It is defined for all W's obeying $\int e^{W(0,a)}\, d\mu_S(a) \neq 0$. The second term ensures that $\Omega(W)(0) = 0$, i.e. that $\Omega(W)(c)$ contains no constant term. Since $\Omega(W) = 0$ for $W = 0$

$$\begin{aligned}\Omega(W)(c) &= \int_0^1 \tfrac{d}{d\varepsilon}\Omega(\varepsilon W)(c)\, d\varepsilon \\ &= \int_0^1 \frac{\int W(c,a)\, e^{\varepsilon W(c,a)}\, d\mu_S(a)}{\int e^{\varepsilon W(c,a)}\, d\mu_S(a)}\, d\varepsilon - \int_0^1 \frac{\int W(0,a)\, e^{\varepsilon W(0,a)}\, d\mu_S(a)}{\int e^{\varepsilon W(0,a)}\, d\mu_S(a)}\, d\varepsilon \qquad (1)\end{aligned}$$

Thus to get bounds on the renormalization group map, it suffices to get bounds on

ii) the **Schwinger functional** $\mathcal{S} : \mathbf{AC} \to \mathbf{C}$, defined by

$$\mathcal{S}(f) \ = \ \tfrac{1}{\mathcal{Z}(c)} \int f(c,a)\ e^{W(c,a)}\ d\mu_S(a)$$

where $\mathcal{Z}(c) \ = \ \int e^{W(c,a)}\ d\mu_S(a)$. Despite our notation, $\mathcal{S}(f)$ is of course a function of W and S as well as f.

iii) Define the linear map $\mathrm{R} : \mathbf{AC} \to \mathbf{AC}$ by

$$\mathrm{R}(f)(c,a) \ = \ \int :e^{W(c,a+b)-W(c,a)} - 1:_b \ f(c,b)\ d\mu_S(b)$$

where $:\ \cdot\ :_b$ denotes Wick ordering of the b–field and is determined by

$$:e^{\Sigma A_i a_i + \Sigma B_i b_i + \Sigma C_i c_i}:_b \ = \ e^{\frac{1}{2}\Sigma B_i S_{ij} B_j}\ e^{\Sigma A_i a_i + \Sigma B_i b_i + \Sigma C_i c_i}$$

Diagramatically, $\frac{1}{\mathcal{Z}(c)}\int e^{W(c,a)}\ f(c,a)\ d\mu_S(a)$ is the sum of all connected (f is viewed as a single, connected, vertex) Feynman diagrams with one f–vertex and arbitrary numbers of W–vertices and S–lines. The operation $\mathrm{R}(f)$ builds parts of those diagrams. It introduces those W–vertices that are connected directly to the f–vertex (i.e that share a common S–line with f) and it introduces those lines that connect f either to itself or to a W–vertex.

To obtain the expansion that will be discussed in these lectures, expand the $(\mathbb{1}-\mathrm{R})^{-1}$ of the following Theorem, which shall be proven shortly, in a power series in R.

Theorem 1 *Suppose that the kernel of* $\mathbb{1}-\mathrm{R}$ *is trivial. Then,*

$$\mathcal{S}(f) \;=\; \int (\mathbb{1}-\mathrm{R})^{-1}(f)\ d\mu_S(a)$$

for every f *in* **AC** *.*

Lemma 2 *For all* f *and* g *in* **AC** *,*

$$\int\!\!\int f(c,b)\ {:}g(c,a+b){:}_b\ d\mu_S(b)\,d\mu_S(a) = \int f(c,a)g(c,a)\,d\mu_S(a)$$

Proof: It suffices to consider $f(c,a)=e^{\Sigma_i A_i a_i+\Sigma_i C_i c_i}$ and $g(c,a) \;=\; e^{\Sigma_i B_i a_i+\Sigma_i D_i c_i}$, with the sources $A_i,\ B_i,\ C_i,\ D_i$ anticommuting amongst themselves and with the original fields $a_i,\ c_i$. Then

$$\begin{aligned} {:}g(c,a+b){:}_b \;&=\; {:}e^{\Sigma_i B_i(a_i+b_i)+\Sigma_i D_i c_i}{:}_b = \; e^{\Sigma_i B_i a_i+\Sigma_i D_i c_i}\ {:}e^{\Sigma_i B_i b_i}{:}_b \\ &=\; e^{\Sigma_i B_i a_i+\Sigma_i D_i c_i} e^{\frac{1}{2}\Sigma\, B_i S_{ij} B_j}\ e^{\Sigma\, B_i b_i} \\ &=\; {:}g(c,a+b){:}_a \end{aligned}$$

so that

$$\int f(c,b)\ {:}g(c,a+b){:}_b\ d\mu_S(a) = e^{\Sigma_i(C_i+D_i)c_i+\Sigma_i(A_i+B_i)b_i} = f(c,b)g(c,b)$$

Integrating this with respect to $d\mu_S(b)$ gives the same answer as integrating $f(c,a)g(c,a) = e^{\Sigma_i(A_i+B_i)a_i+\Sigma_i(C_i+D_i)c_i}$ with respect to $d\mu_S(a)$. ■

Proposition 3 *For all* f *in* **AC**,

$$\int f(c,a)\, e^{W(c,a)}\, d\mu_S(a) \;=\; \int f(c,b)\, d\mu_S(b) \int e^{W(c,a)}\, d\mu_S(a) \;+\; \int \mathrm{R}(f)(c,a)\, e^{W(c,a)}\, d\mu_S(a)$$

Proof: By Lemma 2,

$$\begin{aligned}
\int f(c,b)\, d\mu_S(b) \int e^{W(c,a)}\, d\mu_S(a) \;+\; \int \mathrm{R}(f)(c,a)\, e^{W(c,a)}\, d\mu_S(a) \\
&= \iint :e^{W(c,a+b)-W(c,a)}:_b \; f(c,b)\, d\mu_S(b)\, e^{W(c,a)}\; d\mu_S(a) \\
&= \iint \; :e^{W(c,a+b)}:_b \; f(c,b)\;\; d\mu_S(b)\;\; d\mu_S(a) \\
&= \int \; f(c,a)\;\, e^{W(c,a)}\;\; d\mu_S(a)
\end{aligned}$$

■

Proof of Theorem 1: For all $g(c,a) \in \mathbf{AC}$

$$\int (\mathbb{1} - R)(g)\, e^{W(c,a)}\; d\mu_S(a) \;=\; \mathcal{Z}(c) \int g(c,a)\; d\mu_S(a)$$

by Proposition 3. If the kernel of $\mathbb{1} - R$ is trivial, then we may choose $g = (\mathbb{1} - R)^{-1}(f)$. So

$$\int f(c,a)\, e^{W(c,a)}\; d\mu_S(a) \;=\; \mathcal{Z}(c) \int (\mathbb{1} - R)^{-1}(f)(c,a)\; d\mu_S(a)$$

The left hand side does not vanish for all $f \in \mathbf{AC}$ (for example, for $f = e^{-W}$) so $\mathcal{Z}(c)$ is nonzero and

$$\tfrac{1}{\mathcal{Z}(c)} \int f(c,a)\, e^{W(c,a)}\; d\mu_S(a) \;=\; \int (\mathbb{1} - R)^{-1}(f)(c,a)\; d\mu_S(a)$$

■

Norms

For any function $f : \mathcal{M}_r \to \mathbb{C}$, define

$$\begin{aligned}
\|f\| \;&=\; \max_{1\le i\le r}\; \sup_{1\le k\le n}\; \sum_{\substack{\mathrm{J}\in\mathcal{M}_r \\ j_i = k}} \big|f(\mathrm{J})\big| \\
|||f||| \;&=\; \sum_{\mathrm{J}\in\mathcal{M}_r} \big|f(\mathrm{J})\big|
\end{aligned}$$

The norm $\| \cdot \|$, which is an "L^1 norm with one argument held fixed", is appropriate for kernels, like those appearing in interactions, that become translation invariant when the cutoffs are removed. Any $f(c,a) \in \mathbf{AC}$ has a unique representation

$$f(c,a) \;=\; \sum_{l,r\geq 0} \sum_{\substack{k_1,\cdots,k_l\\ j_1,\cdots,j_r}} f_{l,r}(k_1,\cdots,k_l,j_1,\cdots,j_r)\, c_{k_1}\cdots c_{k_l}\, a_{j_1}\cdots a_{j_r}$$

with each kernel $f_{l,r}(k_1,\cdots,k_l,j_1,\cdots,j_r)$ antisymmetric under separate permutation of its k arguments and its j arguments. Define

$$\|f(c,a)\|_\alpha = \sum_{l,r\geq 0} \alpha^{l+r} \|f_{l,r}\|$$
$$|||f(c,a)|||_\alpha = \sum_{l,r\geq 0} \alpha^{l+r} |||f_{l,r}|||$$

Hypotheses

(HG) $\quad \left| \int b_{\mathrm{H}} \;: b_{\mathrm{J}} :\; d\mu_S(b) \right| \leq \mathrm{F}^{|\mathrm{H}|+|\mathrm{J}|}$

(HS) $\quad \|S\| \leq \mathrm{F}^2\mathrm{D}$

So F is a measure of the "typical size of fields b in the support of the measure $d\mu_S(b)$" and D is a measure of the decay rate of S.

Theorem 4 *Assume Hypotheses (HG) and (HS). Let $\alpha \geq 2$ and $\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}} \leq 1/3$. Then, for all $f \in \mathbf{AC}$,*

$$\|\mathrm{R}(f)\|_{\alpha\mathrm{F}} \;\leq\; \tfrac{3}{\alpha}\,\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}\,\|f\|_{\alpha\mathrm{F}}$$
$$|||\mathrm{R}(f)|||_{\alpha\mathrm{F}} \;\leq\; \tfrac{3}{\alpha}\,\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}\,|||f|||_{\alpha\mathrm{F}}$$

Corollary 5 *Assume Hypotheses (HG) and (HS). Let $\alpha \geq 2$ and $\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}} \leq 1/3$. Then, for all $f \in \mathbf{AC}$,*

$$\|\mathcal{S}(f)(c) - \mathcal{S}(f)(0)\|_{\alpha\mathrm{F}} \;\leq\; \tfrac{\alpha}{\alpha-1}\,\|f\|_{\alpha\mathrm{F}}$$
$$|||\mathcal{S}(f)|||_{\alpha\mathrm{F}} \;\leq\; \tfrac{\alpha}{\alpha-1}\,|||f|||_{\alpha\mathrm{F}}$$
$$\|\Omega(W)\|_{\alpha\mathrm{F}} \;\leq\; \tfrac{\alpha}{\alpha-1}\,\|W\|_{\alpha\mathrm{F}}$$

Let

$$W(c,a) \;=\; \sum_{l,r\in\mathbb{N}} \sum_{\substack{\mathrm{L}\in\mathcal{M}_l\\ \mathrm{J}\in\mathcal{M}_r}} w_{l,r}(\mathrm{L},\mathrm{J})\; c_{\mathrm{L}} a_{\mathrm{J}}$$

where $w_{l,r}(\mathrm{L},\mathrm{J})$ is a function which is separately antisymmetric under permutations of its L and J arguments and that vanishes identically when $l+r$ is zero or odd. With this antisymmetry

$$W(c,a+b) - W(c,a) \;=\; \sum_{\substack{l,r\ge 0\\ s\ge 1}} \sum_{\substack{\mathrm{L}\in\mathcal{M}_l\\ \mathrm{J}\in\mathcal{M}_r\\ \mathrm{K}\in\mathcal{M}_s}} \binom{r+s}{s} w_{l,r+s}(\mathrm{L},\mathrm{J},\mathrm{K}) c_{\mathrm{L}} a_{\mathrm{J}} b_{\mathrm{K}}$$

so

$$:e^{W(c,a+b)-W(c,a)} - 1:_b = \sum_{\ell>0} \tfrac{1}{\ell!} \sum_{\substack{l_i,r_i\ge 0\\ s_i\ge 1}} \sum_{\substack{\mathrm{L}_i\in\mathcal{M}_{l_i}\\ \mathrm{J}_i\in\mathcal{M}_{r_i}\\ \mathrm{K}_i\in\mathcal{M}_{s_i}}} :\prod_{i=1}^{\ell} \binom{r_i+s_i}{s_i} w_{l_i,r_i+s_i}(\mathrm{L}_i,\mathrm{J}_i,\mathrm{K}_i) c_{\mathrm{L}_i} a_{\mathrm{J}_i} b_{\mathrm{K}_i}:_b$$

with the index i in the second and third sums running from 1 to ℓ. Hence

$$\mathrm{R}(f) = \sum_{\ell>0} \sum_{\substack{\mathbf{r},\mathbf{s},l\in\mathbb{N}^\ell\\ s_i\ge 1}} \tfrac{1}{\ell!} \prod_{i=1}^{\ell} \binom{r_i+s_i}{s_i} \mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f)$$

where

$$\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f) = \sum_{\substack{\mathrm{L}_i\in\mathcal{M}_{l_i}\\ \mathrm{J}_i\in\mathcal{M}_{r_i}\\ \mathrm{K}_i\in\mathcal{M}_{s_i}}} \int :\prod_{i=1}^{\ell} w_{l_i,r_i+s_i}(\mathrm{L}_i,\mathrm{J}_i,\mathrm{K}_i) c_{\mathrm{L}_i} a_{\mathrm{J}_i} b_{\mathrm{K}_i}:_b \; f(c,b)\; d\mu_S(b)$$

Proposition 6 *Assume Hypothesis (HG). Let*

$$f^{(p,m)}(c,a) \;=\; \sum_{\substack{\mathrm{H}\in\mathcal{M}_m\\ \mathrm{I}\in\mathcal{M}_p}} f_{p,m}(\mathrm{I},\mathrm{H})\, c_{\mathrm{I}} a_{\mathrm{H}}$$

Let $\mathbf{r},\mathbf{s},l\in\mathbb{N}^\ell$ *with each* $s_i\ge 1$. *If* $m<\ell$, $\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})$ *vanishes. If* $m\ge\ell$

$$\|\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})\|_1 \;\le\; \ell!\binom{m}{\ell} \mathrm{F}^m \|f_{p,m}\| \prod_{i=1}^{\ell} \Big(\|S\|\mathrm{F}^{s_i-2}\; \|w_{l_i,r_i+s_i}\| \Big)$$

$$|||\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})|||_1 \;\le\; \ell!\binom{m}{\ell} \mathrm{F}^m |||f_{p,m}||| \prod_{i=1}^{\ell} \Big(\|S\|\mathrm{F}^{s_i-2}\; \|w_{l_i,r_i+s_i}\| \Big)$$

Proof: We have

$$\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)}) \;=\; \pm \sum_{\mathrm{I}\in\mathcal{M}_p} \sum_{\substack{\mathrm{J}_i\in\mathcal{M}_{r_i}\\ \mathrm{L}_i\in\mathcal{M}_{l_i}\\ 1\le i\le \ell}} f_{l,\mathbf{r},\mathbf{s}}(\mathrm{L}_1,\cdots,\mathrm{L}_\ell,\mathrm{I},\mathrm{J}_1,\cdots,\mathrm{J}_\ell)\; c_{\mathrm{L}_1}\cdots c_{\mathrm{L}_\ell}\, c_{\mathrm{I}}\, a_{\mathrm{J}_1}\cdots a_{\mathrm{J}_\ell}$$

with

$$f_{l,\mathbf{r},\mathbf{s}}(\mathrm{L}_1,\cdots,\mathrm{L}_\ell,\mathrm{I},\mathrm{J}_1,\cdots,\mathrm{J}_\ell) = \sum_{\substack{\mathrm{H}\in\mathcal{M}_m\\ \mathrm{K}_i\in\mathcal{M}_{s_i}}} \int :\prod_{i=1}^{\ell} w_{l_i,r_i+s_i}(\mathrm{L}_i,\mathrm{J}_i,\mathrm{K}_i) b_{\mathrm{K}_i}:_b\; f_{p,m}(\mathrm{I},\mathrm{H})\, b_{\mathrm{H}}\; d\mu_S(b)$$

The integral over b is bounded in Lemma 7 below. It shows that $f_{l,\mathbf{r},\mathbf{s}}(\mathrm{L}_1,\cdots,\mathrm{L}_\ell,\mathrm{I},\mathrm{J}_1,\cdots,\mathrm{J}_\ell)$ vanishes if $m<\ell$ and is, for $m\ge\ell$, bounded by

$$\big|f_{l,\mathbf{r},\mathbf{s}}(\mathrm{L}_1,\cdots,\mathrm{L}_\ell,\mathrm{I},\mathrm{J}_1,\cdots,\mathrm{J}_\ell)\big| \;\le\; \ell!\tbinom{m}{\ell}\; T(\mathrm{L}_1,\cdots,\mathrm{L}_\ell,\mathrm{I},\mathrm{J}_1,\cdots,\mathrm{J}_\ell)\,\mathrm{F}^{m+\Sigma s_i-2\ell}$$

where

$$T(\mathrm{L}_1,\cdots,\mathrm{L}_\ell,\mathrm{I},\mathrm{J}_1,\cdots,\mathrm{J}_\ell) \;= \sum_{\mathrm{H}\in\mathcal{M}_m} \prod_{i=1}^{\ell}\Big(\sum_{k_i=1}^{n} |u_i(\mathrm{L}_i,\mathrm{J}_i,k_i)||S_{k_i,h_i}|\Big)|f_{p,m}(\mathrm{I},\mathrm{H})|$$

and, for each $i=1,\cdots,\ell$

$$u_i(\mathrm{L}_i,\mathrm{J}_i,k_i) \;=\; \sum_{\tilde{\mathrm{K}}_i\in\mathcal{M}_{s_i-1}} |w_{l_i,r_i+s_i}(\mathrm{L}_i,\mathrm{J}_i,\tilde{\mathrm{K}}_i,k_i)|$$

By construction, $\|u_i\| \;=\; \|w_{l_i,r_i+s_i}\|$. By Lemma 8, below

$$\|T\| \;\le\; \|f_{p,m}\|\,\|S\|^{\ell} \prod_{i=1}^{\ell}\|u_i\| \;\le\; \|f_{p,m}\|\,\|S\|^{\ell} \prod_{i=1}^{\ell}\|w_{l_i,r_i+s_i}\|$$

and hence

$$\|f_{l,\mathbf{r},\mathbf{s}}\| \le \ell!\tbinom{m}{\ell}\,\mathrm{F}^{m+\Sigma s_i-2\ell}\,\|f_{p,m}\|\,\|S\|^{\ell}\prod_{i=1}^{\ell}\|w_{l_i,r_i+s_i}\|$$

Similarly, the second bound follows from $|||T||| \;\le\; |||f_{p,m}|||\,\|S\|^{\ell}\,\prod_{i=1}^{\ell}\|u_i\|$. ■

Lemma 7 *Assume Hypothesis (HG). Then*

$$\Big|\int :\prod_{i=1}^{\ell} b_{\mathrm{K}_i}:\; b_{\mathrm{H}}\; d\mu_S(b)\Big| \le\; \mathrm{F}^{|\mathrm{H}|+\Sigma|\mathrm{K}_i|-2\ell} \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le|\mathrm{H}|\\ \text{all different}}} \prod_{i=1}^{\ell} |S_{k_{i1},h_{\mu_i}}|$$

Proof: For convenience, set $j_i = k_{i1}$ and $\tilde{\mathrm{K}}_i = \mathrm{K}_i \setminus \{k_{i1}\}$ for each $i = 1, \cdots, \ell$. By antisymmetry,

$$\int :\prod_{i=1}^{\ell} b_{\mathrm{K}_i}: \; b_{\mathrm{H}} \; d\mu_S(b) \;=\; \pm \int : \prod_{i=1}^{\ell} b_{\tilde{\mathrm{K}}_i} \; b_{j_1} \cdots b_{j_\ell}: \; b_{\mathrm{H}} \; d\mu_S(b)$$

Recall the integration by parts formula

$$\int b_j \, f(b) \; d\mu_S(b) \;=\; \sum_{m=1}^{n} S_{j,m} \int \tfrac{\partial}{\partial b_m} \, f(b) \; d\mu_S(b)$$

for Grassmann Gaussian measures. The left partial derivative is determined by

$$\tfrac{\partial}{\partial b_m} \, b_{\mathrm{H}} \;=\; \begin{cases} 0 & \text{if } m \notin \mathrm{H} \\ (-1)^{|\mathrm{J}|} \, b_{\mathrm{J}} \, b_{\mathrm{K}} & \text{if } b_{\mathrm{H}} = b_{\mathrm{J}} \, b_m \, b_{\mathrm{K}} \end{cases}$$

In the presence of Wick ordering

$$\int : b_{\mathrm{K}} b_j : \; f(b) \; d\mu_S(b) \;=\; \sum_{m=1}^{n} S_{j,m} \int : b_{\mathrm{K}} : \tfrac{\partial}{\partial b_m} \, f(b) \; d\mu_S(b)$$

Integrate by parts successively with respect to $b_{j_\ell} \cdots b_{j_1}$

$$\int :\prod_{i=1}^{\ell} b_{\mathrm{K}_i}: \; b_{\mathrm{H}} \; d\mu_S(b) = \; \pm \int :\prod_{i=1}^{\ell} b_{\tilde{\mathrm{K}}_i}: \; \Big[\prod_{i=1}^{\ell} \Big(\sum_{m=1}^{n} S_{j_i,m} \, \tfrac{\partial}{\partial b_m} \Big) \, b_{\mathrm{H}} \Big] \; d\mu_S(b)$$

and then apply Leibniz's rule

$$\prod_{i=1}^{\ell} \Big(\sum_{m=1}^{n} S_{j_i,m} \, \tfrac{\partial}{\partial b_m} \Big) \, b_{\mathrm{H}} \;=\; \sum_{\substack{1 \le \mu_1, \cdots, \mu_\ell \le |\mathrm{H}| \\ \text{all different}}} \pm \Big(\prod_{i=1}^{\ell} S_{j_i, h_{\mu_i}} \Big) b_{\mathrm{H} \setminus \{h_{\mu_1}, \cdots, h_{\mu_\ell}\}}$$

and Hypothesis (HG). ■

Lemma 8 *Let*

$$T(\mathrm{J}_1, \cdots, \mathrm{J}_\ell, \mathrm{I}) \;= \sum_{\mathrm{H} \in \mathcal{M}_m} \prod_{i=1}^{\ell} \Big(\sum_{k_i=1}^{n} |u_i(\mathrm{J}_i, k_i)| |S_{k_i, h_i}| \Big) |f_{p,m}(\mathrm{I}, \mathrm{H})|$$

with $\ell \le m$. Then

$$\|T\| \;\le\; \|f_{p,m}\| \, \|S\|^{\ell} \prod_{i=1}^{\ell} \|u_i\|$$

$$|||T||| \;\le\; |||f_{p,m}||| \, \|S\|^{\ell} \prod_{i=1}^{\ell} \|u_i\|$$

Proof: For the triple norm,

$$\sum_{\mathrm{I,H,J}_i} \prod_{i=1}^{\ell}\Big(\sum_{k_i=1}^{n} |u_i(\mathrm{J}_i,k_i)||S_{k_i,h_i}|\Big)|f_{p,m}(\mathrm{I,H})| = \sum_{\mathrm{I,H}} \prod_{i=1}^{\ell}\Big(\sum_{k_i}\Big(\sum_{\mathrm{J}_i} |u_i(\mathrm{J}_i,k_i)|\Big)|S_{k_i,h_i}|\Big)|f_{p,m}(\mathrm{I,H})|$$

$$\leq \prod_{i=1}^{\ell}\Big(\sup_{k_i}\sum_{\mathrm{J}_i}|u_i(\mathrm{J}_i,k_i)|\Big)\sum_{\mathrm{I,H}}\prod_{i=1}^{\ell}\Big(\sum_{k_i}|S_{k_i,h_i}|\Big)|f_{p,m}(\mathrm{I,H})|$$

$$\leq \prod_{i=1}^{\ell}\Big(\|u_i\|\sup_{h_i}\sum_{k_i}|S_{k_i,h_i}|\Big)\sum_{\mathrm{I,H}}|f_{p,m}(\mathrm{I,H})|$$

$$\leq |\!|\!| f_{p,m} |\!|\!| \prod_{i=1}^{\ell}\Big(\|u_i\|\,\|S\|\Big)$$

The proof for the double norm, i.e. the norm with one external argument sup'd over rather than summed over, is similar. It repeatedly uses

$$\sup_i \sum_{h,k,\mathrm{H,J}} \big|u(\mathrm{J},k)\,s(k,h)\,v(h,\mathrm{H},i)\big| \leq \|u\|\,\|s\|\,\sup_i\sum_{h,\mathrm{H}}\big|v(h,\mathrm{H},i)\big|$$

in place of

$$\sum_{h,k,\mathrm{H,J}} \big|u(\mathrm{J},k)\,s(k,h)\,v(h,\mathrm{H})\big| \leq \|u\|\,\|s\|\,\sum_{h,\mathrm{H}}\big|v(h,\mathrm{H})\big|$$

■

Lemma 9 *Assume Hypotheses (HG) and (HS). Let* $f^{(p,m)}(c,a) = \sum_{\substack{\mathrm{H}\in\mathcal{M}_m \\ \mathrm{I}\in\mathcal{M}_p}} f_{p,m}(\mathrm{I,H})\,c_{\mathrm{I}}a_{\mathrm{H}}$. *For all* $\alpha \geq 2$ *and* $\ell \geq 1$

$$\sum_{\substack{\mathbf{r,s},l\in\mathbb{N}^{\ell} \\ s_i\geq 1}} \tfrac{1}{\ell!}\prod_{i=1}^{\ell}\tbinom{r_i+s_i}{s_i}\|\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})\|_{\alpha\mathrm{F}} \leq \tfrac{2}{\alpha}\|f^{(p,m)}\|_{\alpha\mathrm{F}}\big[\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}\big]^{\ell}$$

$$\sum_{\substack{\mathbf{r,s},l\in\mathbb{N}^{\ell} \\ s_i\geq 1}} \tfrac{1}{\ell!}\prod_{i=1}^{\ell}\tbinom{r_i+s_i}{s_i}|\!|\!|\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})|\!|\!|_{\alpha\mathrm{F}} \leq \tfrac{2}{\alpha}|\!|\!|f^{(p,m)}|\!|\!|_{\alpha\mathrm{F}}\big[\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}\big]^{\ell}$$

Proof: We prove the bound with the $\|\ \cdot\ \|$ norm. The proof for $|\!|\!|\ \cdot\ |\!|\!|$ is identical. We may assume, without loss of generality, that $m \geq \ell$. By Proposition 6,

$$\tfrac{1}{\ell!}\|\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})\|_{\alpha\mathrm{F}} = \tfrac{1}{\ell!}\alpha^{\Sigma l_i+\Sigma r_i+p}\mathrm{F}^{\Sigma l_i+\Sigma r_i+p}\|\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})\|_1$$

$$\leq \alpha^{\Sigma l_i+\Sigma r_i+p}\mathrm{F}^{\Sigma l_i+\Sigma r_i+p}\tbinom{m}{\ell}\mathrm{F}^m\|f_{p,m}\|\prod_{i=1}^{\ell}\Big(\|S\|\mathrm{F}^{s_i-2}\ \|w_{l_i,r_i+s_i}\|\Big)$$

$$\leq 2^m\alpha^p\mathrm{F}^{m+p}\|f_{p,m}\|\prod_{i=1}^{\ell}\Big(\mathrm{D}\alpha^{l_i+r_i}\mathrm{F}^{l_i+r_i+s_i}\ \|w_{l_i,r_i+s_i}\|\Big)$$

As $\alpha \geq 2$ and $m \geq \ell \geq 1$,

$$
\begin{aligned}
&\sum_{\substack{\mathbf{r},\mathbf{s},l\in\mathbb{N}^\ell \\ s_i\geq 1}} \tfrac{1}{\ell!} \prod_{i=1}^{\ell} \tbinom{r_i+s_i}{s_i} \|\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f^{(p,m)})\|_{\alpha\mathrm{F}} \\
&\qquad \leq \tfrac{2}{\alpha}\|f^{(p,m)}\|_{\alpha\mathrm{F}} \sum_{\substack{\mathbf{r},\mathbf{s},l\in\mathbb{N}^\ell \\ s_i\geq 1}} \prod_{i=1}^{\ell} \left[\tbinom{r_i+s_i}{s_i} \mathrm{D}\alpha^{l_i+r_i}\mathrm{F}^{l_i+r_i+s_i}\ \|w_{l_i,r_i+s_i}\|\right] \\
&\qquad \leq \tfrac{2}{\alpha}\|f^{(p,m)}\|_{\alpha\mathrm{F}} \left[\mathrm{D} \sum_{\substack{r,s,l\in\mathbb{N} \\ s\geq 1}} \tbinom{r+s}{s} \alpha^r \alpha^l \mathrm{F}^{l+r+s}\ \|w_{l,r+s}\|\right]^\ell \\
&\qquad \leq \tfrac{2}{\alpha}\|f^{(p,m)}\|_{\alpha\mathrm{F}} \left[\mathrm{D} \sum_{q\geq 1} (\alpha+1)^q \alpha^l \mathrm{F}^{l+q}\ \|w_{l,q}\|\right]^\ell \\
&\qquad \leq \tfrac{2}{\alpha}\|f^{(p,m)}\|_{\alpha\mathrm{F}} \left[\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}\right]^\ell
\end{aligned}
$$

■

Proof of Theorem 4:

$$
\begin{aligned}
\|\mathrm{R}(f)\|_{\alpha\mathrm{F}} &\leq \sum_{\ell>0} \sum_{\substack{\mathbf{r},\mathbf{s},l\in\mathbb{N}^\ell \\ s_i\geq 1}} \tfrac{1}{\ell!} \prod_{i=1}^{\ell} \tbinom{r_i+s_i}{s_i} \|\mathrm{R}^{\mathbf{s}}_{l,\mathbf{r}}(f)\|_{\alpha\mathrm{F}} \\
&\leq \sum_{\ell>0} \sum_{m,p} \tfrac{2}{\alpha}\|f^{(p,m)}\|_{\alpha\mathrm{F}} \left[\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}\right]^\ell \\
&= \tfrac{2}{\alpha}\|f\|_{\alpha\mathrm{F}}\ \tfrac{\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}}{1-\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}} \\
&\leq \tfrac{3}{\alpha}\|f\|_{\alpha\mathrm{F}}\, \mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}
\end{aligned}
$$

The proof for the other norm is similar. ■

Lemma 10 *Assume Hypothesis (HG). If $\alpha \geq 1$ then, for all $g(a,c) \in$ **AC***

$$
\Big\|\!\Big| \int g(a,c)\, d\mu_S(a) \Big\|\!\Big|_{\alpha\mathrm{F}} \leq |\!|\!| g(a,c) |\!|\!|_{\alpha\mathrm{F}}
$$

$$
\Big\| \int [g(a,c) - g(a,0)]\, d\mu_S(a) \Big\|_{\alpha\mathrm{F}} \leq \|g(a,c)\|_{\alpha\mathrm{F}}
$$

Proof: Let

$$
g(a,c) = \sum_{l,r\geq 0} \sum_{\substack{\mathrm{L}\in\mathcal{M}_l \\ \mathrm{J}\in\mathcal{M}_r}} g_{l,r}(\mathrm{L},\mathrm{J})\ c_{\mathrm{L}} a_{\mathrm{J}}
$$

with $g_{l,r}(\mathrm{L},\mathrm{J})$ antisymmetric under separate permutations of its L and J arguments. Then

$$\begin{aligned}
\Big\vert\!\Big\vert\!\Big\vert \int g(a,c)\, d\mu_S(a) \Big\vert\!\Big\vert\!\Big\vert_{\alpha\mathrm{F}} &= \Big\vert\!\Big\vert\!\Big\vert \sum_{l,r\ge 0} \sum_{\substack{\mathrm{L}\in\mathcal{M}_l\\ \mathrm{J}\in\mathcal{M}_r}} g_{l,r}(\mathrm{L},\mathrm{J})\ c_{\mathrm{L}} \int a_{\mathrm{J}}\, d\mu_S(a) \Big\vert\!\Big\vert\!\Big\vert_{\alpha\mathrm{F}} \\
&= \sum_{l\ge 0} \alpha^l \mathrm{F}^l \sum_{\mathrm{L}\in\mathcal{M}_l} \Big| \sum_{r\ge 0} \sum_{\mathrm{J}\in\mathcal{M}_r} g_{l,r}(\mathrm{L},\mathrm{J}) \int a_{\mathrm{J}}\, d\mu_S(a) \Big| \\
&\le \sum_{l,r\ge 0} \alpha^l \mathrm{F}^{l+r} \sum_{\substack{\mathrm{L}\in\mathcal{M}_l\\ \mathrm{J}\in\mathcal{M}_r}} |g_{l,r}(\mathrm{L},\mathrm{J})| \\
&\le \vert\!\vert\!\vert g(a,c) \vert\!\vert\!\vert_{\alpha\mathrm{F}}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Big\| \int [g(a,c) - g(a,0)\, d\mu_S(a) \Big\|_{\alpha\mathrm{F}} &= \Big\| \sum_{\substack{l\ge 1\\ r\ge 0}} \sum_{\substack{\mathrm{L}\in\mathcal{M}_l\\ \mathrm{J}\in\mathcal{M}_r}} g_{l,r}(\mathrm{L},\mathrm{J})\ c_{\mathrm{L}} \int a_{\mathrm{J}}\, d\mu_S(a) \Big\|_{\alpha\mathrm{F}} \\
&= \sum_{l\ge 1} \alpha^l \mathrm{F}^l \sup_{1\le k\le n} \sum_{\tilde{\mathrm{L}}\in\mathcal{M}_{l-1}} \Big| \sum_{r\ge 0} \sum_{\mathrm{J}\in\mathcal{M}_r} g_{l,r}(k,\tilde{\mathrm{L}},\mathrm{J}) \int a_{\mathrm{J}}\, d\mu_S(a) \Big| \\
&\le \sum_{\substack{l\ge 1\\ r\ge 0}} \alpha^l \mathrm{F}^{l+r} \sup_{1\le k\le n} \sum_{\substack{\tilde{\mathrm{L}}\in\mathcal{M}_{l-1}\\ \mathrm{J}\in\mathcal{M}_r}} |g_{l,r}(k,\tilde{\mathrm{L}},\mathrm{J})| \\
&\le \| g(a,c) \|_{\alpha\mathrm{F}}
\end{aligned}$$

■

Proof of Corollary 5: Set $g = (\mathbb{1} - \mathrm{R})^{-1} f$. Then

$$\begin{aligned}
\|\mathcal{S}(f)(c) - \mathcal{S}(f)(0)\|_{\alpha\mathrm{F}} &= \Big\| \int \big[g(a,c) - g(a,0) \big]\ d\mu_S(a) \Big\|_{\alpha\mathrm{F}} && \text{(Theorem 1)} \\
&\le \|(\mathbb{1} - \mathrm{R})^{-1}(f)\|_{\alpha\mathrm{F}} && \text{(Lemma 10)} \\
&\le \tfrac{1}{1 - 3\mathrm{D}\|W\|_{(\alpha+1)\mathrm{F}}/\alpha} \|f\|_{\alpha\mathrm{F}} \le \tfrac{1}{1-1/\alpha} \|f\|_{\alpha\mathrm{F}} && \text{(Theorem 4)}
\end{aligned}$$

The argument for $\vert\!\vert\!\vert \mathcal{S}(f) \vert\!\vert\!\vert_{\alpha\mathrm{F}}$ is identical.

With the more detailed notation

$$\mathcal{S}(f, W) \ = \ \frac{\int f(a,c)\, e^{\mathrm{W}(a,c)}\ d\mu_S(a)}{\int e^{\mathrm{W}(a,c)}\ d\mu_S(a)}$$

we have

$$\begin{aligned}
\|\Omega(W)\|_{\alpha\mathrm{F}} &= \Big\| \int_0^1 \big[\mathcal{S}(W, \varepsilon W)(c) - \mathcal{S}(W, \varepsilon W)(0) \big] d\varepsilon \Big\|_{\alpha\mathrm{F}} \\
&\le \int_0^1 \tfrac{\alpha}{\alpha-1} \|W\|_{\alpha\mathrm{F}}\ d\varepsilon \ = \ \tfrac{\alpha}{\alpha-1} \|W\|_{\alpha\mathrm{F}}
\end{aligned}$$

■

We now apply the expansion to a few examples. In these examples, the set of fields $\{a_1, \cdots, a_n\}$ is replaced by $\{\ \psi_\sigma(x), \bar\psi_{\sigma'}(x)\ \big|\ x \in \mathbb{R}^{d+1},\ \sigma \in \mathfrak{S}\ \}$ with $\mathfrak{S}$ being a finite set (of spin/colour values). Consequently, sums $\sum\limits_{j=1}^{n}$ are replaced by $\sum\limits_{\sigma\in\mathfrak{S}} \int_{\mathbb{R}_{d+1}} dx$. That our Grassmann algebra is no longer finite dimensional is a technicality that is easily dealt with, using the bounds of Theorem 4 and Corollary 5. We shall not do so. The covariance $S_{i,j}$ will be replaced by a "single scale" propagator that, in each example, will be constructed by substituting a partition of unity of momentum space into the full propagator. The partition of unity will be constructed using a fixed "scale parameter" $M > 1$ and a function $\nu \in C_0^\infty([M^{-2}, M^2])$ that takes values in $[0, 1]$, is identically 1 on $[M^{-1/2}, M^{1/2}]$ and obeys

$$\sum_{j=0}^{\infty} \nu\big(M^{2j} x\big) = 1$$

for $0 < x < 1$.

Example (Gross–Neveu$_2$)

The propagator for the Gross-Neveu model in two space-time dimensions is

$$S(x,y) = \int \frac{d^2p}{(2\pi)^2}\ e^{ip\cdot(y-x)} \frac{\not p + m}{p^2 + m^2} \qquad \not p = \begin{pmatrix} ip_0 & p_1 \\ -p_1 & -ip_0 \end{pmatrix}$$

Set

$$\nu_j(p) = \begin{cases} \nu\big(\frac{M^{2j}}{p^2}\big) & \text{if } j > 0 \\ \nu\big(\frac{M^{2j}}{p^2}\big) & \text{if } j = 0,\ |p| \ge 1 \\ 1 & \text{if } j = 0,\ |p| < 1 \end{cases}$$

Then

$$S(x,y) = \sum_{j=0}^{\infty} S^{(j)}(x,y)$$

with

$$S^{(j)}(x,y) = \int \frac{d^2p}{(2\pi)^2}\ e^{ip\cdot(y-x)} \frac{\not p + m}{p^2 + m}\ \nu_j(p)$$

The integrand of $S^{(j)}$, the propagator of scale j, is supported on $M^{j-1} \le |p| \le M^{j+1}$ for $j > 0$ and $|p| \le M$ for $j = 0$. This is a region of volume at most const M^{2j} and on this region, the integrand is bounded by const $\frac{1}{M^j}$. By Corollary G.2, the value of F for this propagator is bounded by

$$\mathrm{F}_j = \Big(2\int \big\|\tfrac{\not p+m}{p^2+m}\big\|\nu_j(p)\ \tfrac{d^2p}{(2\pi)^2}\Big)^{1/2} \le C_{\mathrm{F}}\big(\tfrac{1}{M^j}M^{2j}\big)^{1/2} = C_{\mathrm{F}} M^{j/2}$$

for some constant $C_{\rm F}$. Here $\big\|\frac{\not p+m}{p^2+m}\big\|$ is the matrix norm of $\frac{\not p+m}{p^2+m}$. Also

$$\sup_{\substack{x,y\\ \sigma,\sigma'}} |S^{(j)}_{\sigma,\sigma'}(x,y)| \le \int \big\|\tfrac{\not p+m}{p^2+m}\big\| \nu_j(p)\, \tfrac{d^2p}{(2\pi)^2} \le {\rm const}\, M^j$$

By the usual integration by parts games (a relatively complex version of which is used in Proposition P.1)

$$|S^{(j)}_{\sigma,\sigma'}(x,y)| \le {\rm const}\, \tfrac{M^j}{[1+M^j|x-y|]^3} \quad\Rightarrow\quad \sup_{x,\sigma} \sum_{\sigma'} \int d^2y\ |S^{(j)}_{\sigma,\sigma'}(x,y)| \le {\rm const}\, \tfrac{1}{M^j}$$

so that the value of D for this propagator is bounded by

$${\rm D}_j = \tfrac{1}{M^{2j}}$$

We can always avoid having a const in ${\rm D}_j$ by increasing the value of the $C_{\rm F}$ in ${\rm F}_j$. To apply Corollary 5 to this model, we fix some $\alpha \ge 2$ and define the norm

$$\|W\|_j = {\rm D}_j \|W\|_{\alpha {\rm F}_j} = \sum_{l,r} (\alpha C_{\rm F})^{l+r} M^{j\frac{l+r-4}{2}} \|w_{l,r}\|$$

Suppose that we have integrated out all scales from some ultraviolet cutoff down to j and have ended up with an interaction that obeys $\|W\|_j \le 1$. To integrate out scale $j-1$ we use

Theorem 11GN *Suppose $\alpha \ge 2$ and $M \ge \frac{\alpha}{\alpha-1}\big(\frac{\alpha+1}{\alpha}\big)^6$. If $\|W\|_j \le \frac{1}{3}$ and $w_{l,r}$ vanishes for $l+r \le 4$, then $\|\Omega_{j-1}(W)\|_{j-1} \le \|W\|_j$.*

Proof: To apply Corollary 5 at scale $j-1$, we need ${\rm D}_{j-1}\|W\|_{(\alpha+1){\rm F}_{j-1}} \le \frac{1}{3}$. But

$$\begin{aligned}
{\rm D}_{j-1}\|W\|_{(\alpha+1){\rm F}_{j-1}} &= \sum_{l,r} ((\alpha+1)C_{\rm F})^{l+r} M^{(j-1)\frac{l+r-4}{2}} \|w_{l,r}\| \\
&= \sum_{\substack{l,r\\ l+r\ge 6}} \Big(\tfrac{\alpha+1}{\alpha} M^{-(\frac{1}{2}-\frac{2}{l+r})}\Big)^{l+r} (\alpha C_{\rm F})^{l+r} M^{j\frac{l+r-4}{2}} \|w_{l,r}\| \\
&\le \sum_{\substack{l,r\\ l+r\ge 6}} \Big(\tfrac{\alpha+1}{\alpha} M^{-\frac{1}{6}}\Big)^{l+r} (\alpha C_{\rm F})^{l+r} M^{j\frac{l+r-4}{2}} \|w_{l,r}\| \\
&\le \big(\tfrac{\alpha+1}{\alpha}\big)^6 \tfrac{1}{M} \|W\|_j \le \tfrac{1}{3}
\end{aligned}$$

as $M > \left(\frac{\alpha+1}{\alpha}\right)^6$ and $\|W\|_j \le \frac{1}{3}$. By Corollary 5,

$$\begin{aligned}\|\Omega_{j-1}(W)\|_{j-1} &= \mathrm{D}_{j-1}\|\Omega_{j-1}(W)\|_{\alpha\mathrm{F}_{j-1}} \le \tfrac{\alpha}{\alpha-1}\mathrm{D}_{j-1}\|W\|_{\alpha\mathrm{F}_{j-1}} \le \tfrac{\alpha}{\alpha-1}\left(\tfrac{\alpha+1}{\alpha}\right)^6\tfrac{1}{M}\|W\|_j \\ &\le \|W\|_j\end{aligned}$$

■

It is no surprise that two and four-legged vertices cannot be handled. We have not built in any renormalization.

Example (Many-fermion$_2$ – without sectorization)

The propagator, or covariance, for many–fermion models is the Fourier transform of

$$C_{\sigma,\sigma'}(k) = \frac{\delta_{\sigma,\sigma'}}{ik_0 - e(\mathbf{k})}$$

where $k = (k_0, \mathbf{k})$ and $e(\mathbf{k})$ is the one particle dispersion relation minus the chemical potential. The subscript on many-fermion$_2$ signifies that the number of space dimensions is two (i.e. $\mathbf{k} \in \mathbb{R}^2,\ k \in \mathbb{R}^3$). We assume that $e(\mathbf{k})$ is a reasonably smooth function (for example, C^4) that has a nonempty, compact, strictly convex zero set, called the Fermi curve and denoted $\mathcal{F}$. We further assume that $\nabla e(\mathbf{k})$ does not vanish for $\mathbf{k} \in \mathcal{F}$, so that $\mathcal{F}$ is itself a reasonably smooth curve. At low temperatures only those momenta with $k_0 \approx 0$ and $\mathbf{k}$ near $\mathcal{F}$ are important, so we replace the above propagator with

$$C(k) = \frac{U(k)}{ik_0 - e(\mathbf{k})}\delta_{\sigma,\sigma'}$$

The precise ultraviolet cutoff, $U(k)$, shall be chosen shortly. It is a C_0^∞ function which takes values in $[0,1]$, is identically 1 for $k_0^2 + e(\mathbf{k})^2 \le 1$ and vanishes for $k_0^2 + e(\mathbf{k})^2$ larger than some constant.

We slice momentum space into shells around the Fermi curve. The j^{th} shell is defined to be the support of

$$\nu^{(j)}(k) = \nu\left(M^{2j}(k_0^2 + e(\mathbf{k})^2)\right)$$

By construction, the j^{th} shell is a subset of

$$\left\{\ k \ \middle|\ \tfrac{1}{M^{j+1}} \le |ik_0 - e(\mathbf{k})| \le \tfrac{1}{M^{j-1}}\ \right\}$$

As the scale parameter $M > 1$, the shells near the Fermi curve have j near $+\infty$. Setting

$$C^{(j)}(k) = C(k)\nu^{(j)}(k)$$

and $U(k) = \sum\limits_{j=0}^{\infty} \nu^{(j)}(k)$ we have

$$C(k) = \sum_{j=0}^{\infty} C^{(j)}(k)$$

The propagator $C^{(j)}(k)$ is supported on a region of volume at most const M^{-2j} (k_0 is restricted to an interval of length const M^{-j} and $\mathbf{k}$ must remain within a distance const M^{-j} of $\mathcal{F}$) and is bounded by const M^j. By Corollary G.2, the value of F for this propagator is bounded by

$$\mathrm{F}_j = \Big(2 \int \tfrac{\nu^{(j)}(k)}{|ik_0 - e(\mathbf{k})|} \tfrac{d^3k}{(2\pi)^3}\Big)^{1/2} \le C_{\mathrm{F}} \big(M^j \tfrac{1}{M^{2j}}\big)^{1/2} = C_{\mathrm{F}} \tfrac{1}{M^{j/2}} \tag{1}$$

for some constant C_{F}. Also

$$\sup_{\substack{x,y \\ \sigma,\sigma'}} |C^{(j)}_{\sigma,\sigma'}(x,y)| \le \int \tfrac{\nu^{(j)}(k)}{|ik_0 - e(\mathbf{k})|} \tfrac{d^3k}{(2\pi)^3} \le \text{const}\, \tfrac{1}{M^j}$$

Each derivative $\frac{\partial}{\partial k_i}$ acting on $\frac{\nu^{(j)}(k)}{ik_0 - e(\mathbf{k})}$ increases the supremum of its magnitude by a factor of order M^j. So, naively, it looks like

$$|C^{(j)}_{\sigma,\sigma'}(x,y)| \le \text{const}\, \tfrac{1/M^j}{[1+M^{-j}|x-y|]^4} \qquad \Rightarrow \qquad \sup_{x,\sigma} \sum_{\sigma'} \int d^3y\, |C^{(j)}_{\sigma,\sigma'}(x,y)| \le \text{const}\, M^{2j}$$

In fact, using Corollary P.3, with $\mathfrak{l}_j = \frac{1}{M^{j/2}}$, yields the better bound

$$\sup_{x,\sigma} \sum_{\sigma'} \int d^3y\, |C^{(j)}_{\sigma,\sigma'}(x,y)| \le \text{const}\, \tfrac{1}{\mathfrak{l}_j} M^j \le \text{const}\, M^{3j/2} \tag{2}$$

Here, the factor $\frac{1}{\mathfrak{l}_j}$ is the number of terms in the partition of unity used to write $C^{(j)}$ as a sum of $C^{(j)}_\chi$'s, each term of which is bounded using Corollary P.3. So the value of D for this propagator is bounded by

$$\mathrm{D}_j = M^{5j/2}$$

This time we define the norm

$$\|W\|_j = \mathrm{D}_j \|W\|_{\alpha \mathrm{F}_j} = \sum_{l,r} (\alpha C_{\mathrm{F}})^{l+r} M^{-j\frac{l+r-5}{2}} \|w_{l,r}\|$$

If we have integrated out all scales from the ultraviolet cutoff, which in this (infrared) problem is at scale 0, to j and we have ended up with some interaction that obeys $\|W\|_j \le 1$, then we integrate out scale $j+1$using the following analog of Theorem 11GN.

Theorem 11MB$_1$ *Suppose $\alpha \ge 2$ and $M \ge \left(\frac{\alpha}{\alpha-1}\right)^2\left(\frac{\alpha+1}{\alpha}\right)^{12}$. If $\|W\|_j \le \frac{1}{3}$ and $w_{l,r}$ vanishes for $l+r<6$, then $\|\Omega_{j+1}(W)\|_{j+1} \le \|W\|_j$.*

Proof: To apply Corollary 5 at scale $j+1$, we need $\mathrm{D}_{j+1}\|W\|_{(\alpha+1)\mathrm{F}_{j+1}} \le \frac{1}{3}$. But

$$\begin{aligned}
\mathrm{D}_{j+1}\|W\|_{(\alpha+1)\mathrm{F}_{j+1}} &= \sum_{l,r}((\alpha+1)C_{\mathrm{F}})^{l+r}M^{-(j+1)\frac{l+r-5}{2}}\|w_{l,r}\| \\
&= \sum_{\substack{l,r\\ l+r\ge 6}} \left(\tfrac{\alpha+1}{\alpha}M^{-\frac{1}{2}(1-\frac{5}{l+r})}\right)^{l+r}(\alpha C_{\mathrm{F}})^{l+r}M^{-j\frac{l+r-5}{2}}\|w_{l,r}\| \\
&\le \sum_{\substack{l,r\\ l+r\ge 6}} \left(\tfrac{\alpha+1}{\alpha}M^{-\frac{1}{12}}\right)^{l+r}(\alpha C_{\mathrm{F}})^{l+r}M^{-j\frac{l+r-5}{2}}\|w_{l,r}\| \\
&\le \left(\tfrac{\alpha+1}{\alpha}\right)^6\tfrac{1}{M^{1/2}}\|W\|_j \le \|W\|_j \le \tfrac{1}{3}
\end{aligned}$$

By Corollary 5

$$\begin{aligned}
\|\Omega_{j+1}(W)\|_{j+1} &= \mathrm{D}_{j+1}\|\Omega_{j+1}(W)\|_{\alpha\mathrm{F}_{j+1}} \le \tfrac{\alpha}{\alpha-1}\mathrm{D}_{j+1}\|W\|_{\alpha\mathrm{F}_{j+1}} \le \tfrac{\alpha}{\alpha-1}\left(\tfrac{\alpha+1}{\alpha}\right)^6\tfrac{1}{M^{1/2}}\|W\|_j \\
&\le \|W\|_j
\end{aligned}$$

■

It looks, in Theorem 11MB$_1$, like five-legged vertices are marginal and all vertices with five or fewer legs have to be renormalized. Of course, by evenness, there are no five–legged vertices so only vertices with two or four legs have to be renormalized. But it still looks, contrary to the behaviour of perturbation theory, like four–legged vertices are worse than marginal. Fortunately, this is not the case. Our bounds can be tightened still further.

In the bounds (1) and (2) the momentum k runs over a shell around the Fermi curve. Effectively, the estimates we have used to count powers of M^j assume that all momenta entering an $l+r$–legged vertex run independently over the shell. Thus the estimates fail to take into account conservation of momentum. As a simple illustration of this, observe that for the two–legged diagram $B(x,y) = \int d^3z\, C^{(j)}_{\sigma,\sigma}(x,z)C^{(j)}_{\sigma,\sigma}(z,y)$, (2) yields the bound

$$\begin{aligned}
\sup_x \int d^3y\,|B(x,y)| &\le \sup_x \int d^3z\,\left|C^{(j)}_{\sigma,\sigma}(x,z)\right| \int d^3y\,\left|C^{(j)}_{\sigma,\sigma}(z,y)\right| \\
&\le \mathrm{const}\, M^{3j/2}M^{3j/2} = \mathrm{const}\, M^{3j}
\end{aligned}$$

But $B(x,y)$ is the Fourier transform of $W(k) = \frac{\nu^{(j)}(k)^2}{[ik_0 - e(\mathbf{k})]^2} = C^{(j)}(k)C^{(j)}(p)\big|_{p=k}$. Conservation of momentum forces the momenta in the two lines to be the same. Plugging this $W(k)$ and $\mathfrak{l}_j = \frac{1}{M^{j/2}}$ into Corollary P.2 yields

$$\sup_x \int d^3y\ |B(x,y)| \le \text{const}\ \tfrac{1}{\mathfrak{l}_j} M^{2j} \le \text{const}\ M^{5j/2}$$

We exploit conservation of momentum by partitioning the Fermi curve into "sectors".

Example (Many-fermion$_2$ – with sectorization)

We start by describing precisely what sectors are, as subsets of momentum space. Let, for $k = (k_0, \mathbf{k})$, $\mathbf{k}'(k)$ be any reasonable "projection" of $\mathbf{k}$ onto the Fermi curve. In the event that $\mathcal{F}$ is a circle of radius $k_{\mathcal{F}}$ centered on the origin, it is natural to choose $\mathbf{k}'(k) = \frac{k_{\mathcal{F}}}{|\mathbf{k}|}\mathbf{k}$. For general $\mathcal{F}$, one can always construct, in a tubular neighbourhood of $\mathcal{F}$, a C^∞ vector field that is transverse to $\mathcal{F}$, and then define $\mathbf{k}'(k)$ to be the unique point of $\mathcal{F}$ that is on the same integral curve of the vector field as $\mathbf{k}$ is.

Let $j > 0$ and set

$$\nu^{(\ge j)}(k) = \begin{cases} 1 & \text{if } k \in \mathcal{F} \\ \sum\limits_{i \ge j} \nu^{(i)}(k) & \text{otherwise} \end{cases}$$

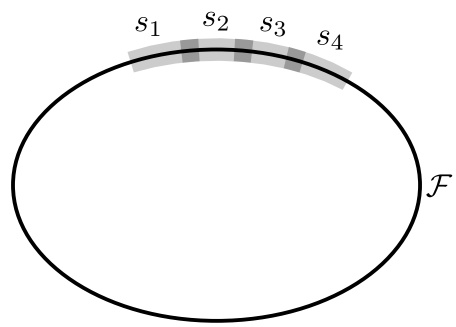
Let I be an interval on the Fermi surface $\mathcal{F}$. Then

$$s = \left\{\ k\ \middle|\ \mathbf{k}'(k) \in I,\ \ k \in \operatorname{supp} \nu^{(\ge j-1)}\ \right\}$$

is called a sector of length length(I) at scale j. Two different sectors s and s' are called neighbours if $s' \cap s \ne \emptyset$. A sectorization of length $\mathfrak{l}_j$ at scale j is a set Σ_j of sectors of length $\mathfrak{l}_j$ at scale j that obeys

- the set Σ_j of sectors covers the Fermi surface
- each sector in Σ_j has precisely two neighbours in Σ_j, one to its left and one to its right
- if $s,\ s' \in \Sigma_j$ are neighbours then $\frac{1}{16}\mathfrak{l}_j \le \text{length}(s \cap s' \cap \mathcal{F}) \le \frac{1}{8}\mathfrak{l}_j$

Observe that there are at most $2\,\text{length}(\mathcal{F})/\mathfrak{l}_j$ sectors in Σ_j. In these notes, we fix $\mathfrak{l}_j = \frac{1}{M^{j/2}}$ and a sectorization Σ_j at scale j.



Next we describe how we "sectorize" an interaction

$$W_n = \sum_{\substack{\sigma_i \in \{\uparrow,\downarrow\} \\ \kappa_i \in \{0,1\}}} \int w_n\big((x_1,\sigma_1,\kappa_1),\cdots,(x_n,\sigma_n,\kappa_n)\big)\ \psi_{\sigma_1}(x_1,\kappa_1)\cdots\psi_{\sigma_n}(x_n,\kappa_n)\ \prod_{i=1}^n dx_i$$

where

$$\psi_{\sigma_i}(x_i) = \psi_{\sigma_i}(x_i,\kappa_i)\big|_{\kappa_i=0} \qquad\qquad \bar\psi_{\sigma_i}(x_i) = \psi_{\sigma_i}(x_i,\kappa_i)\big|_{\kappa_i=1}$$

To save writing, we are temporarily ignoring the distinction between "internal" (a_i–type) fields and "external" (c_i–type) fields, so that $w_{l,r}$ is being replaced by w_n. Let $\mathcal{F}(n,\Sigma_j)$ denote the space of all translation invariant functions

$$f_n\big((x_1,\sigma_1,\kappa_1,s_1),\cdots,(x_n,\sigma_n,\kappa_n,s_n)\big) : \big(\mathbb{R}^3\times\{\uparrow,\downarrow\}\times\{0,1\}\times\Sigma_j\big)^n \to \mathbb{C}$$

whose Fourier transform, $\hat f_n\big((k_1,\sigma_1,\kappa_1,s_1),\cdots,(k_n,\sigma_n,\kappa_n,s_n)\big)$, vanishes unless $k_i\in s_i$. An $f_n\in\mathcal{F}(n,\Sigma_j)$ is said to be a sectorized representative for w_n if

$$\hat w_n\big((k_1,\sigma_1,\kappa_1),\cdots,(k_n,\sigma_n,\kappa_n)\big) = \sum_{\substack{s_i\in\Sigma_j \\ 1\le i\le n}} \hat f_n\big((k_1,\sigma_1,\kappa_1,s_1),\cdots,(k_n,\sigma_n,\kappa_n,s_n)\big)$$

for all $k_1,\cdots,k_n \in \operatorname{supp}\nu^{(\ge j)}$. It is easy to construct a sectorized representative for w_n by introducing (in momentum space) a partition of unity of $\operatorname{supp}\nu^{(\ge j)}$ subordinate to Σ_j. Furthermore, if f_n is a sectorized representative for w_n, then

$$\begin{aligned}\int w_n\big((x_1,\sigma_1,\kappa_1),\cdots,(x_n,\sigma_n,\kappa_n)\big)\ \psi_{\sigma_1}(x_1,\kappa_1)\cdots\psi_{\sigma_n}(x_n,\kappa_n)\ \prod_{i=1}^n dx_i \\ = \sum_{\substack{s_i\in\Sigma_j \\ 1\le i\le n}} \int f_n\big((x_1,\sigma_1,\kappa_1,s_1),\cdots,(x_n,\sigma_n,\kappa_n,s_n)\big)\ \psi_{\sigma_1}(x_1,\kappa_1)\cdots\psi_{\sigma_n}(x_n,\kappa_n)\ \prod_{i=1}^n dx_i\end{aligned}$$

for all $\psi_{\sigma_i}(x_n,\kappa_i)$ "in the support of" $d\mu_{C^{(\ge j)}}$, i.e. provided ψ is integrated out using a Gaussian Grassmann measure whose propagator is supported in supp $\nu^{(\ge j)}(k)$. Furthermore, by the momentum space support property of f_n,

$$\begin{aligned}\int f_n\big((x_1,\sigma_1,\kappa_1,s_1),\cdots,(x_n,\sigma_n,\kappa_n,s_n)\big)\ \psi_{\sigma_1}(x_1,\kappa_1)\cdots\psi_{\sigma_n}(x_n,\kappa_n)\ \prod_{i=1}^n dx_i \\ =\int f_n\big((x_1,\sigma_1,\kappa_1,s_1),\cdots,(x_n,\sigma_n,\kappa_n,s_n)\big)\ \psi_{\sigma_1}(x_1,\kappa_1,s_1)\cdots\psi_{\sigma_n}(x_n,\kappa_n,s_n)\ \prod_{i=1}^n dx_i\end{aligned}$$

where

$$\psi_\sigma(x,b,s)=\int d^3y\ \psi_\sigma(y,b,s)\hat\chi_s^{(j)}(x-y)$$

and $\hat\chi_s^{(j)}$ is the Fourier transform of a function that is identically one on the sector s. This function is chosen shortly before Proposition P.1.

We have expressed the interaction

$$W_n=\sum_{\substack{s_i\in\Sigma_j\\ \sigma_i\in\{\uparrow,\downarrow\}\\ \kappa_i\in\{0,1\}}}\int f_n\big((x_1,\sigma_1,\kappa_1,s_1),\cdots,(x_n,\sigma_n,\kappa_n,s_n)\big)\ \prod_{i=1}^n\psi_{\sigma_i}(x_i,\kappa_i,s_i)\ \prod_{i=1}^n dx_i$$

in terms of a sectorized kernel f_n and new "sectorized" fields, $\psi_\sigma(x,\kappa,s)$, that have propagator

$$\begin{aligned}C^{(j)}_{\sigma,\sigma'}\big((x,s),(y,s')\big)&=\int\psi_\sigma(x,0,s)\psi_{\sigma'}(y,1,s')\,d\mu_{C^{(j)}}(\psi)\\ &=\delta_{\sigma,\sigma'}\int\frac{d^3k}{(2\pi)^3}\ e^{ik\cdot(y-x)}\ \frac{\nu^{(j)}(k)\chi_s^{(j)}(k)\chi_{s'}^{(j)}(k)}{ik_0-e(\mathbf{k})}\end{aligned}$$

The momentum space propagator

$$C^{(j)}_{\sigma,\sigma'}(k,s,s')=\delta_{\sigma,\sigma'}\frac{\nu^{(j)}(k)\chi_s^{(j)}(k)\chi_{s'}^{(j)}(k)}{ik_0-e(\mathbf{k})}$$

vanishes unless s and s' are equal or neighbours, is supported in a region of volume const $\mathfrak{l}_j\frac{1}{M^{2j}}$ and has supremum bounded by const M^j. By Corollary G.3, the value of F for this propagator is bounded by

$$\mathrm{F}_j\le C_{\mathrm{F}}\big(\tfrac{1}{M^{2j}}\,M^j\mathfrak{l}_j\big)^{1/2}=C_{\mathrm{F}}\sqrt{\tfrac{\mathfrak{l}_j}{M^j}}$$

for some constant C_{F}. By Corollary P.3,

$$\sup_{x,\sigma,s}\sum_{\sigma',s'}\int d^3y\ |C^{(j)}_{\sigma,\sigma'}\big((x,s),(y,s')\big)|\le\text{const}\,M^j$$

so the value of D for this propagator is bounded by

$$\mathrm{D}_j = \tfrac{1}{\mathfrak{l}_j} M^{2j}$$

We are now almost ready to define the norm on interactions that replaces the unsectorized norm $\|W\|_j = \mathrm{D}_j \|W\|_{\alpha \mathrm{F}_j}$ of the last example. We define a norm on $\mathcal{F}(l+r, \Sigma_j)$ by

$$\|f\| = \max_{1 \le i \le l+r} \max_{x_i, \sigma_i, \kappa_i, s_i} \sum_{\substack{\sigma_k, \kappa_k, s_k \\ k \ne i}} \int \prod_{\ell \ne i} dx_\ell \, \big| f\big((x_1, \sigma_1, \kappa_1, s_1), \cdots, (x_{l+r}, \sigma_{l+r}, \kappa_{l+r}, s_{l+r})\big) \big|$$

and for any translation invariant function

$$w_{l,r}\big((x_1, \sigma_1, \kappa_1), \cdots, (x_{l+r}, \sigma_{l+r}, \kappa_{l+r})\big) : \big(\mathrm{I\!R}^3 \times \{\uparrow, \downarrow\} \times \{0,1\}\big)^{l+r} \to \mathbb{C}$$

we define

$$\|w_{l,r}\|_{\Sigma_j} = \inf \Big\{ \, \|f\| \, \Big| \, f \in \mathcal{F}(l+r, \Sigma_j) \text{ a representative for } W \Big\}$$

The sectorized norm on interactions is

$$\|W\|_{\alpha, j} = \mathrm{D}_j \sum_{l,r} (\alpha \mathrm{F}_j)^{l+r} \|w_{l,r}\|_{\Sigma_j} = \sum_{l,r} (\alpha C_{\mathrm{F}})^{l+r} \mathfrak{l}_j^{\frac{l+r-2}{2}} M^{-j \frac{l+r-4}{2}} \|w_{l,r}\|_{\Sigma_j}$$

Proposition 12 (Change of Sectorization) *Let $j' > j \ge 0$. There is a constant C_S, independent of M, j and j', such that for all $l + r \ge 4$*

$$\|w_{l,r}\|_{\Sigma_{j'}} \le \big[C_S \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j'}} \big]^{l+r-3} \|w_{l,r}\|_{\Sigma_j}$$

Proof: Write $l + r = n$. Also the spin indices σ_i and bar/unbar indices κ_i play no role, so we supress them. Let $\epsilon > 0$ and choose $f_n \in \mathcal{F}(n, \Sigma_j)$ such that

$$w_{l,r}(k_1, \cdots, k_n) = \sum_{\substack{s_i \in \Sigma_j \\ 1 \le i \le n}} f_n\big((k_1, s_1), \cdots, (k_n, s_n)\big)$$

for all $k_1, \cdots, k_n$ in the support of $\operatorname{supp} \nu^{(\ge j)}$ and

$$\|w_{l,r}\|_{\Sigma_j} \ge \|f_n\| - \epsilon$$

Let

$$1 = \sum_{s' \in \Sigma_{j'}} \chi_{s'}(\mathbf{k}')$$

be a partition of unity of the Fermi curve $\mathcal{F}$ subordinate to the set $\left\{ \ s' \cap \mathcal{F} \ \middle| \ s' \in \Sigma_j \ \right\}$ of intervals that obeys

$$\sup_{\mathbf{k}'} \left| \partial_{\mathbf{k}'}^m \chi_{s'} \right| \le \tfrac{\mathrm{const}_m}{l_{j'}^m}$$

Fix a function $\varphi \in C_0^\infty\big([0,2)\big)$, independent of j, j' and M, which takes values in $[0,1]$ and which is identically 1 for $0 \le x \le 1$. Set

$$\varphi_{j'}(k) = \varphi\big(M^{2(j'-1)}[k_0^2 + e(\mathbf{k})^2]\big)$$

Observe that $\varphi_{j'}$ is identically one on the support of $\nu^{(\ge j')}$ and is supported in the support of $\nu^{(\ge j'-1)}$. Define $g_n \in \mathcal{F}(n, \Sigma_{j'})$ by

$$\begin{aligned} g_n\big((k_1,s_1'),\cdots,(k_n,s_n')\big) &= \sum_{\substack{s_\ell \in \Sigma \\ 1\le \ell\le n}} f_n\big((k_1,s_1),\cdots,(k_n,s_n)\big) \prod_{m=1}^n \big[\chi_{s_m'}(\mathbf{k}_m)\varphi_{j'}(k_m)\big] \\ &= \sum_{\substack{s_\ell \cap s_\ell' \ne \emptyset \\ 1\le \ell\le n}} f_n\big((k_1,s_1),\cdots,(k_n,s_n)\big) \prod_{m=1}^n \big[\chi_{s_m'}(\mathbf{k}_m)\varphi_{j'}(k_m)\big] \end{aligned}$$

Clearly

$$w_{l,r}(k_1,\cdots,k_n) = \sum_{\substack{s_\ell' \in \Sigma_{j'} \\ 1\le \ell\le n}} g_n\big((k_1,s_1'),\cdots,(k_n,s_n')\big)$$

for all k_ℓ in the support of $\operatorname{supp} \nu^{(\ge j')}$. Define

$$\begin{aligned} \mathrm{Mom}_i(s') = \Big\{ \ (s_1',\cdots,s_n') \in \Sigma_{j'}^n \ \Big| \ s_i' = s' \text{ and there exist } k_\ell \in s_\ell', \ 1\le \ell\le n \\ \text{such that } \sum_\ell (-1)^\ell k_\ell = 0 \ \Big\} \end{aligned}$$

Here, I am assuming, without loss of generality, that the even (respectively, odd) numbered legs of $w_{l,r}$ are hooked to ψ's (respectively $\bar\psi$'s). Then

$$\|g_n\| = \max_{1\le i\le n} \sup_{\substack{x_i \in \mathbb{R}^3 \\ s' \in \Sigma_{j'}}} \sum_{\mathrm{Mom}_i(s')} \int \prod_{\ell \ne i} dx_\ell \ \big| g_n\big((x_1,s_1'),\cdots,(x_n,s_n')\big)\big|$$

Fix any $1 \le i \le n$, $s' \in \Sigma_j'$ and $x_i \in \mathbb{R}^3$. Then

$$\begin{aligned} &\sum_{\mathrm{Mom}_i(s')} \int \prod_{\ell\ne i} dx_\ell \ \big|g_n\big((x_1,s_1'),\cdots,(x_n,s_n')\big)\big| \\ &\le \sum_{\mathrm{Mom}_i(s')} \sum_{\substack{s_1,\cdots,s_n \\ s_\ell \cap s_\ell' \ne \emptyset}} \int \prod_{\ell \ne i} dx_\ell \ \big|f_n\big((x_1,s_1),\cdots,(x_n,s_n)\big)\big| \max_{s'' \in \Sigma_{j'}} \|\hat\chi_{s''} * \hat\varphi_{j'}\|^n \end{aligned}$$

By Proposition P.1, with $j = j'$ and $\phi^{(j)} = \hat{\varphi}_{j'}$, $\max_{s''\in\Sigma_{j'}} \|\hat{\chi}_{s''} * \hat{\varphi}_{j'}\|^n$ is bounded by a constant independent of M, j' and $\mathfrak{l}_{j'}$. Observe that

$$\begin{aligned}
&\sum_{\mathrm{Mom}_i(s')} \sum_{\substack{s_1,\cdots,s_n \\ s_\ell\cap s'_\ell\neq\emptyset}} \int \prod_{\ell\neq i} dx_\ell \, \big|f_n\big((x_1,s_1),\cdots,(x_n,s_n)\big)\big| \\
&\qquad \le \sum_{\substack{s_1,\cdots,s_n \\ s_i\cap s'\neq\emptyset}} \sum_{\substack{\mathrm{Mom}_i(s') \\ s_\ell\cap s'_\ell\neq\emptyset \\ 1\le\ell\le n}} \int \prod_{\ell\neq i} dx_\ell \, \big|f_n\big((x_1,s_1),\cdots,(x_n,s_n)\big)\big|
\end{aligned}$$

I will not prove the fact that, for any fixed $s_1, \cdots, s_n \in \Sigma_j$, there are at most $\big[C'_S \frac{\mathfrak{l}_j}{\mathfrak{l}_{j'}}\big]^{n-3}$ elements of $\mathrm{Mom}_i(s')$ obeying $s_\ell \cap s'_\ell \neq \emptyset$ for all $1 \le \ell \le n$, but I will try to motivate it below. As there are at most two sectors $s \in \Sigma_j$ that intersect s',

$$\begin{aligned}
&\sum_{\substack{s_1,\cdots,s_n \\ s_i\cap s'\neq\emptyset}} \sum_{\substack{\mathrm{Mom}_i(s') \\ s_\ell\cap s'_\ell\neq\emptyset \\ 1\le\ell\le n}} \int \prod_{\ell\neq i} dx_\ell \, \big|f_n\big((x_1,s_1),\cdots,(x_n,s_n)\big)\big| \\
&\le 2\big[C'_S \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j'}}\big]^{n-3} \sup_{s\in\Sigma_j} \sum_{\substack{s_1,\cdots,s_n \\ s_i=s}} \int \prod_{\ell\neq i} dx_\ell \, \big|f_n\big((x_1,s_1),\cdots,(x_n,s_n)\big)\big| \\
&\le 2\big[C'_S \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j'}}\big]^{n-3} \|f_n\|
\end{aligned}$$

and

$$\begin{aligned}
\|w_{l,r}\|_{\Sigma_{j'}} \le \|g_n\| &\le 2 \max_{s''\in\Sigma_{j'}} \|\hat{\chi}_{s''} * \hat{\varphi}_{j'}\|^n \big[C'_S \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j'}}\big]^{n-3} \|f_n\| \\
&\le \big[C_S \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j'}}\big]^{n-3} \|\big(\|w_{l,r}\|_{\Sigma_j} + \epsilon\big)
\end{aligned}$$

with $C_S = 2 \max_{s''\in\Sigma_{j'}} \|\hat{\chi}_{s''} * \hat{\varphi}_{j'}\|^4 \, C'_S$.

Now, I will try to motivate the fact that, for any fixed $s_1, \cdots s_n \in \Sigma_j$, there are at most $\big[C'_S \frac{\mathfrak{l}_j}{\mathfrak{l}_{j'}}\big]^{n-3}$ elements of $\mathrm{Mom}_i(s')$ obeying $s_\ell \cap s'_\ell \neq \emptyset$ for all $1 \le \ell \le n$. We may assume that $i = 1$. Then s'_1 must be s'. Denote by I_ℓ the interval on the Fermi curve $\mathcal{F}$ that has length $\mathfrak{l}_j + 2\mathfrak{l}_{j'}$ and is centered on $s_\ell \cap \mathcal{F}$. If $s' \in \Sigma_{j'}$ intersects s_ℓ, then $s' \cap \mathcal{F}$ is contained in I_ℓ. Every sector in $\Sigma_{j'}$ contains an interval of $\mathcal{F}$ of length $\frac{3}{4}\mathfrak{l}_{j'}$ that does not intersect any other sector in $\Sigma_{j'}$. At most $[\frac{4}{3}\frac{\mathfrak{l}_j+2\mathfrak{l}_{j'}}{\mathfrak{l}_{j'}}]$ of these "hard core" intervals can be contained in I_ℓ. Thus there are at most $[\frac{4}{3}\frac{\mathfrak{l}_j}{\mathfrak{l}_{j'}} + 3]^{n-3}$ choices for $s'_2, \cdots, s'_{n-2}$.

Fix $s'_1, s'_2, \cdots, s'_{n-2}$. Once s'_{n-1} is chosen, s'_n is essentially uniquely determined by conservation of momentum. But the desired bound on $\mathrm{Mom}_i(s')$ demands more. It says, roughly speaking, that both s'_{n-1} and s'_n are essentially uniquely determined. As k_ℓ runs over s'_ℓ for $1 \le \ell \le n-2$, the sum $\sum_{\ell=1}^{n-2}(-1)^\ell k_\ell$ runs over a small set centered on some point $\mathbf{p}$. In order for $(s'_1, \cdots, s'_n)$ to be in $\mathrm{Mom}_1(s')$, there must exist $\mathbf{k}_{n-1} \in s'_{n-1} \cap \mathcal{F}$ and $\mathbf{k}_n \in s'_n \cap \mathcal{F}$ with $\mathbf{k}_n - \mathbf{k}_{n-1}$ very close to $\mathbf{p}$. But $\mathbf{k}_n - \mathbf{k}_{n-1}$ is a secant joining two points of the Fermi curve $\mathcal{F}$. We have assumed that $\mathcal{F}$ is convex. Consequently, for any given $\mathbf{p} \neq 0$ in $\mathbb{R}^2$ there exist at most two pairs $(\mathbf{k}', \mathbf{q}') \in \mathcal{F}^2$ with $\mathbf{k}' - \mathbf{q}' = \mathbf{p}$. So, if $\mathbf{p}$ is not near the origin, s'_{n-1} and s'_n are almost uniquely determined. If $\mathbf{p}$ is close to zero, then $\sum_{\ell=1}^{n-2}(-1)^\ell k_\ell$ must be close to zero and the number of allowed $s'_1, s'_2, \cdots, s'_{n-2}$ is reduced.

■

Theorem 11MB$_2$ *Suppose $\alpha \ge 2$ and $M \ge \big(\frac{\alpha}{\alpha-1}\big)^2 \big(C_S \frac{\alpha+1}{\alpha}\big)^{12}$. If $\|W\|_{\alpha,j} \le \frac{1}{3}$ and $w_{l,r}$ vanishes for $l+r \le 4$, then $\|\Omega_{j+1}(W)\|_{\alpha,j+1} \le \|W\|_{\alpha,j}$.*

Proof: We first verify that $\|W\|_{\alpha+1,j+1} \le \frac{1}{3}$.

$$
\begin{aligned}
\|W\|_{\alpha+1,j+1} &= \sum_{l,r} ((\alpha+1)C_{\mathrm{F}})^{l+r}\, \mathfrak{l}_{j+1}^{(l+r-2)/2}\, M^{-(j+1)\frac{l+r-4}{2}} \|w_{l,r}\|_{\Sigma_{j+1}} \\
&\le \sum_{\substack{l,r \\ l+r\ge 6}} \big(\tfrac{\alpha+1}{\alpha}\big)^{l+r} \big(\tfrac{\mathfrak{l}_{j+1}}{\mathfrak{l}_j}\big)^{\frac{l+r-2}{2}} M^{-\frac{l+r-4}{2}} \big(C_S \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\big)^{l+r-3} \quad (\alpha C_{\mathrm{F}})^{l+r}\, \mathfrak{l}_j^{(l+r-2)/2}\, M^{-j\frac{l+r-4}{2}} \|w_{l,r}\|_{\Sigma_j} \\
&\le \sum_{\substack{l,r \\ l+r\ge 6}} \big(C_S \tfrac{\alpha+1}{\alpha}\big)^{l+r} M^{-\frac{l+r-4}{2}} \big(\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\big)^{\frac{l+r-4}{2}} \quad (\alpha C_{\mathrm{F}})^{l+r}\, \mathfrak{l}_j^{(l+r-2)/2}\, M^{-j\frac{l+r-4}{2}} \|w_{l,r}\|_{\Sigma_j} \\
&= \sum_{\substack{l,r \\ l+r\ge 6}} \big(C_S \tfrac{\alpha+1}{\alpha}\big)^{l+r} M^{-\frac{l+r-4}{4}} \quad (\alpha C_{\mathrm{F}})^{l+r}\, \mathfrak{l}_j^{(l+r-2)/2}\, M^{-j\frac{l+r-4}{2}} \|w_{l,r}\|_{\Sigma_j} \\
&= \sum_{\substack{l,r \\ l+r\ge 6}} \big(C_S \tfrac{\alpha+1}{\alpha} M^{-\frac{1}{4}(1-\frac{4}{l+r})}\big)^{l+r} \quad (\alpha C_{\mathrm{F}})^{l+r}\, \mathfrak{l}_j^{(l+r-2)/2}\, M^{-j\frac{l+r-4}{2}} \|w_{l,r}\|_{\Sigma_j} \\
&\le \big(C_S \tfrac{\alpha+1}{\alpha}\big)^6 \tfrac{1}{M^{1/2}} \|W\|_{\alpha,j} \le \tfrac{1}{3}
\end{aligned}
$$

By Corollary 5,

$$
\|\Omega_{j+1}(W)\|_{\alpha,j+1} \le \tfrac{\alpha}{\alpha-1} \|W\|_{\alpha,j+1} \le \tfrac{\alpha}{\alpha-1} \big(C_S \tfrac{\alpha+1}{\alpha}\big)^6 \tfrac{1}{M^{1/2}} \|W\|_{\alpha,j} \le \|W\|_{\alpha,j}
$$

■

Gram Bounds

Let X be a finite set and A a function on $X \times X$. Let $\psi(\ell,\kappa),\ \ell \in X,\ \kappa \in \{0,1\}$ be the generators of a Grassmann algebra $\mathcal{A}$. In conventional notation, $\psi(\ell,0)$ is written $\psi(\ell)$ and $\psi(\ell,1)$ is written $\bar\psi(\ell)$. Let $d\mu_A(\psi)$ be the Grassmann Gaussian measure on $\mathcal{A}$ with

$$\int \psi(\ell,\kappa)\psi(\ell',\kappa')\ d\mu_A(\psi) = \begin{cases} 0 & \text{if } \kappa=\kappa'=0 \\ A(\ell,\ell') & \text{if } \kappa=0,\ \kappa'=1 \\ -A(\ell',\ell) & \text{if } \kappa=1,\ \kappa'=0 \\ 0 & \text{if } \kappa=\kappa'=1 \end{cases}$$

and denote by $:\ \cdot\ :$ Wick ordering with respect to $d\mu_A$.

Proposition G.1 *Assume that there is a Hilbert space $\mathcal{H}$ and vectors $f_\ell, g_\ell,\ \ell \in X$ in $\mathcal{H}$ such that*

$$A(\ell,\ell') = \langle f_\ell, g_{\ell'}\rangle_{\mathcal{H}} \qquad \text{for all } \ell,\ell' \in X$$

Then

$$\Big| \int \prod_{i=1}^n :\prod_{\mu=1}^{e_i} \psi\big(\ell_{i,\mu},\kappa_{i,\mu}\big):\ d\mu_A(\psi)\Big| \le \prod_{\substack{1\le i\le n \\ 1\le\mu\le e_i \\ \kappa_{i,\mu}=0}} \sqrt{2}\|f_{\ell_{i,\mu}}\|_{\mathcal{H}} \prod_{\substack{1\le i\le n \\ 1\le\mu\le e_i \\ \kappa_{i,\mu}=1}} \sqrt{2}\|g_{\ell_{i,\mu}}\|_{\mathcal{H}}$$

Proof: Define

$$S = \big\{\ (i,\mu)\ \big|\ 1\le i\le n,\ 1\le\mu\le e_i,\ \kappa_{i,\mu}=0\ \big\}$$
$$\bar S = \big\{\ (i,\mu)\ \big|\ 1\le i\le n,\ 1\le\mu\le e_i,\ \kappa_{i,\mu}=1\ \big\}$$

If the integral does not vanish, the cardinality of S and $\bar S$ coincide and there is a sign $\pm$ such that

$$\int \prod_{i=1}^n :\prod_{\mu=1}^{e_i} \psi\big(\ell_{i,\mu},\kappa_{i,\mu}\big):\ d\mu_A(\psi) = \pm \det\Big(M_{\alpha,\beta}\Big)_{\substack{\alpha\in S \\ \beta\in\bar S}}$$

where

$$M_{(i,\mu),(i',\mu')} = \begin{cases} 0 & \text{if } i=i' \\ A(\ell_{i,\mu},\ell_{i',\mu'}) & \text{if } i\ne i' \end{cases}$$

Define the vectors u^α, $\alpha \in S$ and v^β, $\beta \in \bar{S}$ in $\mathbb{C}^{n+1}$ by

$$u_i^\alpha = \begin{cases} 1 & \text{if } i = n+1 \\ 1 & \text{if } \alpha = (i,\mu) \text{ for some } 1 \le \mu \le e_i \\ 0 & \text{otherwise} \end{cases}$$

$$v_i^\beta = \begin{cases} 1 & \text{if } i = n+1 \\ -1 & \text{if } \beta = (i,\mu) \text{ for some } 1 \le \mu \le e_i \\ 0 & \text{otherwise} \end{cases}$$

Observe that, for all $\alpha \in S$ and $\beta \in \bar{S}$,

$$\|u^\alpha\| = \|v^\beta\| = \sqrt{2}$$

$$u^\alpha \cdot v^\beta = \begin{cases} 1 & \text{if } \alpha = (i,\mu),\ \beta = (i',\mu') \text{ with } i \ne i' \\ 0 & \text{if } \alpha = (i,\mu),\ \beta = (i',\mu') \text{ with } i = i' \end{cases}$$

Hence, setting

$$F_\alpha = u^\alpha \otimes f_{\ell_{i,\mu}} \in \mathbb{C}^{n+1} \otimes \mathcal{H} \qquad \text{for } \alpha = (i,\mu) \in S$$

$$G_\beta = v^\beta \otimes g_{\ell_{i,\mu}} \in \mathbb{C}^{n+1} \otimes \mathcal{H} \qquad \text{for } \beta = (i,\mu) \in \bar{S}$$

we have

$$M_{\alpha,\beta} = \langle F_\alpha, G_\beta \rangle_{\mathbb{C}^{n+1}\otimes\mathcal{H}}$$

and consequently, by Gram's inequality,

$$\begin{aligned} \left| \int \prod_{i=1}^n : \prod_{\mu=1}^{e_i} \psi\big(\ell_{i,\mu}, \kappa_{i,\mu}\big) :\ d\mu_A(\psi) \right| &= \left| \det \Big(M_{\alpha,\beta} \Big)_{\substack{\alpha\in S \\ \beta \in \bar{S}}} \right| \\ &\le \prod_{\alpha \in S} \|F_\alpha\|_{\mathbb{C}^{n+1}\otimes\mathcal{H}} \prod_{\beta\in\bar{S}} \|G_\beta\|_{\mathbb{C}^{n+1}\otimes\mathcal{H}} \\ &\le \prod_{\alpha\in S} \sqrt{2}\|f_{\ell_\alpha}\|_{\mathcal{H}} \prod_{\beta\in\bar{S}} \sqrt{2}\|g_{\ell_\beta}\|_{\mathcal{H}} \end{aligned}$$

■

Let $\mathfrak{S}$ be a finite set (of spin/colour values). Let $E_{\sigma,\sigma'}(k) \in L^1\big(\mathbb{R}^{d+1}, \frac{dk}{(2\pi)^{d+1}}\big)$, for each $\sigma, \sigma' \in \mathfrak{S}$, and let $d\mu_C$ be the Grassmann, Gaussian measure with covariance

$$C_{\sigma,\sigma'}(x,y) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\ e^{ik\cdot(y-x)} E_{\sigma,\sigma'}(k)$$

That is, for all $x, y \in \mathbb{R}^{d+1}$ and $\sigma, \sigma' \in \mathfrak{S}$

$$\int \psi_\sigma(x,\kappa)\psi_{\sigma'}(y,\kappa')\ d\mu_C(\psi) = \begin{cases} 0 & \text{if } \kappa = \kappa' = 0 \\ C_{\sigma,\sigma'}(x,y) & \text{if } \kappa = 0,\ \kappa' = 1 \\ -C_{\sigma',\sigma}(y,x) & \text{if } \kappa = 1,\ \kappa' = 0 \\ 0 & \text{if } \kappa = \kappa' = 1 \end{cases}$$

Corollary G.2

$$\sup_{x_{i,\mu},\sigma_{i,\mu},\kappa_{i,\mu}} \Big| \int \prod_{i=1}^{n} :\psi_{\sigma_{i,1}}(x_{i,1},\kappa_{i,1})\cdots\psi_{\sigma_{i,e_i}}(x_{i,e_i},\kappa_{i,e_i}) : d\mu_C(\psi) \Big| \le \Big(2 \int \|E(k)\| \tfrac{dk}{(2\pi)^{d+1}} \Big)^{\Sigma_i e_i/2}$$

Here $\|E(k)\|$ *denotes the norm of the matrix* $\Big(E_{\sigma,\sigma'}(k)\Big)_{\sigma,\sigma'\in\mathfrak{S}}$ *as an operator on* $\ell^2\big(\mathbb{C}^{|\mathfrak{S}|}\big)$.

Proof: Define

$$X = \big\{ \ (i,\mu) \ \big|\ 1 \le i \le n,\ 1 \le \mu \le e_i \ \big\}$$

$$A\big((i,\mu),(i',\mu')\big) = C_{\sigma_{i,\mu},\sigma_{i',\mu'}}(x_{i,\mu},x_{i',\mu'})$$

Let $\Psi\big((i,\mu),\kappa\big),\ (i,\mu)\in X,\ \kappa\in\{0,1\}$ be generators of a Grassmann algebra and let $d\mu_A(\Psi)$ be a Grassmann Gaussian measure on the Grassmann algebra with covariance A. This construction has been arranged so that

$$\int \psi_{\sigma_{i,\mu}}(x_{i,\mu},\kappa_{i,\mu})\psi_{\sigma_{i',\mu'}}(x_{i',\mu'},\kappa_{i',\mu'})\, d\mu_C(\psi) = \int \Psi\big((i,\mu),\kappa_{i,\mu}\big)\Psi\big((i',\mu'),\kappa_{i',\mu'}\big)\big)\, d\mu_A(\Psi)$$

and consequently

$$\begin{aligned}\int \prod_{i=1}^{n} &: \psi_{\sigma_{i,1}}(x_{i,1},\kappa_{i,1})\cdots\psi_{\sigma_{i,e_i}}(x_{i,e_i},\kappa_{i,e_i}) :\ d\mu_C(\psi) \\ &= \int \prod_{i=1}^{n} : \Psi\big((i,1),\kappa_{i,1}\big)\cdots\Psi\big((i,e_i),\kappa_{i,e_i}\big) :\ d\mu_A(\Psi)\end{aligned}$$

Let $\mathcal{H} = L^2\big(\mathbb{R}^{d+1},\tfrac{dk}{(2\pi)^{d+1}}\big)\otimes\mathbb{C}^{|\mathfrak{S}|}$ and

$$f_{i,\mu}(k,\sigma) = e^{ik\cdot x_{i,\mu}}\sqrt{\|E(k)\|}\,\delta_{\sigma,\sigma_{i,\mu}} \qquad g_{i,\mu}(k,\sigma) = e^{ik\cdot x_{i,\mu}}\frac{E_{\sigma,\sigma_{i,\mu}}(k)}{\sqrt{\|E(k)\|}}$$

If $\|E(k)\| = 0$, set $g_{i,\mu}(k,\sigma) = 0$. Then

$$A\big((i,\mu),(i',\mu')\big) = \langle f_{i,\mu}, g_{i',\mu'}\rangle_{\mathcal{H}}$$

and, since $\sum_{\sigma\in\mathfrak{S}} \big|E_{\sigma,\sigma_{i,\mu}}(k)\big|^2 \le \|E(k)\|^2$,

$$\big\|f_{i,\mu}\big\|_{\mathcal{H}},\ \big\|g_{i,\mu}\big\|_{\mathcal{H}} = \big\|\sqrt{\|E(k)\|}\,\big\|_{L^2} = \Big(\int \|E(k)\| \tfrac{dk}{(2\pi)^{d+1}}\Big)^{1/2}$$

The Corollary now follows from Proposition G.1. ■

Corollary G.3 *For any L^2 function χ, define $\psi_{\chi,\sigma}(x,\kappa) = \int d^{d+1}y\ \psi_\sigma(y,\kappa)\hat{\chi}(x-y)$. Then, for all $x_{i,\mu},\ \sigma_{i,\mu},\ \kappa_{i,\mu}$,*

$$\Big| \int \prod_{i=1}^{n} :\psi_{\chi_{i,1},\sigma_{i,1}}(x_{i,1},\kappa_{i,1})\cdots\psi_{\chi_{i,e_i},\sigma_{i,e_i}}(x_{i,e_i},\kappa_{i,e_i}):\ d\mu_C(\psi)\Big| \\ \le \prod_{i=1}^{n}\prod_{\mu=1}^{e_i}\Big(2\int \chi_{i,\mu}^2(k)\|E(k)\|\,\tfrac{dk}{(2\pi)^{d+1}}\Big)^{1/2}$$

Proof: The proof is identical to that of Corollary G.2 once the replacements

$$\begin{aligned} A\big((i,\mu),(i',\mu')\big) &= \delta_{\sigma_{i,\mu},\sigma_{i',\mu'}}\int \tfrac{dk}{(2\pi)^{d+1}} e^{ik\cdot(x_{i',\mu'}-x_{i,\mu})}\chi_{i,\mu}(k)E(k)\chi_{i',\mu'}(k) \\ f_{i,\mu}(k,\sigma) &= e^{ik\cdot x_{i,\mu}}\chi_{i,\mu}(k)\sqrt{\|E(k)\|}\,\delta_{\sigma,\sigma_{i,\mu}} \\ g_{i,\mu}(k,\sigma) &= e^{ik\cdot x_{i,\mu}}\chi_{i,\mu}(k)\tfrac{E_{\sigma,\sigma_{i,\mu}}(k)}{\sqrt{\|E(k)\|}} \end{aligned}$$

have been made. ■

Propagator Bounds

The propagator, or covariance, for many–fermion models is the Fourier transform of

$$C_{\sigma,\sigma'}(k) = \frac{\delta_{\sigma,\sigma'}}{ik_0 - e(\mathbf{k})}$$

where $k = (k_0, \mathbf{k})$ and $e(\mathbf{k})$ is the one particle dispersion relation minus the chemical potential. For this appendix, the spins σ, σ' play no role, so we suppress them completely. We also restrict our attention to two space dimensions (i.e. $\mathbf{k} \in \mathbb{R}^2,\ k \in \mathbb{R}^3$) though it is trivial to extend the results of this appendix to any number of space dimensions. We assume that $e(\mathbf{k})$ is a reasonably smooth function (for example, C^4) that has a nonempty, compact zero set $\mathcal{F}$, called the Fermi curve. We further assume that $\nabla e(\mathbf{k})$ does not vanish for $\mathbf{k} \in \mathcal{F}$, so that $\mathcal{F}$ is itself a reasonably smooth curve. At low temperatures only those momenta with $k_0 \approx 0$ and $\mathbf{k}$ near $\mathcal{F}$ are important, so we replace the above propagator with

$$C(k) = \frac{U(k)}{ik_0 - e(\mathbf{k})}$$

The precise ultraviolet cutoff, $U(k)$, shall be chosen shortly. It is a C_0^∞ function which takes values in $[0, 1]$, is identically 1 for $k_0^2 + e(\mathbf{k})^2 \le 1$ and vanishes for $k_0^2 + e(\mathbf{k})^2$ larger than some constant.

We slice momentum space into shells around the Fermi surface. To do this, we fix $M > 1$ and choose a function $\nu \in C_0^\infty([M^{-2}, M^2])$ that takes values in $[0, 1]$, is identically 1 on $[M^{-1/2}, M^{1/2}]$ and obeys

$$\sum_{j=0}^{\infty} \nu\left(M^{2j} x\right) = 1$$

for $0 < x < 1$. The j^{th} shell is defined to be the support of

$$\nu^{(j)}(k) = \nu\left(M^{2j}\left(k_0^2 + e(\mathbf{k})^2\right)\right)$$

By construction, the j^{th} shell is a subset of

$$\left\{\ k \ \middle|\ \tfrac{1}{M^{j+1}} \le |ik_0 - e(\mathbf{k})| \le \tfrac{1}{M^{j-1}}\ \right\}$$

As the scale parameter $M > 1$, the shells near the Fermi curve have j near $+\infty$. Setting

$$C^{(j)}(k) = C(k)\nu^{(j)}(k)$$

and $U(k) = \sum_{j=0}^{\infty} \nu^{(j)}(k)$ we have

$$C(k) = \sum_{j=0}^{\infty} C^{(j)}(k)$$

To analyze the Fourier transform of $C^{(j)}(k)$, we further decompose the j^{th} shell into more or less rectangular "sectors". To do so, we fix $\mathfrak{l}_j \in \left[\frac{1}{M^j}, \frac{1}{M^{j/2}}\right]$ and choose a partition of unity

$$1 = \sum_{s \in \Sigma^{(j)}} \chi_s^{(j)}(\mathbf{k}')$$

of the Fermi curve $\mathcal{F}$ with each $\chi_s^{(j)}$ supported on an interval of length $\mathfrak{l}_j$ and obeying

$$\sup_{\mathbf{k}'} \left|\partial_{\mathbf{k}'}^m \chi_s^{(j)}\right| \le \frac{\text{const}_m}{\mathfrak{l}_j{}^m}$$

Given any function $\chi(\mathbf{k}')$ on the Fermi curve $\mathcal{F}$, we define

$$C_\chi^{(j)}(k) = C^{(j)}(k)\chi(\mathbf{k}'(k))$$

where, for $k = (k_0, \mathbf{k})$, $\mathbf{k}'(k)$ is any reasonable "projection" of $\mathbf{k}$ onto the Fermi curve. In the event that $\mathcal{F}$ is a circle of radius $k_{\mathcal{F}}$ centered on the origin, it is natural to choose $\mathbf{k}'(k) = \frac{k_{\mathcal{F}}}{|\mathbf{k}|}\mathbf{k}$. For general $\mathcal{F}$, one can always construct, in a tubular neighbourhood of $\mathcal{F}$, a C^∞ vector field that is transverse to $\mathcal{F}$, and then define $\mathbf{k}'(k)$ to be the unique point of $\mathcal{F}$ that is on the same integral curve of the vector field as $\mathbf{k}$ is.

Proposition P.1 *Let $\chi(\mathbf{k}')$ be a C_0^∞ function on the Fermi curve $\mathcal{F}$ which takes values in $[0,1]$, which is supported on an interval of length $\mathfrak{l}_j \in \left[\frac{1}{M^j}, \frac{1}{M^{j/2}}\right]$ and whose derivatives obey*

$$\sup_{\mathbf{k}'} \left|\partial_{\mathbf{k}'}^n \chi(\mathbf{k}')\right| \le \frac{\text{const}_n}{\mathfrak{l}_j^n}$$

Fix any point $\mathbf{k}'_c$ in the support of χ. Let $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ be unit tangent and normal vectors to the Fermi curve at $\mathbf{k}'_c$ and set

$$\rho(x,y) \;=\; 1 + M^{-j}|x_0 - y_0| + M^{-j}|\mathbf{x}_\perp - \mathbf{y}_\perp| + \mathfrak{l}_j|\mathbf{x}_\parallel - \mathbf{y}_\parallel|$$

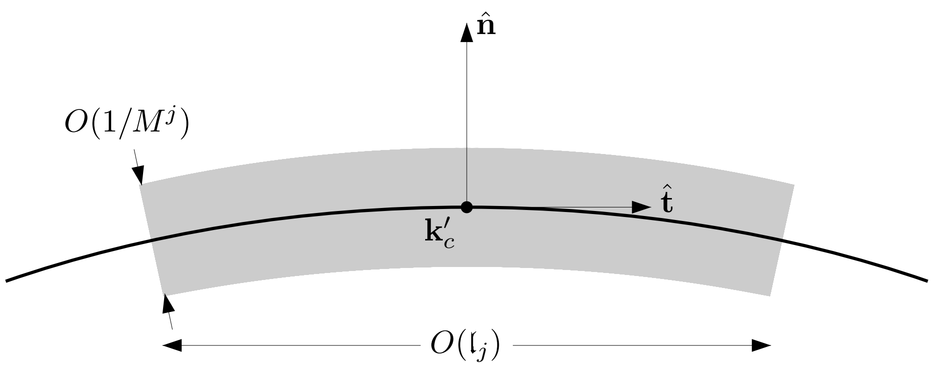
where $\mathbf{x}_\parallel$ *is the component of* $\mathbf{x}$ *parallel to* $\hat{\mathbf{t}}$ *and* $\mathbf{x}_\perp$ *is the component parallel to* $\hat{\mathbf{n}}$*.*

Let ϕ *be a* C_0^∞ *function which takes values in* $[0,1]$ *and set* $\phi^{(j)} = \phi\big(M^{2j}[k_0^2+e(\mathbf{k})^2]\big)$*. For any function* $W(k)$ *define*

$$W^{(j)}_{\chi,\phi}(x,y) = \int \tfrac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)} W(k)\phi^{(j)}(k)\,\chi\big(\mathbf{k}'(k)\big)$$

Let $\gamma \in \mathbb{N}$*. If* $e(\mathbf{k})$ *has bounded* $\max\{2,\gamma\}^{\rm th}$ *derivatives, then there is a constant,* const *, depending on* γ, $\text{const}_0, \cdots, \text{const}_\gamma$ *,* ϕ *and* $e(\mathbf{k})$*, but independent of* j*,* x *and* y *such that*

$$|W^{(j)}_{\chi,\phi}(x,y)| \le \text{const}\ \tfrac{\mathfrak{l}_j}{M^{2j}}\rho(x,y)^{-\gamma} \max_{\substack{\alpha\in\mathbb{N}^3\\|\alpha|\le\gamma}} \sup_{k\in\text{supp}\,\chi\phi^{(j)}} \tfrac{\mathfrak{l}_j^{\alpha_2}}{M^{j(\alpha_0+\alpha_1)}} \Big|\partial_{k_0}^{\alpha_0}\big(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\big)^{\alpha_1}\big(\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\big)^{\alpha_2}W(k)\Big|$$



Proof: Use S to denote the support of $\phi^{(j)}(k)\chi\big((\mathbf{k}'(k)\big)$. Observe that S has volume at most $\text{const}\,M^{-2j}\mathfrak{l}_j$, since k_0 is supported in an interval of length $\text{const}\,M^{-j}$, the component of $\mathbf{k}$ normal to $\mathcal{F}$ is supported in an interval of length $\text{const}\,M^{-j}$ and the component of $\mathbf{k}$ tangential to $\mathcal{F}$ runs over an interval of length $\text{const}\,\mathfrak{l}_j$. Hence

$$\sup_{x,y}|W^{(j)}_{\chi,\phi}(x,y)| \le \text{vol}(S)\sup_{k\in S}|W(k)| \le \text{const}\ \tfrac{\mathfrak{l}_j}{M^{2j}}\sup_{k\in S}|W(k)|$$

which is the desired bound for $\gamma = 0$.

To bound $\sup_{x,y}\rho(x,y)^\gamma|W^{(j)}_{\chi,\phi}(x,y)|$ by $4^\gamma C$, it suffices to bound

$$\Big|\big(\tfrac{x_0-y_0}{M^j}\big)^{\beta_0}\big(\tfrac{\mathbf{x}_\perp-\mathbf{y}_\perp}{M^j}\big)^{\beta_1}\big(\mathfrak{l}_j(\mathbf{x}_\parallel-\mathbf{y}_\parallel)\big)^{\beta_2}W^{(j)}_{\chi,\phi}(x,y)\Big|$$
$$= \Big|\int \tfrac{d^3k}{(2\pi)^3}e^{ik\cdot(x-y)}\big(\tfrac{1}{M^j}\partial_{k_0}\big)^{\beta_0}\big(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_1}\big(\mathfrak{l}_j\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_2}\Big(W(k)\phi^{(j)}(k)\,\chi\big(\mathbf{k}'(k)\big)\Big)\Big|$$

by C for all $x, y \in \mathbb{R}^3$ and $\beta \in \mathbb{N}^3$ with $|\beta| \le \gamma$. The volume of the domain of integration is still bounded by const $\frac{\mathfrak{l}_j}{M^{2j}}$, so by the product rule, to prove the desired bound it suffices to prove that

$$\max_{|\beta|\le\gamma} \sup_{k} \left| \big(\tfrac{1}{M^j}\partial_{k_0}\big)^{\beta_0} \big(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_1} \big(\mathfrak{l}_j\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_2} \Big(\phi^{(j)}(k)\,\chi\big(\mathbf{k}'(k)\big)\Big)\right| \le \text{const}$$

Since $\mathfrak{l}_j \ge \frac{1}{M^j}$ and all derivatives of $\mathbf{k}'(k)$ to order γ are bounded,

$$\max_{|\beta|\le\gamma} \sup_{k} \left| \big(\tfrac{1}{M^j}\partial_{k_0}\big)^{\beta_0} \big(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_1} \big(\mathfrak{l}_j\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_2} \chi\big(\mathbf{k}'(k)\big)\right| \le \text{const} \max_{\beta_1+\beta_2\le\gamma} \frac{\mathfrak{l}_j^{\beta_2}}{M^{j\beta_1}} \frac{1}{\mathfrak{l}_j^{\beta_1+\beta_2}} \le \text{const}$$

so, by the product rule, it suffices to prove

$$\max_{|\beta|\le\gamma} \sup_{k\in S} \left| \big(\tfrac{1}{M^j}\partial_{k_0}\big)^{\beta_0} \big(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_1} \big(\mathfrak{l}_j\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_2} \phi^{(j)}(k)\right| \le \text{const}$$

Set $I = \{1, \cdots, |\beta|\}$,

$$d_i = \begin{cases} \frac{1}{M^j}\partial_{k_0} & \text{if } 1 \le i \le \beta_0 \\ \frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}} & \text{if } \beta_0+1 \le i \le \beta_0+\beta_1 \\ \mathfrak{l}_j\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}} & \text{if } \beta_0+\beta_1+1 \le i \le |\beta| \end{cases}$$

and, for each $I' \subset I$, $d^{I'} = \prod_{i\in I'} d_i$. By the product and chain rules

$$d^I\phi^{(j)}(k) = \sum_{m=1}^{|\beta|} \sum_{(I_1,\cdots,I_m)\in\mathcal{P}_m} \tfrac{d^m\phi}{dx^m}\big(M^{2j}\,(k_0^2+e(\mathbf{k})^2)\big) \prod_{i=1}^{m} M^{2j} d^{I_i}\,\big(k_0^2+e(\mathbf{k})^2\big)$$

where $\mathcal{P}_m$ is the set of all partitions of I into m nonempty subsets $I_1, \cdots, I_m$ with, for all $i < i'$, the smallest element of I_i smaller than the smallest element of $I_{i'}$. For all m, $\big|\frac{d^m\phi}{dx^m}\big(M^{2j}\,(k_0^2+e(\mathbf{k})^2)\big)\big|$ is bounded by a constant independent of j, so to prove the Proposition, it suffices to prove that

$$\max_{|\beta|\le\gamma} \sup_{k\in S} \left| M^{2j}\big(\tfrac{1}{M^j}\partial_{k_0}\big)^{\beta_0} \big(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_1} \big(\mathfrak{l}_j\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_2}\,\big(k_0^2+e(\mathbf{k})^2\big)\right| \le \text{const}$$

If $\beta_0 \ne 0$

$$M^{2j}\big(\tfrac{1}{M^j}\partial_{k_0}\big)^{\beta_0} \big(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_1} \big(\mathfrak{l}_j\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\big)^{\beta_2}\,\big(k_0^2+e(\mathbf{k})^2\big) = \begin{cases} 2k_0M^j & \text{if } \beta_0=1,\ \beta_1=\beta_2=0 \\ 2 & \text{if } \beta_0=2,\ \beta_1=\beta_2=0 \\ 0 & \text{otherwise} \end{cases}$$

is bounded, independent of j since $|k_0| \le \mathrm{const}\,\frac{1}{M^j}$ on S. Thus it suffices to consider $\beta_0 = 0$. Applying the product rule once again, this time to the derivatives acting on $M^{2j} e(\mathbf{k})^2 = [M^j e(\mathbf{k})]\,[M^j e(\mathbf{k})]$, it suffices to prove

$$\max_{|\beta|\le\gamma}\ \sup_{k\in S} \left| M^j \left(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\beta_1} \left(\mathfrak{l}_j \hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\right)^{\beta_2} e(\mathbf{k})\right| \le \mathrm{const}$$

If $\beta_1 = \beta_2 = 0$, this follows from the fact that $|e(\mathbf{k})| \le \mathrm{const}\,\frac{1}{M^j}$ on S. If $\beta_1 \ge 1$ or $\beta_2 \ge 2$, it follows from $\frac{M^j}{M^{\beta_1 j}\mathfrak{l}_j^{\beta_2}} \le 1$. (Recall that $\mathfrak{l}_j \ge \frac{1}{M^{j/2}}$.) This leaves only $\beta_1 = 0,\ \beta_2 = 1$.

If $\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}} e(\mathbf{k})$ is evaluated at $\mathbf{k} = \mathbf{k}'_c$, it vanishes, since $\nabla_{\mathbf{k}} e(\mathbf{k}'_c)$ is parallel to $\hat{\mathbf{n}}$. The second derivative of e is bounded so that,

$$M^j \mathfrak{l}_j \sup_{k\in S} \left|\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}} e(\mathbf{k})\right| \le \mathrm{const}\, M^j \mathfrak{l}_j \sup_{k\in S} |\mathbf{k} - \mathbf{k}'_c| \le \mathrm{const}\, M^j \mathfrak{l}_j^2 \le \mathrm{const}$$

since $\mathfrak{l}_j \le \frac{1}{M^{j/2}}$. ■

Corollary P.2 *Under the hypotheses of Proposition P.1,*

$$\sup \left|W^{(j)}_{\chi,\phi}(x,y)\right| \le \mathrm{const}\ \tfrac{\mathfrak{l}_j}{M^{2j}} \sup_{k\in \mathrm{supp}\,\chi\phi^{(j)}} |W(k)|$$

and, if $e(\mathbf{k})$ *has bounded fourth derivatives,*

$$\begin{aligned} &\sup_x \int dy \left|W^{(j)}_{\chi,\phi}(x,y)\right|,\ \sup_y \int dx \left|W^{(j)}_{\chi,\phi}(x,y)\right| \\ &\qquad \le \mathrm{const}\ \max_{\substack{\alpha\in\mathbb{N}^3\\ |\alpha|\le 4}}\ \sup_{k\in\mathrm{supp}\,\chi\phi^{(j)}} \tfrac{\mathfrak{l}_j^{\alpha_2}}{M^{j(\alpha_0+\alpha_1)}} \left|\partial_{k_0}^{\alpha_0}\left(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\alpha_1}\left(\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\right)^{\alpha_2} W(k)\right| \end{aligned}$$

Proof: The first claim is simply a restatement of Proposition P.1 with $\gamma = 0$, For the second statement just use

$$\sup_x \int dy\, \tfrac{1}{\rho(x,y)^4},\ \sup_y \int dx\, \tfrac{1}{\rho(x,y)^4} = \int dx\, \tfrac{1}{\rho(x,0)^4} \le \mathrm{const}\, M^{2j}\tfrac{1}{\mathfrak{l}_j}$$

For the last inequality, just make the change of variables $x_0 = M^j z_0,\ \mathbf{x}_\perp = M^j z_1,\ \mathbf{x}_\parallel = \frac{1}{\mathfrak{l}_j} z_2$. ■

Corollary P.3 *Under the hypotheses of Proposition P.1,*

$$\sup \left|C_\chi^{(j)}(x,y)\right| \le \text{const } \tfrac{\mathfrak{l}_j}{M^j}$$

and, if $e(\mathbf{k})$ *has bounded fourth derivatives,*

$$\sup_x \int dy \left|C_\chi^{(j)}(x,y)\right|, \ \sup_y \int dx \left|C_\chi^{(j)}(x,y)\right| \le \text{const } \ M^j$$

Proof: Apply Corollary P.2 with $W(k) = \frac{1}{ik_0 - e(\mathbf{k})}$ and $\phi = \nu$. To achieve the desired bounds, we need

$$\max_{|\alpha|\le 4} \sup_{k \in \text{supp}\, \chi\nu^{(j)}} \left| \left(\tfrac{1}{M^j}\partial_{k_0}\right)^{\alpha_0} \left(\tfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\alpha_1} \left(\mathfrak{l}_j \hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\right)^{\alpha_2} \frac{1}{ik_0 - e(\mathbf{k})} \right| \le \text{const}\, M^j$$

In the notation of the proof of Proposition P.1, with β replaced by α,

$$d^I \nu^{(j)}(k) = M^j \sum_{m=1}^{|\alpha|} (-1)^m m! \sum_{(I_1,\cdots,I_m)\in\mathcal{P}_m} \left(\tfrac{1/M^j}{ik_0 - e(\mathbf{k})}\right)^{m+1} \prod_{i=1}^m M^j d^{I_i}(ik_0 - e(\mathbf{k}))$$

On the support of $\chi\nu^{(j)}$, $|ik_0 - e(\mathbf{k})| \ge \text{const}\,\frac{1}{M^j}$ so that $\left(\frac{1/M^j}{ik_0 - e(\mathbf{k})}\right)^{m+1}$ is bounded uniformly in j. That $M^j d^{I_i}(ik_0 - e(\mathbf{k}))$ is bounded uniformly in j was proven during the course of the proof of Proposition P.1. ■

References

The "abstract" fermionic expansion discussed in the first part of these lectures is essentially identical to that in

- J. Feldman, H. Knörrer, E. Trubowitz, **A Nonperturbative Representation for Fermionic Correlation Functions**, Communications in Mathematical Physics, to appear.

and is a simplified version of the expansion in

- J. Feldman, J. Magnen, V. Rivasseau and E.Trubowitz, **An Infinite Volume Expansion for Many Fermion Green's Functions**, Helvetica Physica Acta, **65** (1992) 679-721.

The second reference also contains a discussion of sectors. Both references are available over the web at

http://www.math.ubc.ca/~feldman/research.html