The Geometric Hypotheses

These axioms concern a class of marked Riemann surfaces $(X; A_1, B_1, \cdots)$, with $A_1, B_1, \cdots$ a canonical homology basis for X, that are "asymptotic to" a finite number of complex planes $\mathbb{C}$ joined by infinitely many handles. The notation

$$X = X^{\text{com}} \cup X^{\text{reg}} \cup X^{\text{han}}$$

$$X^{\text{reg}} = \bigcup_{\nu=1}^{m} X^{\text{reg}}_{\nu} \qquad X^{\text{han}} = \bigcup_{j \geq g+1} Y_j$$

denotes a marked Riemann surface with a decomposition into a compact, connected submanifold $X^{\text{com}} \subset X$ with smooth boundary and genus $g \geq 0$, a finite number of open "regular pieces" $X^{\text{reg}}_{\nu} \subset X$, $\nu = 1, \cdots, m$, and an infinite number of closed "handles" $Y_j \subset X$, $j \geq g+1$, with $X^{\text{com}} \cap \left(X^{\text{reg}} \cup X^{\text{han}}\right) = \emptyset$. Each handle will be biholomorphic to the model handle

$$\text{H}(t) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; z_1 z_2 = t \text{ and } |z_1|, |z_2| \leq 1 \right\}$$

for some $0 < t < \frac{1}{2}$.

(GH1) (Regular pieces)

(i) For all $1 \le \mu \ne \nu \le m$, $\overline{X_\mu^{\rm reg}} \cap \overline{X_\nu^{\rm reg}} = \emptyset$.

(ii) For each $1 \le \nu \le m$ there is a compact simply connected neighborhood $K_\nu \subset \mathbb{C}$ of 0 with smooth boundary. There is also an infinite discrete subset $S_\nu \subset \mathbb{C}$ and, for each $s \in S_\nu$, there is a compact, simply connected neighborhood $D_\nu(s)$ with smooth boundary $\partial D_\nu(s)$ such that

$$D_\nu(s) \cap D_\nu(s') = \emptyset \quad \forall s, s' \in S_\nu \text{ with } s \ne s'$$

$$K_\nu \cap D_\nu(s) = \emptyset \quad \forall s \in S_\nu$$

(iii) Set $G_\nu = \mathbb{C} \setminus \left(\operatorname{int} K_\nu \cup \bigcup_{s \in S_\nu} \operatorname{int} D_\nu(s) \right)$. There is a biholomorphic map Φ_ν ,

$$\Phi_\nu : G_\nu \to \overline{X_\nu^{\rm reg}}$$

between G_ν and $\overline{X_\nu^{\rm reg}}$.

(GH2) (Handles)

(i) For all $i \neq j$ with $i, j \geq g+1$, $Y_i \cap Y_j = \emptyset$.

(ii) For each $j \geq g+1$ there is a $0 < t_j < \frac{1}{2}$ and a biholomorphic map ϕ_j

$$\phi_j : \mathrm{H}(t_j) \rightarrow Y_j$$

between the model handle $\mathrm{H}(t_j)$ and Y_j.

(iii) For all $j \geq g+1$, A_j is the homology class represented by the oriented loop

$$\phi_j\Big(\big\{ (\sqrt{t_j}\, e^{i\theta}, \sqrt{t_j}\, e^{-i\theta}) \,\big|\, 0 \leq \theta \leq 2\pi \big\}\Big)$$

(iv) For every $\beta > 0$

$$\sum_{j \geq g+1} t_j^{\beta} < \infty$$

(GH3) (Glueing handles and regular pieces)

(i) For each $j \geq g+1$ the intersection $Y_j \cap X^{\mathrm{reg}}$ consists of two components Y_{j1}, Y_{j2}:

$$Y_j \cap X^{\mathrm{reg}} = Y_{j1} \cup Y_{j2}$$

For each pair (j,μ) with $j \geq g+1$ and $\mu = 1,2$ there is a radius $\tau_\mu(j) \in \left(\sqrt{t_j}\,,\,1\right)$ and a sheet number $\nu_\mu(j) \in \{1,\ldots,m\}$ such that

$$Y_{j\mu} = \phi_j\left(\left\{(z_1,z_2) \in H(t_j) \,\middle|\, \tau_\mu(j) < |z_\mu| \leq 1\right\}\right) \subset X^{\mathrm{reg}}_{\nu_\mu(j)}$$

There is a bijective map

$$\begin{aligned} (j,\mu) \ &\mapsto\ s_\mu(j) \\ \left\{ j \in \mathbb{Z} \,\middle|\, j \geq g+1 \right\} \times \{1,2\} \ &\to\ \bigsqcup_{\nu=1}^{m} S_\nu \end{aligned}$$

(disjoint union) such that

$$\phi_j\left(\left\{(z_1,z_2) \in H(t_j) \,\middle|\, |z_\mu| = \tau_\mu(j)\right\}\right) = \Phi_{\nu_\mu(j)}\left(\partial D_{\nu_\mu(j)}(s_\mu(j))\right)$$

(ii) For each $j \geq g+1$ and $\mu = 1, 2$ there are

$$R_\mu(j) \; > \; 4r_\mu(j) \; > \; 0$$

such that the biholomorphic map

$$g_{j\mu} : \mathcal{A}_{j\mu} \;=\; \left\{ z \in \mathbb{C} \;\middle|\; \tau_\mu(j) \leq |z| \leq 1 \right\} \longrightarrow \mathbb{C}$$

defined by

$$g_{j\mu}(z) \;=\; \begin{cases} \Phi^{-1}_{\nu_1(j)} \circ \phi_j(z, \frac{t_j}{z}) \;, & \mu = 1 \\ \Phi^{-1}_{\nu_2(j)} \circ \phi_j(\frac{t_j}{z}, z) \;, & \mu = 2 \end{cases}$$

satisfies

$$\begin{aligned} \left| g_{j\mu}(4\tau_\mu(j) e^{i\theta}) \; - \; s_\mu(j) \right| \; &< \; r_\mu(j) \\ \left| g_{j\mu}(e^{i\theta}/4) \; - \; s_\mu(j) \right| \; &> \; R_\mu(j)/4 \\ \left| g_{j\mu}(e^{i\theta}/2) \; - \; s_\mu(j) \right| \; &< \; R_\mu(j) \\ R_\mu(j) \; &< \; \left| g_{j\mu}(e^{i\theta}) \; - \; s_\mu(j) \right| \end{aligned}$$

for all $0 \leq \theta \leq 2\pi$.

(GH4) (Glueing in the compact piece)

$$\partial X^{\text{com}} \;=\; \Phi_1(\partial K_1) \cup \cdots \cup \Phi_m(\partial K_m)$$

Furthermore $A_1, B_1, \cdots, A_g, B_g$ is the image of a canonical homology basis of X^{com} under the map

$$H_1(X^{\text{com}}, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$

induced by inclusion.

(GH5) (Estimates on the Glueing Maps)

(i) For each $j \geq g+1$ and $\mu = 1,2$

$$R_\mu(j) \;<\; \tfrac{1}{4} \min_{\substack{s \in S_{\nu_\mu(j)} \\ s \neq s_\mu(j)}} |s - s_\mu(j)|$$

$$R_\mu(j) \;<\; \tfrac{1}{4}\mathrm{dist}\big(s_\mu(j), K_{\nu_\mu(j)}\big)$$

(ii) There are $0 < \delta < d$ such that

$$\sum_{j,\mu} \frac{1}{|s_\mu(j)|^{d-4\delta-2}} < \infty$$

and such that, for all $j \geq g+1$ and $\mu = 1,2$

$$r_\mu(j) < \frac{1}{|s_\mu(j)|^d} \qquad R_\mu(j) > \frac{1}{|s_\mu(j)|^\delta}$$

$$|s_1(j) - s_2(j)| > \frac{1}{|s_\mu(j)|^\delta}$$

(iii) For all $j \geq g+1$

$$\big||s_1(j)| - |s_2(j)|\big| \leq \tfrac{1}{4} \min_{\mu=1,2} \min_{\substack{s \in S_{\nu_\mu(j)} \\ s \neq s_\mu(j)}} |s - s_\mu(j)|$$

For $\mu = 1,2$

$$\sum_j \frac{\big||s_1(j)| - |s_2(j)|\big|}{|s_\mu(j)|} < \infty$$

(iv) For $\mu = 1, 2$

$$\lim_{j\to\infty} \frac{\log |s_\mu(j)|}{|\log t_j|} = 0$$

(v) For $\mu = 1, 2$

$$\lim_{j\to\infty} \frac{R_\mu(j)}{\min\limits_{\substack{s\in S_{\nu_\mu(j)} \\ s\neq s_\mu(j)}} |s - s_\mu(j)|} \log |s_\mu(j)| = 0$$

(vi) For each $j \geq g+1$ and $\mu = 1, 2$ we define $\alpha_{j,\mu}(z)$ by

$$\alpha_{j,\mu}(z)dz = (g_{j,\mu})_*\Big(\tfrac{1}{2\pi i}\tfrac{dz_1}{z_1}\Big) - \tfrac{(-1)^{\mu+1}}{2\pi i}\tfrac{1}{z-s_\mu(j)}dz$$

We assume

$$\sup_{j,\mu} \Big\| \alpha_{j,\mu}(z)dz\big|_{\{z\in\mathbb{C}\,|\, r_\mu(j)<|z-s_\mu(j)|<R_\mu(j)\}} \Big\|_2 < \infty$$

and, for $\mu = 1, 2$

$$\lim_{j\to\infty} R_\mu(j) \sup_{|z-s_\mu(j)|=R_\mu(j)} |\alpha_{j,\mu}(z)| = 0$$

(GH6) (Distribution of s_ν)

For all $\nu = 1, \cdots, m$ such that

$$\#\{\ (j,\mu)\ \big|\ \nu_\mu(j) = \nu,\ \nu_1(j) \neq \nu_2(j)\ \} < \infty$$

that is, such that the sheet X_ν^{reg} is joined to other sheets by only finitely many handles, one has

$$\limsup_{\substack{j \to \infty \\ \nu_1(j) = \nu_2(j) = \nu}} |s_1(j) - s_2(j)| = \infty$$