Infinite Genus Riemann Surfaces

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Abstract This survey introduces a class of infinite genus Riemann surfaces, specified by means of a number of geometric axioms, to which the classical theory of compact Riemann surfaces up to and including the Torelli Theorem extends. The axioms are flexible enough to include a number of interesting examples, such as the heat curve. We discuss this example and its connection to the periodic Kadomcev-Petviashvilli equation.

[1] Research supported in part by the Natural Sciences and Engineering Research Council of Canada, the Forschungsinstitut für Mathematik, ETH Zürich and the Schweizerischer Nationalfonds zur Förderung der wissenschaftlichen Forschung.

This survey is intended to introduce a class of infinite genus Riemann surfaces to which the classical theory of compact Riemann surfaces up to and including the Torelli Theorem extends [FKT1,2,3]. The Torelli Theorem states that two Riemann surfaces having the same period matrices are biholomorphically equivalent. The class is specified by means of a number of geometric axioms, that we list in the Appendix. The axioms are not only restrictive enough to allow recovery of much of the classical theory, but also flexible enough to include a number of interesting examples. We start with an example, in fact the example which motivated this project, the heat curve $\mathcal{H}(q)$ with source q. It will be defined precisely in §I.

To explain the significance of this particular example, we briefly discuss its connection to the periodic Kadomcev-Petviashvilli (KP) equation. This partial differential equation is related to the Korteweg-de Vries (KdV) equation and, like KdV, originally arose in the study of shallow water waves. Let Γ be a lattice in $\mathbb{R}^2$ and q a function on $\mathbb{R}^2$ that is periodic with respect to Γ. The periodic Kadomcev-Petviashvilli equation refers to the initial value problem

$$\begin{aligned} u_{tx_2}(x_1,x_2;t) &= \left(3uu_{x_2} - \tfrac{1}{2}\,u_{x_2x_2x_2}\right)_{x_2} - \tfrac{3}{2}u_{x_1x_1} \\ u(x_1,x_2;0) &= q(x_1,x_2) \end{aligned}$$

The importance of heat curves in the analysis of KP arises from the facts that $\mathcal{H}\big(u(\cdot,\cdot\,;t)\big)$ is independent of t and that, at least formally, there is a formula

$$u(x_1,x_2;t) \;=\; -2\,\frac{\partial^2}{\partial x_2^2}\log\theta(\,\vec{U}x_2+\vec{V}x_1-\tfrac{1}{2}\,\vec{W}t+\vec{Z}\,)+c$$

expressing the solution in terms of the theta function (defined in §II) of the heat curve $\mathcal{H}(q)$. The formula implies that solutions of the initial value problem for the spatially periodic KP equation are almost periodic in time. It is rigorous and well-known [K, M] for finite zone potentials. Then (the normalization of) $\mathcal{H}(q)$ is of finite genus. However for generic q, $\mathcal{H}(q)$ is of infinite genus. For such a formula to be true in this case, it is necessary at the very least to be able to define a theta function on an infinite genus Riemann surface.

Conversely, KP is also extremely important in the theory of Riemann surfaces. Firstly, because a $g\times g$ matrix R is a period matrix (defined in §II) of some Riemann surface of genus g if and only if R is symmetric, has positive definite imaginary part and

$$-2\,\frac{\partial^2}{\partial x_2^2}\log\theta(\vec{U}x_2+\vec{V}x_1-\tfrac{1}{2}\vec{W}t+\vec{Z}\,;R)+c$$

with $\theta(\,\cdot\,;R)$ being the theta function determined by R, gives a solution of KP for some constants $\vec{U},\vec{V},\vec{W},\vec{Z},c$ [AdC,S]. Secondly, because, for each $g\geq 1$, the set of Riemann surfaces of genus g that are normalizations of heat curves $\mathcal{H}(q)$, with $q\in L^2(\mathbb{R}^2/\Gamma)$ for some rectangular lattice Γ, is dense in the moduli space of all Riemann surfaces of genus g [BEKT].

So first we describe $\mathcal{H}(q)$ as an example of an infinite genus Riemann surface satisfying axioms (GH1-6) of the Appendix. Other examples, like Fermi curves or spectral curves of one-dimensional periodic differential operators, are described in [FKT3]. Then we discuss the definition and some properties of theta functions for such surfaces.

§I Heat Curves

For concreteness fix $\Gamma=2\pi\mathbb{Z}^2$. Let $q\in C^\infty(\mathbb{R}^2/\Gamma)$. The heat operator with source q is $H(q)=\left(\frac{\partial}{\partial x_1}-\frac{\partial^2}{\partial x_2^2}\right)+q$. Because q is periodic with respect to Γ, $H(q)$ commutes with all the operators, $(T_\gamma\psi)(x)=\psi(x-\gamma)$, in the natural unitary representation of Γ on $L^2(\mathbb{R}^2)$. Hence $H(q)$ and all the T_γ's can be simultaneously diagonalized and the spectrum of $H(q)$ is encoded in

$$\mathcal{H}(q)=\left\{\ (\xi_1,\xi_2)\in\mathbb{C}^*\times\mathbb{C}^*\ \middle|\ \begin{aligned}&\exists\ \psi(x_1,x_2)\neq 0 \text{ obeying } H\psi=0\\ &\psi(x_1+2\pi,x_2)=\xi_1\psi(x_1,x_2)\\ &\psi(x_1,x_2+2\pi)=\xi_2\psi(x_1,x_2)\end{aligned}\right\}$$

The requirement $H\psi=0$ makes ψ an eigenfunction of H of eigenvalue zero. (Patience - we'll get to nonzero eigenvalues shortly.) The requirements $\psi(x_1+2\pi,x_2)=\xi_1\psi(x_1,x_2)$ and $\psi(x_1,x_2+2\pi)=\xi_2\psi(x_1,x_2)$ make ψ an eigenfunction for $T_{(2\pi,0)}$ and $T_{(0,2\pi)}$. It is convenient to replace these "twisted periodic" boundary conditions with true periodic boundary conditions by writing $\xi_1 = e^{2\pi i k_1}$, $\xi_2 = e^{2\pi i k_2}$ and studying

$$\widehat{\mathcal{H}}(q) = \Big\{ \ k \in \mathbb{C}^2 \ \Big| \ \exists\ \phi = e^{-i<k,x>}\psi \neq 0 \text{ obeying } H_k(q)\phi = 0 \\ \phi(x_1 + 2\pi, x_2) = \phi(x_1, x_2) \\ \phi(x_1, x_2 + 2\pi) = \phi(x_1, x_2) \ \Big\}$$

where

$$\begin{aligned} H_k(q) &= e^{-i<k,x>}\left(\tfrac{\partial}{\partial x_1} - \tfrac{\partial^2}{\partial x_2^2} + q\right) e^{i<k,x>} \\ &= \tfrac{\partial}{\partial x_1} - 2ik_2 \tfrac{\partial}{\partial x_2} - \tfrac{\partial^2}{\partial x_2^2} + ik_1 + k_2^2 + q \\ &= i\left(-i\tfrac{\partial}{\partial x_1} + k_1\right) + \left(-i\tfrac{\partial}{\partial x_2} + k_2\right)^2 + q \end{aligned}$$

Even though we only look for eigenfunctions of eigenvalue zero of $H_k(q)$, with periodic boundary conditions, we in fact determine the full spectrum of $H(q)$ because λ is in the spectrum of $H(q)$ if and only if

$$\widehat{\mathcal{H}}(q - \lambda) = \left\{ \ k \in \mathbb{C}^2 \ \middle| \ (k_1 + i\lambda, k_2) \in \widehat{\mathcal{H}}(q) \ \right\} = \widehat{\mathcal{H}}(q) - (i\lambda, 0)$$

has nonempty intersection with $\mathbb{R}^2$.

To get an idea of what $\widehat{\mathcal{H}}(q)$ looks like, first set $q = 0$. Then we, of course, observe that $\left\{ \ e^{i<x,b>} \ \middle| \ b \in \Gamma^\# = \mathbb{Z}^2 \ \right\}$ is a basis of $L^2(\mathbb{R}^2/\Gamma)$ of eigenfunctions of H_k. Precisely,

$$H_k e^{i<x,b>} = P_b(k) e^{i<x,b>}$$
$$\text{where } P_b(k) = i(k_1 + b_1) + (k_2 + b_2)^2$$

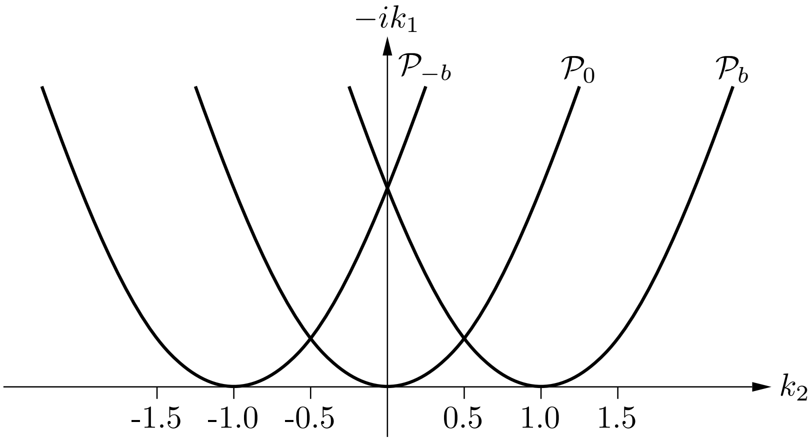
and

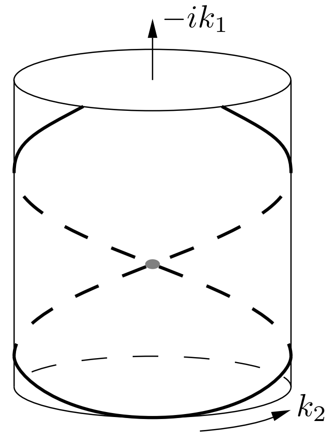
$$\widehat{\mathcal{H}}(q = 0) = \left\{ \ k \in \mathbb{C}^2 \ \middle| \ \exists\ b \in \Gamma^\# \text{ such that } P_b(k) = 0 \ \right\}$$

is the union $\bigcup_{b \in \Gamma^\#} \mathcal{P}_b$ of parabolas

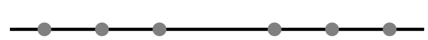
$$\mathcal{P}_b = \left\{ \ k \in \mathbb{C}^2 \ \middle| \ -ik_1 = (k_2 + b_2)^2 + ib_1 \ \right\}$$

On the left below is a sketch of the set of $(k_1, k_2) \in \widehat{\mathcal{H}}(q = 0)$ for which both ik_1 and k_2 are real. Recall that points (k_1, k_2) in $\widehat{\mathcal{H}}$ that differ by elements of $\Gamma^\# = \mathbb{Z}^2$ correspond



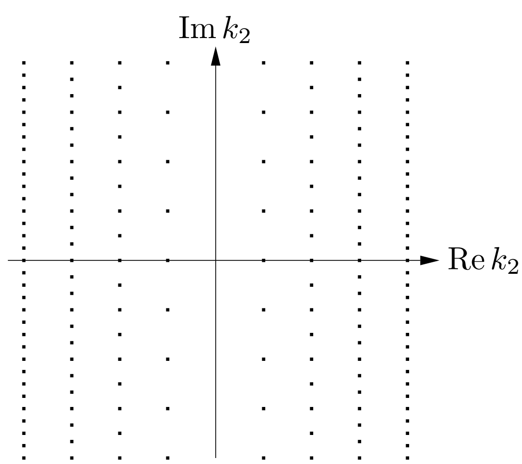


to the same point (ξ_1, ξ_2) in $\mathcal{H}$. So, in the sketch on the left, we should identify the lines $k_2 = -1/2$ and $k_2 = 1/2$ to get a single "parabolic vine" climbing up the outside of a cylinder, as illustrated by the figure on the right above. This vine intersects itself twice on each cycle of the cylinder – once on the front half of the cylinder and once on the back half. So viewed as a manifold, the vine is just $\mathbb{R}$ with pairs of points that correspond to



self-intersections of the vine identified. We can use k_2 as a coordinate on this manifold and then the pairs of identified points are $k_2 = \pm\frac{n}{2},\ n = 1, 2, 3\cdots$.

So far we have only considered k_2 real. The full $\mathcal{H}(q = 0)$ is just $\mathbb{C}$, with k_2 as a coordinate, provided we identify $\frac{n}{2} + i\frac{m}{n}$ and $-\frac{n}{2} + i\frac{m}{n}$ for all $(m, n) \in \Gamma^{\#} \setminus \{(0, 0)\} = \mathbb{Z}^2 \setminus \{(0, 0)\}$.



Now, we have to turn on the source term q. We can determine what $\widehat{\mathcal{H}}(q)$ looks like by studying the function $F_q(k)$ in

Theorem [FKT3, Theorems 15.8, 16.9] *There is an entire function F_q of finite order such that*

$$\widehat{\mathcal{H}}(q) = \left\{ \ k \in \mathbb{C}^2 \ \middle| \ F_q(k) = 0 \ \right\}$$

$\mathcal{H}(q)$ is a reduced and irreducible one dimensional complex analytic variety.

If $L^2(\mathbb{R}^2/\Gamma)$ were a finite dimensional vector space with $H_k(q)$ a matrix acting on it, we would just have $F_q(k) = \det\big(H_k(q)\big)$. Of course, this is not the case. But $H_k(q)$ does have purely discrete spectrum and its large eigenvalues are close to those of $H_k(0)$. So it easy to define a regularized determinant $\det_{\rm reg}\big(H_k(q)\big)$ and set $F_q(k) = \det_{\rm reg}\big(H_k(q)\big)$. In fact, there is a perturbation expansion that (rigorously) determines $F_q(k)$ to any desired degree of accuracy provided k is large enough (depending on the desired accuracy). This expansion shows that, for large k_2, $\widehat{\mathcal{H}}(q)$ looks a lot like $\widehat{\mathcal{H}}(0)$ except near the double points (i.e. self-intersection points) of $\widehat{\mathcal{H}}(0)$. Indeed, except near the double points, one can solve $F_q(k) = 0$ for k_1 as a function of k_2. The solution obeys

$$-ik_1 = k_2^2 + O\left(\tfrac{1}{1+|k_2|^2}\right)$$

Just as in the free case, we may use k_2 as a coordinate on $\mathcal{H}(q)$ away from the double points.

Near the double point $k_2 = \frac{n}{2} + i\frac{m}{n}$, $\widehat{\mathcal{H}}(q=0)$ is given by the equation

$$P_0(k)P_{-(m,n)}(k) = \left(ik_1 + k_2^2\right)\left(i(k_1 - m) + (k_2 - n)^2\right) = 0$$

In the same region, for general q and large (m,n), the equation becomes

$$\left(P_0(k) - D(k)_{1,1}\right)\left(P_{-(m,n)}(k) - D(k)_{2,2}\right) = \left(\hat{q}(m,n) - D(k)_{1,2}\right)\left(\hat{q}(-m,-n) - D(k)_{2,1}\right)$$

with the functions D obeying

$$|D(k)_{i,j}| \le \frac{\text{const}}{1+|k_2|^2} \qquad \text{for } i = j$$
$$|D(k)_{i,j}| \le \frac{\text{const}_{\,\alpha}}{1+(m^2+n^2)^\alpha} \qquad \text{for } i \ne j$$

For smooth q, the Fourier coefficients $\hat{q}(m,n)$ decay rapidly with (m,n) and α can be chosen arbitrarily large, so the equation is of the form

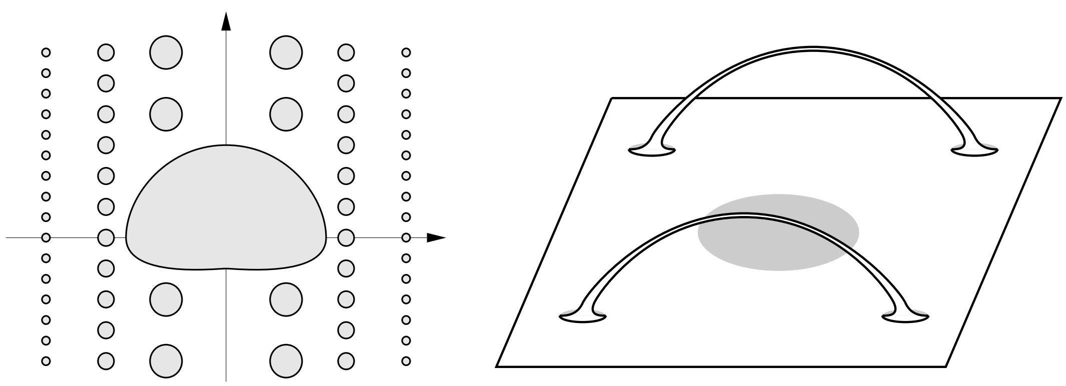
$$(P_0(k) - \text{small})\big(P_{-(m,n)}(k) - \text{small}\big) = \text{very small}$$

Under a suitable change of coordinates, which for $q = 0$ reduces to $z_1 = P_0(k),\ z_2 = P_{-(m,n)}(k)$, the equation becomes

$$z_1 z_2 = t_{(m,n)}$$

with the constant $t_{(m,n)}$ decaying rapidly with (m,n). When $t \neq 0$, and this is generically the case, the set $\left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; z_1 z_2 = t,\ |z_1|, |z_2| \le 1 \right\}$ or equivalently $\left\{ z_1 \in \mathbb{C} \;\middle|\; |t| \le |z_1| \le 1 \right\}$ is topologically a cylinder or handle. Generically, the free double point $z_1 z_2 = 0$, which is the union of two intersecting complex lines, opens up when q is nonzero to become a handle $z_1 z_2 = t$.

We now have the following picture of $\mathcal{H}(q)$. Take the complex k_2–plane $\mathbb{C}$. Cut out a compact simply connected neighbourhood of the origin. Glue in its place a compact Riemann surface with boundary. This is the part of $F_q(k) = 0$ with k_2 too small for us to be able to do an accurate analysis. Also cut out of the k_2 plane a small disk around each $\pm\frac{n}{2} + i\frac{m}{n}$ with $(m,n) \in \mathbb{Z},\ n > 0$. Glue in a handle joining each matching pair of disks.



This picture fulfills the qualitative aspects of the axioms referred to in the first paragraph. The axioms are given in the Appendix and are numbered (GH1-6). They require that the Riemann surface have a decomposition

$$X \;=\; X^{\rm com} \cup X^{\rm reg} \cup X^{\rm han}$$

into a compact, connected submanifold $X^{\rm com} \subset X$ with smooth boundary and genus $g \ge 0$, a finite number of open "regular sheets" $X_\nu^{\rm reg} \subset X$, $\nu = 1, \cdots, m$,

$$X^{\rm reg} = \bigcup_{\nu=1}^{m} X_\nu^{\rm reg}$$

and an infinite number of closed "handles" $Y_j \subset X$, $j \ge g+1$,

$$X^{\rm han} = \bigcup_{j \ge g+1} Y_j$$

There is a biholomorphism

$$\Phi_\nu : G_\nu \subset \mathbb{C} \to \overline{X_\nu^{\rm reg}}$$

between the $\nu^{\rm th}$ regular sheet and a copy of the complex plane with a bunch of holes cut in it and there is a biholomorphism

$$\phi_j : {\rm H}(t_j) \to Y_j$$

between the $j^{\rm th}$ handle and the model handle

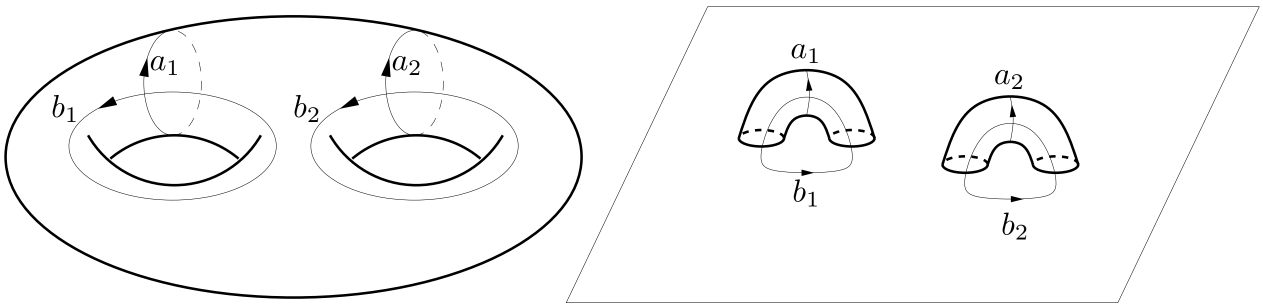
$${\rm H}(t_j) = \Big\{ (z_1, z_2) \in \mathbb{C}^2 \ \Big|\ z_1 z_2 = t_j \text{ and } |z_1|, |z_2| \le 1 \Big\}$$

There are restrictions on the density, sizing and positioning of handles. Heat curves obey all the restrictions.

Theorem [FKT3, Theorem 17.2] *Suppose that $q \in C^\infty(\mathbb{R}^2/\Gamma)$ and that $\mathcal{H}(q)$ is smooth. Then $\mathcal{H}(q)$ obeys (GH1-6).*

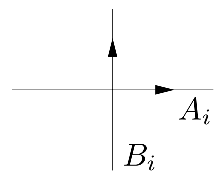
§II Theta Functions

We now discuss some properties of surfaces obeying (GH1-6), in parallel with a brief review of some of the classical theory [FK] of finite genus Riemann surfaces. Let X be a Riemann surface (one complex dimensional manifold) of genus g. For $g = 2$, X looks like the two-holed donut on the left. For our purposes, the noncompact Riemann surface of



genus two shown on the right is more appropriate. It is gotten by deleting one point from the donut and treating the deleted point as infinity. The topology of X is characterized by the existence of cycles $A_1, \cdots, A_g, B_1, \cdots, B_g$ with intersection numbers

$$A_i \times A_j = 0$$
$$A_i \times B_j = \delta_{i,j}$$
$$B_i \times B_j = 0$$



$A_i \times B_i = 1$

that provide a basis for the homology of X. If $C \times A_i = C \times B_i = 0$ for all i then $C \times D = 0$ for all closed curves D. There is a basis $\omega_1, \cdots, \omega_g$ for the vector space of holomorphic one forms that is dual to the A–cyles in the sense that

$$\int_{A_i} \omega_j = \delta_{i,j}$$

In the infinite genus case, one also has A– and B–cycles and a basis for the space of all L^2 holomorphic forms that is dual to the A–cycles, provided the surface is parabolic in the sense of Ahlfors and Nevanlinna [AS]. One traditional test for parabolicity is the existence of a harmonic exhaustion function, that is, a continuous, nonnegative, proper function on X that obeys $d * dh = 0$ outside a compact set. For example

$$h(z) = \begin{cases} 0 & \text{if } |z| \le 1 \\ \log |z| & \text{if } |z| \ge 1 \end{cases}$$

is a harmonic exhaustion function for $\mathbb{C}$. One can weaken this test [FKT1, Proposition 3.6] to the existence of an exhaustion function with $\int_X |d * dh| < \infty$. We use (GH5 i,ii,iii) to construct such an exhaustion function. The harmonic function $\log \left|\Phi_\nu^{-1}(x)\right|$, with Φ_ν being the coordinate map on the $\nu^{\rm th}$ sheet, is used on the part of the $\nu^{\rm th}$ sheet that does not overlap with a handle. The harmonic function $c_2(j)\log|z_1| + c_1(j)\log|z_2|$, with $x = \phi_j(z_1, z_2)$ being the coordinate map on the $j^{\rm th}$ handle, is used on the $j^{\rm th}$ handle, except where is overlaps the sheets. A nonharmonic interpolating function is used in the overlaps.

In general, parabolicity only ensures the existence of at least one homology basis for which there is a corresponding dual basis of ω_k's. It need not allow you to use any A_k's you like. In [FKT1, Theorem 3.8] we show that if the exhaustion function h obeys $\int_X |d*dh| < \infty$ and is consistent with a given homology basis $A_k, B_k,\ k \ge 1$ in the sense that, for all sufficiently large $t > 0$

- there is an $n \ge 1$ such that $A_1, B_1, \cdots, A_n, B_n$ are homologous in X to a canonical basis for $h^{-1}([0,t])$ and
- every component of $\partial h^{-1}([0,t])$ is homologous to a finite linear combination of A_k's and dividing cycles

then there is a unique basis $\omega_k,\ k \ge 1$ for the space of all square integrable holomorphic one forms that is dual to the given A_k's.

The integrals of the ω_j's over the B–cycles form the Riemann period matrix

$$R_{i,j} = \int_{B_j} \omega_i$$

which, in turn is used to define the theta function,

$$\theta(\vec{z}) = \sum_{\vec{n}\in\mathbb{Z}^g} e^{2\pi i<\vec{n},\vec{z}>} e^{\pi i<\vec{n},R\vec{n}>} : \mathbb{C}^g \to \mathbb{C}$$

on which much of the theory of finite genus Riemann surfaces depends. In the finite genus case, the convergence of the theta series for all $\vec{z} \in \mathbb{C}^g$ is an immediate consequence of the fact that the imaginary part of the Riemann period matrix is a strictly positive definite matrix. Even in the infinite genus case, one may use

$$\langle \vec{n}, (\mathrm{Im}\, R)\vec{n}\rangle = \Big\| \sum_{i\ge1} n_i\omega_i \Big\|^2_{L^2(X)} \ge \sum_{j\ge1} \Big\| \sum_{i\ge1} n_i\omega_i \Big\|^2_{L^2(Y_j)}$$

and a local computation in each handle Y_j to prove [FKT1, Proposition 4.4]

$$\sum_{i,j} n_i\,(\mathrm{Im}\, R_{i,j})\,n_j \ge \frac{1}{2\pi}\sum_j |\log t_j|^2 n_j^2 \tag{II.1}$$

Combining this with (GH2 iv) and the following Theorem gives the existence and usual periodicity properties of the theta function for all Riemann surfaces obeying (GH1-6).

Theorem [FKT1, Theorem 4.6, Proposition 4.12] *Suppose that R obeys* (II.1) *with the sequence $t_j \in (0,1)$, $j \geq 1$ obeying $\sum_{j\geq 1} t_j^\beta < \infty$ for some $0 < \beta < \frac{1}{2}$. Then*

$$\theta(\vec{z}) = \sum_{\substack{\vec{n}\in\mathbb{Z}^\infty \\ |\vec{n}|<\infty}} e^{2\pi i <\vec{n},\vec{z}>} e^{\pi i <\vec{n},R\vec{n}>}$$

converges absolutely and uniformly on bounded subsets of the Banach space

$$B = \left\{ \ \vec{z} \in \mathbb{C}^\infty \ \middle| \ \lim_{j\to\infty} \frac{z_j}{|\ln t_j|} = 0 \ \right\}$$
$$\|\vec{z}\|_B = \sup_{j\geq 1} \frac{z_j}{|\ln t_j|}$$

to an entire function that does not vanish identically. Furthermore, for all $\vec{n} \in \mathbb{Z}^\infty \cap B$ and all columns $\vec{R}_j$ of R with $\vec{R}_j \in B$

$$\theta(\vec{z} + \vec{n}) = \theta(\vec{z}) \tag{II.2}$$
$$\theta(\vec{z} + \vec{R}_j) = e^{-2\pi i (z_j + R_{jj}/2)} \theta(\vec{z}) \tag{II.3}$$

§III Zeroes of the Theta Function and Riemann's Vanishing Theorem

One of Riemann's many classical theorems is that, for each fixed $e \in \mathbb{C}^g$ and $x_0 \in X$,

$$\theta\left(e + \int_{x_0}^{x} \vec{\omega}\right)$$

either vanishes identically or has exactly g roots. The integral $\int_{x_0}^{x} \vec{\omega}$ does depend on the path from x_0 to x. But any two such paths differ by $\sum_{j=1}^{g}(n_j A_j + m_j B_j)$ with $\vec{n},\vec{m} \in \mathbb{Z}^g$ and, by the periodicity properties (II.2,3) of the theta function, $\theta(\vec{z}) = 0$ if and only if $\theta(\vec{z} + \vec{n} + \sum_{j=1}^{g} m_j \vec{R}_j) = 0$.

To generalize Riemann's result to our class of infinite genus Riemann surfaces we must first show that for any path joining x_1 to x_2 on X the infinite component vector

$$\int_{x_1}^{x_2} \vec{\omega} \ = \ \left(\int_{x_1}^{x_2} \omega_1\,, \int_{x_1}^{x_2} \omega_2\,, \cdots\right)$$

lies in the domain of definition of the theta function. This, and much of the further development of the theory, depend on bounds on the frame $\omega_1, \omega_2, \cdots$. Suppose that one end of the j^{th} handle is glued into the $\nu_1(j)^{\text{st}}$ sheet near $s_1(j)$ and that the other end of the j^{th} handle is glued into the $\nu_2(j)^{\text{nd}}$ sheet near $s_2(j)$. We prove [FKT2, Theorem 8.4] that, when $\nu_1(j) = \nu_2(j)$, the pull back $w_j^\nu(z)dz = \Phi_\nu^* \omega_j$ of ω_j to the ν^{th} sheet obeys

$$\begin{aligned}
&\left| w_j^\nu(z) - \frac{1}{2\pi i}\frac{1}{z - s_1(j)} + \frac{1}{2\pi i}\frac{1}{z - s_2(j)} \right| \le \frac{\text{const}}{1 + |z^2|} && \text{if } \nu = \nu_1(j) = \nu_2(j) \\
&\left| w_j^\nu(z) \right| \le \frac{\text{const}}{1 + |z^2|} && \text{if } \nu \ne \nu_1(j) = \nu_2(j)
\end{aligned}$$

On the other hand, when $\nu_1(j) \ne \nu_2(j)$,

$$\begin{aligned}
&\left| w_j^\nu(z) - \frac{1}{2\pi i}\frac{1}{z - s_1(j)} + \frac{1}{2\pi i}\frac{1}{z} \right| \le \frac{\text{const}}{1 + |z^2|} && \text{if } \nu = \nu_1(j) \\
&\left| w_j^\nu(z) - \frac{1}{2\pi i}\frac{1}{z} + \frac{1}{2\pi i}\frac{1}{z - s_2(j)} \right| \le \frac{\text{const}}{1 + |z^2|} && \text{if } \nu = \nu_2(j) \\
&\left| w_j^\nu(z) \right| \le \frac{\text{const}}{1 + |z^2|} && \text{if } \nu \ne \nu_1(j), \nu_2(j)
\end{aligned}$$

The const is independent of j. We also make a detailed investigation of the pull backs of ω_j to the handles. For example [FKT2, Proposition 8.16]

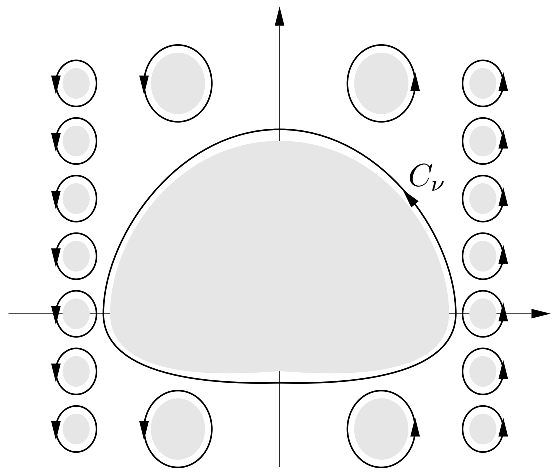
$$\left| \frac{\phi_j^* \omega_j(z)}{dz_1/z_1} - \frac{1}{2\pi i} \right| \le \frac{2}{5\pi}(|z_1| + |z_2|)$$

Here is a rough idea of how these bounds are proven. The ν^{th} sheet is biholomorphic to a complex plane from which a compact neighbourhood of the origin and an infinite set of other small holes have been cut. Draw a contour C_ν around the first hole and circles $|z - s| = r(s)$, $s \in S_\nu$ around the other small holes. By the Cauchy integral formula

$$w_j^\nu(z) = w_{j,\text{com}}^\nu(z) + \sum_{s \in S_\nu} w_{j,s}^\nu(z)$$

where

$$\begin{aligned}
w_{j,s}^\nu(z) &= -\frac{1}{2\pi i} \int_{|z-s|=r(s)} \frac{w_j^\nu(\zeta)}{\zeta - z} d\zeta \\
w_{j,\text{com}}^\nu(z) &= -\frac{1}{2\pi i} \int_{C_\nu} \frac{w_j^\nu(\zeta)}{\zeta - z} d\zeta
\end{aligned}$$



First, by writing

$$w^{\nu}_{j,s}(z) = -\frac{1}{2\pi r(s) i}\int_{r(s)}^{2r(s)}\left[\int_{|z-s|=r}\frac{w^{\nu}_j(\zeta)}{\zeta - z}d\zeta\right]dr$$

and using Cauchy-Schwarz, one may get an upper bound [FKT2, Propositions 8.5,8.12,8.14 and Lemma 8.9] on $w^{\nu}_{j,s}(z)$ for $|z-s|\ge 3r(s)$ in terms of the L^2 norm of ω_j restricted to the annulus $\Phi_\nu\big(\{r(s)\le |z-s|\le 2r(s)\big)$. To get a bound that decays like $\frac{1}{|z-s|^2}$ rather than $\frac{1}{|z-s|}$ one exploits the fact that $\int_{|z-s|=r} w^{\nu}_j(\zeta)\,d\zeta = 0$, unless the circle $|z-s| = r$ happens to be homologous to $\pm A_j$. If so, one works with $w^{\nu}_j(\zeta)\mp\frac{1}{2\pi i}\frac{1}{\zeta-s}$ instead of $w^{\nu}_j(\zeta)$. One also gets the analogous bound on $w^{\nu}_{j,\mathrm{com}}(z)$. Second, consider the handle Y_i. One end of it is glued into sheet $\nu_1(i)$ near the point $s_1(i)$ and the other end is glued into sheet $\nu_2(i)$ near the point $s_2(i)$. Denote by Y_i' the part of the handle Y_i bounded by $\Phi_{\nu_1(i)}\big(\{|z-s_1(i)| = R_1(i)\}\big)$ and $\Phi_{\nu_2(i)}\big(\{|z-s_2(i)| = R_2(i)\}\big)$. The radii $R_\mu(i)$ are chosen in (GH3,5) to be much larger than the corresponding radii $r_\mu(i)$. Using Stoke's Theorem and the holomorphicity of ω_j [FKT2, Corollary 8.7] one gets a bound on the L^2 norm of the restriction of ω_j to Y_i' in terms of the values of $w_j^{\nu_\mu(i)}$ on $\{|z-s_\mu(i)| = R_\mu(i)\}$, $\mu = 1,2$. Substituting the first family of bounds into the second family gives a system of inequalities relating the L^2 norms of ω_j restricted to the $Y_i's$. This family of inequalities may be "solved" to get inequalities on the L^2 norms themselves. Finally, bounds on the L^2 norms are turned into pointwise bounds on the sheets by the "first" bounds above and on the handles by a similar method [FKT2, Lemma 8.8].

The above bounds on $\vec{\omega}$ imply that for any path joining x_1 to x_2 on X, the integral $\int_{x_1}^{x_2} \vec{\omega}$ is in the Banach space B and that this remains the case in the limit as x_1 tends to infinity along a reasonable path. In the event that X has m sheets we can choose m such paths each approaching infinity on a different sheet. Call the limits $\int_{\infty_\nu}^{x_2} \vec{\omega}$, $1 \le \nu \le m$. The precise statement that $\theta\left(e + \int_{\infty_1}^{x} \vec{\omega}\right)$ has exactly "genus(X)" roots is

Theorem [FKT2, Theorem 9.11] *Let $e \in B$ be such that $\theta(e) \neq 0$ and $\theta\left(e + \int_{\infty_1}^{\infty_\nu} \vec{\omega}\right) \neq 0$ for all $1 < \nu \le m$. Then, there is a compact submanifold Y with boundary such that the multivalued, holomorphic function*

$$\theta\left(e + \int_{\infty_1}^{x} \vec{\omega}\right)$$

has

(i) exactly genus(Y) *roots in Y*

(ii) exactly one root in each in each handle of X outside of Y

(iii) and no other roots.

That $\theta\left(e + \int_{\infty_1}^{x} \vec{\omega}\right)$ has no zeroes near ∞_ν, except in handles, is a consequence of the facts that $\theta\left(e + \int_{\infty_1}^{\infty_\nu} \vec{\omega}\right) \neq 0$ by hypothesis, that $\left\|\int_{x}^{\infty_\nu} \vec{\omega}\right\|_B$ is small for all sufficiently large x in the ν^{th} sheet and that θ is continuous in the norm of the Banach space B. The proof that there is exactly one zero in each, sufficiently far out, handle is based on the argument principle and the computation

$$\begin{aligned}\int_{A_j B_j A_j^{-1} B_j^{-1}} d\log\theta\left(e + \int_{\infty_1}^{x} \vec{\omega}\right) = &\int_{A_j} d\log\theta\left(e + \int_{\infty_1}^{x} \vec{\omega}\right) + \int_{B_j} d\log\theta\left(e + \vec{e}_j + \int_{\infty_1}^{x} \vec{\omega}\right) \\ &- \int_{A_j} d\log\theta\left(e + \vec{R}_j + \int_{\infty_1}^{x} \vec{\omega}\right) - \int_{B_j} d\log\theta\left(e + \int_{\infty_1}^{x} \vec{\omega}\right) \\ =&2\pi i \int_{A_j} d\left(e_j + \int_{\infty_1}^{x} \omega_j + \tfrac{1}{2} R_{jj}\right) \\ =&2\pi i\end{aligned}$$

where $\vec{e}_j$ is the vector whose k^{th} component is δ_{ik} and $\vec{R}_j$ is the j^{th} column of the Riemann period matrix. The periodicity properties of the theta function are used twice in this computation. We also used

$$\int_{B_j^{-1}} d\log\theta\left(e + \vec{R}_j + \int_{\infty_1}^{x} \vec{\omega}\right) = -\int_{B_j} d\log\theta\left(e + \int_{\infty_1}^{x} \vec{\omega}\right)$$

and an analogous formula for A_j^{-1}.

In preparation for Riemann's vanishing theorem, we introduce the notion of a divisor of degree "genus(X)" on the universal cover of X. This is done by fixing an auxiliary point $\hat{e} \in B$ with $\theta(\hat{e}) \neq 0$ and comparing sequences of points to the "genus(X)" many roots $\hat{x}_1$, $\hat{x}_2 \cdots$ of $\theta\left(\hat{e} + \int_{\infty_1}^{x} \vec{\omega}\right) = 0$. Precisely, let $\pi : \tilde{X} \to X$ be the universal cover of X and choose $\tilde{x}_j \in \pi^{-1}(\hat{x}_j)$. A sequence y_j, $j \geq 1$, on $\tilde{X}$ represents a divisor of degree "genus(X)" if eventually y_j lies in the same component of $\pi^{-1}(Y_j)$ as $\tilde{x}_j$ and the vector

$$\left(\int_{\tilde{x}_1}^{y_1} \omega_1 \,,\, \int_{\tilde{x}_2}^{y_2} \omega_2 \,,\, \cdots \right)$$

lies in B. The space $W^{(0)}$ of all these sequences is given the structure of a complex Banach manifold modelled on B. The quotient $S^{(0)}$ of $W^{(0)}$ by the group of all finite permutations is the manifold of divisors of degree "genus(X)". The construction is independent of the auxiliary point $\hat{e}$. We similarly construct Banach manifolds $S^{(-n)}$ of divisors of index n, that is, of degree "genus$(X) - n$", by deleting the first n components in a sequence $y_1, y_2, \cdots$ belonging to $W^{(0)}$.

Fix $\hat{e}$ as above. The right hand side of

$$(y_1\,,\, y_2\,, \cdots) \;\mapsto\; \hat{e} - \sum_{i \geq 1} \int_{\tilde{x}_i}^{y_i} \vec{\omega}$$

is invariant under permutations of the y_i's and induces the analog

$$f^{(0)} \,:\, S^{(0)} \;\longrightarrow\; B$$

of the Abel-Jacobi map. The map $f^{(0)}$ is holomorphic [FKT2, Proposition 10.1] and indeed is a biholomorphism between $f^{(0)\,-1}\,(B \setminus \Theta)$ and $B \setminus \Theta$ where

$$\Theta \;=\; \left\{\, e \in B \,\middle|\, \theta(e) = 0 \,\right\}$$

is the theta divisor of X.

Similarly, the map

$$f^{(-1)} \,:\, S^{(-1)} \;\longrightarrow\; B$$

is induced by

$$(y_2\,,\, y_3\,, \cdots) \;\mapsto\; \hat{e} \,-\, \int_{\tilde{x}_1}^{\infty} \vec{\omega} \,-\, \sum_{i \geq 2} \int_{\tilde{x}_i}^{y_i} \vec{\omega}$$

The analogue of the Riemann vanishing theorem is

Theorem [FKT2, Theorem 10.4]

$$f^{(-1)}\left(S^{(-1)}\right) \subset \Theta$$

and

$$\left\{ e \in \Theta \;\middle|\; \theta\left(e - \textstyle\int_{\infty}^{x} \vec{\omega}\right) \neq 0 \text{ for some } x \text{ in } X \right\} \subset f^{(-1)}\left(S^{(-1)}\right)$$

In contrast to the case of compact Riemann surfaces, one can construct zeroes of the theta function that are not in the range of $f^{(-1)}$ by taking limits of $f^{(-1)}([y_1, y_2, \cdots])$ as some of the y_i's tend to infinity. The set

$$\left\{ e \in \Theta \;\middle|\; \theta\left(e - \textstyle\int_{\infty}^{x} \vec{\omega}\right) = 0 \text{ for all } x \text{ in } X \right\}$$

is stratified and studied in [FKT2, Theorem 11.1].

The Torelli theorem for compact Riemann surfaces states that two Riemann surfaces that have the same period matrices are biholomorphically equivalent. We prove

Theorem [FKT2, Theorem 13.1] *Let X and X' be two Riemann surfaces that fulfill the hypotheses (GH1-6) of the Appendix. Denote their canonical homology bases by $A_1, B_1, A_2, B_2, \cdots$ and $A_1', B_1', A_2', B_2', \cdots$. Let R_{ij} and R_{ij}' be the associated period matrices. If $R_{ij} = R_{ij}'$ for all $i, j \in \mathbb{Z}$ then there is a biholomorphic map $\psi : X \to X'$ and $\epsilon \in \{\pm 1\}$ such that*

$$\psi_*(A_j) = \epsilon A_j' \qquad \psi_*(B_j) = \epsilon B_j'$$

The proof mimics the argument of Andreotti [An,GH] for the compact case. We look at how the tangent space $T_e\Theta$ varies as e moves in directions $v \in T_e\Theta$. In particular, we look for directions v such that $T_e\Theta$ is stationary, equivalently such that $\mathbb{C}\nabla\theta(e)$ is stationary. In other words, we investigate the ramification locus of the Gauss map on the theta divisor. Stationarity is given by the conditions

$$\nabla\theta(e) \neq 0,\ \nabla\theta(e) \cdot v = 0,\ \frac{d}{d\lambda} \nabla\theta(e + \lambda v)\Big|_{\lambda=0} \in \mathbb{C}\nabla\theta(e) \tag{III.1}$$

For generic $e = f^{(-1)}(y)$ we find, in [FKT2, Proposition 11.8], necessary and sufficient conditions that the set of v's satisfying III.1 is of dimension 1 and determine precisely what the set is. The conditions are that the form $\omega_e(z) = \sum_{k\geq 1} \nabla\theta(e)_k \omega_k(z)$ have a zero of precisely the right order, namely $\sharp\{i \mid \pi(y_i) = \pi(y_j)\}$, at each y_j $j \geq 2$ with one exception, say $y_j = x$. And that $\omega_e(z)$ have one excess zero, in other words a zero of order $\sharp\{i \mid \pi(y_i) = x\} + 1$, at $z = x$. Then the set of stationary directions $v \in T_e\Theta$ is precisely $\mathbb{C}\vec{\omega}(x)$.

Note that the conditions (III.1) are stated purely in terms of the function θ. They do not involve the Riemann surface that gave rise to θ. On the other hand the statement "the set of stationary directions $v \in T_e\Theta$ is precisely $\mathbb{C}\vec{\omega}(x)$" does involve the Riemann surface and indeed assigns, in the nonhyperelliptic case, a unique point $x \in X$ to the given $e \in \Theta$ [FKT1, Proposition 3.26]. In [FKT2, Proposition 11.10] we find, in the nonhyperelliptic case, a set $E \subset \Theta$ of such e's, that is dense in a subset of codimension 1 in Θ. Furthermore, for x in a dense subset of X, the set $\left\{\, e \in E \;\middle|\; e \text{ is paired with } x \,\right\}$ is, roughly speaking, of codimesion 2 in Θ. The pairing of points e in E with points $x \in X$ is the principal ingredient in the proof of the Torelli Theorem for the nonhyperelliptic case.

For hyperelliptic Riemann surfaces, the map $x \in X \mapsto \mathbb{C}\vec{\omega}(x)$ is of degree two. Except for a discrete set of points x, $\#\left\{\, x' \in X \;\middle|\; \vec{\omega}(x') \parallel \vec{\omega}(x) \,\right\} = 2$. At the exceptional points, called Weierstrass points, $\#\left\{\, x' \in X \;\middle|\; \vec{\omega}(x') \parallel \vec{\omega}(x) \,\right\} = 1$. For each Weierstrass point $b \in X$, we find a set $H^{(b)}$ which is dense in a subset of codimension 1 in Θ with every point $e \in H^{(b)}$ paired, as above, with b.

Using these observations it is possible to recover the Riemann surface X from Θ, which in turn is completely determined by the period matrix of X.

Appendix. The Geometric Hypotheses

In this Appendix we list the axioms used in [FKT1-3]. They concern a class of marked Riemann surfaces $(X; A_1, B_1, \cdots)$ that are "asymptotic to" a finite number of complex planes $\mathbb{C}$ joined by infinitely many handles. Here, X is a Riemann surface and $A_1, B_1, \cdots$ is a canonical homology basis for X. To be precise, the notation

$$X \;=\; X^{\mathrm{com}} \cup X^{\mathrm{reg}} \cup X^{\mathrm{han}}$$

denotes a marked Riemann surface $(X;\, A_1, B_1, \cdots)$ with a decomposition into a compact, connected submanifold $X^{\mathrm{com}} \subset X$ with smooth boundary and genus $g \geq 0$, a finite number of open "regular pieces" $X^{\mathrm{reg}}_\nu \subset X$, $\nu = 1, \cdots, m$,

$$X^{\mathrm{reg}} \;=\; \bigcup_{\nu=1}^{m} X^{\mathrm{reg}}_\nu$$

and an infinite number of closed "handles" $Y_j \subset X$, $j \geq g+1$,

$$X^{\mathrm{han}} \;=\; \bigcup_{j \geq g+1} Y_j$$

with $X^{\mathrm{com}} \cap \left(X^{\mathrm{reg}} \cup X^{\mathrm{han}} \right) = \emptyset$. Each handle will be biholomorphic to the model handle

$$\mathrm{H}(t) \;=\; \left\{ \, (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; z_1 z_2 = t \text{ and } |z_1|, |z_2| \leq 1 \, \right\}$$

for some $0 < t < \frac{1}{2}$.

(GH1) (Regular pieces)

(i) For all $1 \leq \mu \neq \nu \leq m$, $\overline{X^{\mathrm{reg}}_\mu} \cap \overline{X^{\mathrm{reg}}_\nu} \;=\; \emptyset$.

(ii) For each $1 \leq \nu \leq m$ there is a compact simply connected neighborhood $K_\nu \subset \mathbb{C}$ of 0 with smooth boundary. There is also an infinite discrete subset $S_\nu \subset \mathbb{C}$ and, for each $s \in S_\nu$, there is a compact, simply connected neighborhood $D_\nu(s)$ with smooth boundary $\partial D_\nu(s)$ such that

$$D_\nu(s) \cap D_\nu(s') \;=\; \emptyset \qquad \text{for all } s, s' \in S_\nu \text{ with } s \neq s'$$

$$K_\nu \cap D_\nu(s) \;=\; \emptyset \qquad \text{for all } s \in S_\nu$$

(iii) Set $G_\nu \;=\; \mathbb{C} \setminus \Big(\mathrm{int}\, K_\nu \cup \bigcup_{s \in S_\nu} \mathrm{int}\, D_\nu(s) \Big)$. There is a biholmorphic map Φ_ν ,

$$\Phi_\nu \,:\, G_\nu \;\to\; \overline{X^{\mathrm{reg}}_\nu}$$

between G_ν and $\overline{X^{\mathrm{reg}}_\nu}$.

Remark. Informally, the closure of the regular piece $X_\nu^{\rm reg}$, $\nu = 1, \cdots, m$, is biholomorphic to a copy of $\mathbb{C}$ minus an open, simply connected neighborhood around each point of S_ν and an additional compact set K_ν . One end of a closed cylindrical handle will be glued to a closed "annular" region surrounding $D_\nu(s)$ in G_ν . One connected component of $\partial X^{\rm com}$ will be glued to ∂K_ν .

(GH2) (Handles)

(i) For all $i \neq j$ with $i, j \geq g+1$, $Y_i \cap Y_j \; = \; \emptyset$.

(ii) For each $j \geq g+1$ there is a $0 < t_j < \frac{1}{2}$ and a biholomorphic map ϕ_j

$$\phi_j \; : \; {\rm H}(t_j) \; \to \; Y_j$$

between the model handle ${\rm H}(t_j)$ and Y_j .

(iii) For all $j \geq g+1$, A_j is the homology class represented by the oriented loop

$$\phi_j\Big(\big\{ \, (\sqrt{t_j}\, e^{i\theta}, \sqrt{t_j}\, e^{-i\theta}) \, \big| \, 0 \leq \theta \leq 2\pi \, \big\}\Big)$$

(iv) For every $\beta > 0$

$$\sum_{j \geq g+1} t_j^\beta < \infty$$

(GH3) (Glueing handles and regular pieces together)

(i) For each $j \geq g+1$ the intersection $Y_j \cap X^{\rm reg}$ consists of two components Y_{j1}, Y_{j2}:

$$Y_j \cap X^{\rm reg} = Y_{j1} \cup Y_{j2}$$

For each pair (j, μ) with $j \geq g+1$ and $\mu = 1, 2$ there is a radius $\tau_\mu(j) \in \big(\sqrt{t_j}\,,\, 1\big)$ and a sheet number $\nu_\mu(j) \in \{1, \ldots, m\}$ such that

$$Y_{j\mu} \;\; = \;\; \phi_j\left(\big\{(z_1, z_2) \in H(t_j) \;\big|\; \tau_\mu(j) < |z_\mu| \leq 1\big\}\right) \subset \; X^{\rm reg}_{\nu_\mu(j)}$$

There is a bijective map

$$(j, \mu) \; \in \; \big\{ \, j \in \mathbb{Z} \; \big| \; j \geq g+1 \, \big\} \times \{1, 2\} \;\; \mapsto \;\; s_\mu(j) \; \in \; \bigsqcup_{\nu=1}^{m} S_\nu \quad \text{(disjoint union)}$$

such that

$$\phi_j\left(\{(z_1,z_2)\in H(t_j)\,\big|\,|z_\mu|=\tau_\mu(j)\}\right) \;=\; \Phi_{\nu_\mu(j)}\left(\partial D_{\nu_\mu(j)}(s_\mu(j))\right)$$

(ii) For each $j \ge g+1$ and $\mu = 1,2$ there are

$$R_\mu(j) \;>\; 4r_\mu(j) \;>\; 0$$

such that the biholomorphic map $g_{j\mu} : \mathcal{A}_{j\mu} \;=\; \left\{z\in\mathbb{C}\;\big|\;\tau_\mu(j)\le|z|\le 1\right\} \longrightarrow \mathbb{C}$ defined by

$$g_{j\mu}(z) \;=\; \begin{cases} \Phi^{-1}_{\nu_1(j)}\circ\phi_j(z,\frac{t_j}{z})\;, & \mu=1 \\ \Phi^{-1}_{\nu_2(j)}\circ\phi_j(\frac{t_j}{z},z)\;, & \mu=2 \end{cases}$$

satisfies

$$\left|g_{j\mu}(4\tau_\mu(j)e^{i\theta}) \;-\; s_\mu(j)\right| \;<\; r_\mu(j)$$

$$\left|g_{j\mu}(e^{i\theta}/4) \;-\; s_{\mu(j)}\right| \;>\; R_{\mu(j)}/4$$

$$\left|g_{j\mu}(e^{i\theta}/2) \;-\; s_{\mu(j)}\right| \;<\; R_{\mu(j)} \;<\; \left|g_{j\mu}(e^{i\theta}) \;-\; s_{\mu(j)}\right|$$

for all $0\le\theta\le 2\pi$.

Remark. The parameters that control the overlap $Y_{j1}\cup Y_{j2}$ between X^{reg} and the handle Y_j are introduced in hypothesis (GH3). The map $s_\mu(j)$ specifies that the end of the handle $\mathrm{H}(t_j)$ containing $\{\,(z_1,z_2)\in\mathrm{H}(t_j)\,\big|\,|z_\mu|=1\,\}$ is glued to the annular region $\Phi^{-1}_{\nu_\mu(j)}(Y_{j\mu})$ in $G_{\nu_\mu(j)}$. The component $Y_{j\mu}$ of the overlap is specified in two charts; in the plane region $G_{\nu_\mu(j)}$, and on the model handle $H(t_j)$. The map $g_{j,\mu}$ describes how $Y_{j\mu}$ is glued to $G_{\nu_\mu(j)}$.

The preimage $\phi_j^{-1}(Y_{j\mu})$ in $H(t_j)$ of the overlap $Y_{j\mu}$ is

$$\left\{\,(z_1,z_2)\in H(t_j)\,\big|\,\tau_\mu(j)<|z_\mu|\le 1\,\right\}$$

We imagine that $\tau_\mu(j)$ is relatively close to the radius of the waist $\sqrt{t_j}$ so that the overlap on $H(t_j)$ is large.

The preimage $\Phi^{-1}_{\nu_\mu(j)}\big(Y_{j\mu}\big)$ of the overlap $Y_{j\mu}$ in the other chart $G_{\nu_\mu(j)}$ is the annular plane region surrounding $s_\mu(j)$ whose inner boundary is $\partial D_{\nu_\mu(j)}(s_\mu(j))$ and whose outer boundary is $\Phi^{-1}_{\nu_\mu(j)} \circ \phi_j\left(\left\{ (z_1,z_2) \in H(t_j) \,\middle|\, |z_\mu| = 1 \right\}\right)$. It strictly contains $\left\{ z \in \mathbb{C} \,\middle|\, r_\mu(j) \le |z - s_\mu(j)| \le R_\mu(j) \right\}$. We will assume, (GH5)(ii), that $r_\mu(j)$ and $\frac{r_\mu(j)}{R_\mu(j)}$ are both asymptotically small. Consequently, the "holes" $D_\nu(s)$, $s \in S_\nu$, in G_ν are also asymptotically small and the overlap is asymptotically big.

Passing from the chart $G_{\nu_\mu(j)}$ to $H(t_j)$, the image of the circle $|z - s_\mu(j)| = R_\mu(j)$ lies in $\left\{ (z_1,z_2) \in H(t_j) \,\middle|\, \frac{1}{2} \le |z_\mu| \le 1 \right\}$ and the image of the circle $|z - s_\mu(j)| = r_\mu(j)$ lies outside $\left\{ (z_1,z_2) \in H(t_j) \,\middle|\, |z_\mu| = 4\tau_\mu(j) \right\}$

If $\partial D_{\nu_\mu(j)}$ is counterclockwise oriented, then, by construction, $\Phi\left(\partial D_{\nu_\mu(j)}\right)$ is homologous to $(-1)^{\mu+1} A_j$.

(GH4) (Glueing in the compact piece)

$$\partial X^{\mathrm{com}} \;=\; \Phi_1(\partial K_1) \cup \cdots \cup \Phi_m(\partial K_m)$$

Furthermore $A_1, B_1, \cdots, A_g, B_g$ is the image of a canonical homology basis of X^{com} under the map $H_1(X^{\mathrm{com}}, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ induced by inclusion.

(GH5) (Estimates on the Glueing Maps)

(i) For each $j \ge g+1$ and $\mu = 1,2$

$$R_\mu(j) \;<\; \tfrac{1}{4} \min_{\substack{s \in S_{\nu_\mu(j)} \\ s \ne s_\mu(j)}} |s - s_\mu(j)|$$

$$R_\mu(j) \;<\; \tfrac{1}{4} \mathrm{dist}\big(s_\mu(j), K_{\nu_\mu(j)}\big)$$

(ii) There are $0 < \delta < d$ such that

$$\sum_{j,\mu} \frac{1}{|s_\mu(j)|^{d-4\delta-2}} < \infty$$

and such that, for all $j \ge g+1$ and $\mu = 1,2$

$$r_\mu(j) < \frac{1}{|s_\mu(j)|^{d}} \qquad R_\mu(j) > \frac{1}{|s_\mu(j)|^{\delta}}$$

$$|s_1(j) - s_2(j)| > \frac{1}{|s_\mu(j)|^{\delta}}$$

(iii) For all $j \geq g+1$

$$\big||s_1(j)| - |s_2(j)|\big| \leq \tfrac{1}{4} \min_{\mu=1,2} \min_{\substack{s \in S_{\nu_\mu(j)} \\ s \neq s_\mu(j)}} |s - s_\mu(j)|$$

For $\mu = 1,2$

$$\sum_j \frac{\big||s_1(j)| - |s_2(j)|\big|}{|s_\mu(j)|} < \infty$$

(iv) For $\mu = 1,2$

$$\lim_{j\to\infty} \frac{\log |s_\mu(j)|}{|\log t_j|} = 0$$

(v) For $\mu = 1,2$

$$\lim_{j\to\infty} \frac{R_\mu(j)}{\min\limits_{\substack{s \in S_{\nu_\mu(j)} \\ s \neq s_\mu(j)}} |s - s_\mu(j)|} \log |s_\mu(j)| = 0$$

(vi) For each $j \geq g+1$ and $\mu = 1,2$ we define $\alpha_{j,\mu}(z)$ by

$$\alpha_{j,\mu}(z)dz = (g_{j,\mu})_* \left(\tfrac{1}{2\pi i}\tfrac{dz_1}{z_1}\right) - \tfrac{(-1)^{\mu+1}}{2\pi i}\tfrac{1}{z - s_\mu(j)}dz$$

We assume

$$\sup_{j,\mu} \left\| \alpha_{j,\mu}(z)dz\big|_{\{z\in\mathbb{C}\,|\, r_\mu(j) < |z - s_\mu(j)| < R_\mu(j)\}} \right\|_2 < \infty$$

and, for $\mu = 1,2$

$$\lim_{j\to\infty} R_\mu(j) \sup_{|z - s_\mu(j)| = R_\mu(j)} |\alpha_{j,\mu}(z)| = 0$$

Remark. Morally, (GH5)(i) says that the holes $D_\nu(s)$ are separated from each other and from the compact region K_ν. The first and last parts of (GH5)(ii) and (GH5)(v) bound their density at infinity. The second part of (GH5)(ii) implies that $r_\mu(j)$ and $\frac{r_\mu(j)}{R_\mu(j)}$ are both asymptotically small. The condition (GH5)(iii) forces the two ends of a handle to be attached at approximately the same distance from the origin on the regular pieces. (GH5)(iv) relates the size of the waist of the handle Y_j to the distance from the origin at which it is attached to the regular piece. Finally, (GH)(vi) measures the derivative of the glueing map $g_{j,\mu}$ by pulling back the holomorphic form $\frac{dz_1}{z_1}$ on $H(t_j)$ and comparing it to the meromorphic form $\frac{1}{z-s_\mu(j)}dz$. The intuition is that both of these forms have the same A_j period and should be the leading part of ω_j.

(GH6) (Distribution of s_ν)

For all $\nu = 1, \cdots, m$ such that

$$\#\{\ (j,\mu)\ |\ \nu_\mu(j) = \nu,\ \nu_1(j) \neq \nu_2(j)\ \} < \infty$$

that is, such that the sheet $X_\nu^{\rm reg}$ is joined to other sheets by only finitely many handles, one has

$$\limsup_{\substack{j\to\infty \\ \nu_1(j)=\nu_2(j)=\nu}} |s_1(j) - s_2(j)| = \infty$$

Remark. Many of these Hypotheses, particularly in (GH5), may be substantially weakened. However, at this point in time, the weakened versions are even uglier than the versions here. See [FKT 2] for the details.

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