# Periodic Schrödinger Operators

Let $\Gamma$ be a lattice in $\mathbb{R}^2$. Let $V \in C^\infty_{\mathbb{R}}(\mathbb{R}^2/\Gamma)$.

**Proposition S.9** *The spectrum of the self–adjoint boundary value problem*

$$\Big[\big(i\boldsymbol{\nabla}\big)^2 + V(\mathbf{x})\Big]\phi = \lambda\phi \qquad \phi(\mathbf{x}+\boldsymbol{\gamma}) = e^{i\mathbf{k}\cdot\boldsymbol{\gamma}}\phi(\mathbf{x}) \ \forall\boldsymbol{\gamma}\in\Gamma$$

*or, equivalently, the spectrum of the self–adjoint boundary value problem*

$$\Big[\big(i\boldsymbol{\nabla}_{\mathbf{x}} - \mathbf{k}\big)^2 + V(\mathbf{x})\Big]\psi = \lambda\psi \qquad \psi(\mathbf{x}+\boldsymbol{\gamma}) = \psi(\mathbf{x}) \ \forall\boldsymbol{\gamma}\in\Gamma$$

*consists of a sequence of eigenvalues*

$$e_1(\mathbf{k}) \le e_2(\mathbf{k}) \le e_3(\mathbf{k}) \le \cdots$$

*with, for each $n$, $e_n(\mathbf{k})$ continuous in $\mathbf{k}$ and periodic with respect to $\Gamma^\#$ and $\lim_{n\to\infty} e_n(\mathbf{k}) = \infty$.*

**Theorem S.10** *The spectrum of $-\Delta + V(\mathbf{x})$ is*

$$\big\{\ e_n(\mathbf{k}) \ \big|\ \mathbf{k} \in \mathbb{R}^d/\Gamma^\#,\ n \in \mathbb{N}\ \big\}$$

## Fermi Curves Defined

The (real "lifted") Fermi curve for energy $\lambda$ is

$$\widehat{\mathcal{F}}_{\lambda,\mathbb{R}}(V) = \left\{ \ \mathbf{k} \in \mathbb{R}^2 \ \middle| \ e_n(\mathbf{k}) = \lambda \text{ for some } n \in \mathbb{N} \ \right\}$$

$$= \left\{ \mathbf{k} \in \mathbb{R}^2 \,\middle|\, \left[ \left( i\boldsymbol{\nabla}_{\mathbf{x}} - \mathbf{k} \right)^2 + V(\mathbf{x}) - \lambda \right] \psi = 0 \right.$$

$$\left. \text{for some } 0 \neq \psi \in H^2(\mathbb{R}^2/\Gamma) \right\}$$

We may absorb the $\lambda$ in $V$.

$$\widehat{\mathcal{F}}_{\mathbb{R}}(V) = \widehat{\mathcal{F}}_{0,\mathbb{R}}(V)$$

$$\mathcal{F}_{\mathbb{R}}(V) = \widehat{\mathcal{F}}_{\mathbb{R}}(V)/\Gamma^{\#}$$

The complexifications

$$\widehat{\mathcal{F}}(V) = \left\{ \mathbf{k} \in \mathbb{C}^2 \,\middle|\, \left[ \left( i\boldsymbol{\nabla}_{\mathbf{x}} - \mathbf{k} \right)^2 + V(\mathbf{x}) \right] \psi = 0 \right.$$

$$\left. \text{for some } 0 \neq \psi \in H^2(\mathbb{R}^2/\Gamma) \right\}$$

$$\mathcal{F}(V) = \widehat{\mathcal{F}}(V)/\Gamma^{\#}$$

## The Free Fermi Curve

First set $V = 0$. Then

$$\left\{ \ e^{i\,\mathbf{x}\cdot\mathbf{b}} \ \middle| \ \mathbf{b} \in \Gamma^{\#} \ \right\}$$

is a basis of $L^2(\mathbb{R}^2/\Gamma)$ of eigenfunctions of $\big(i\boldsymbol{\nabla}_{\mathbf{x}} - \mathbf{k}\big)^2$.

$$\begin{aligned}
\big(i\boldsymbol{\nabla}_{\mathbf{x}} - \mathbf{k}\big)^2 e^{i\,\mathbf{x}\cdot\mathbf{b}} &= (-\mathbf{b} - \mathbf{k})^2 \ e^{i\,\mathbf{x}\cdot\mathbf{b}} \\
&= N_{\mathbf{b}}(\mathbf{k}) \ e^{i\,\mathbf{x}\cdot\mathbf{b}} \\
&= N_{\mathbf{b},1}(\mathbf{k}) N_{\mathbf{b},2}(\mathbf{k}) \ e^{i\,\mathbf{x}\cdot\mathbf{b}} \\
\text{where } N_{\mathbf{b}}(\mathbf{k}) &= (k_1 + b_1)^2 + (k_2 + b_2)^2 \\
N_{\mathbf{b},\nu}(\mathbf{k}) &= (k_1 + b_1) + i(-1)^{\nu}(k_2 + b_2)
\end{aligned}$$

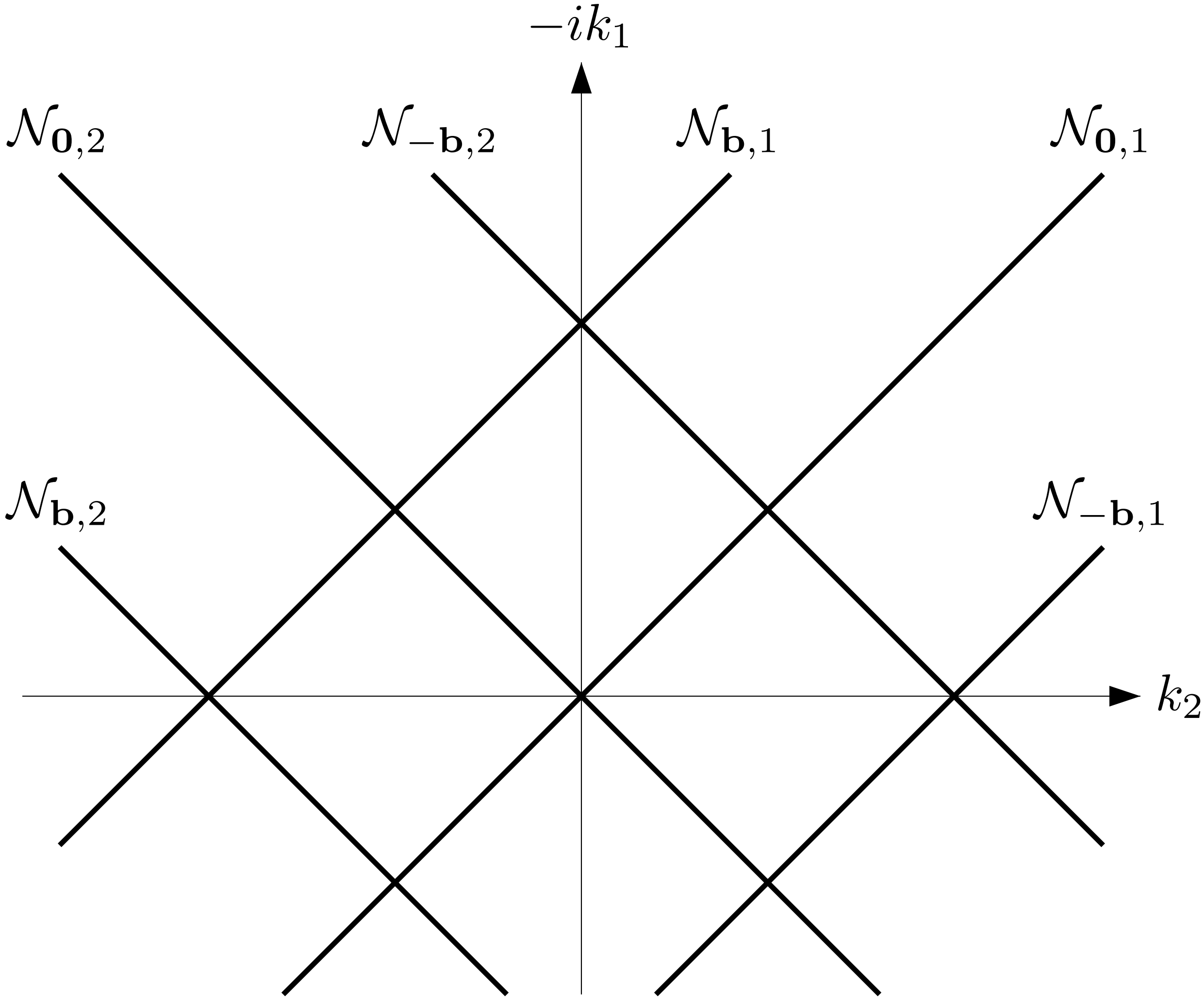
Hence

$$\begin{aligned}
\widehat{\mathcal{F}}(V = 0) &= \left\{ \ \mathbf{k} \in \mathbb{C}^2 \ \middle| \ \exists \ \mathbf{b} \in \Gamma^{\#} \text{ with } N_{\mathbf{b}}(\mathbf{k}) = 0 \ \right\} \\
&= \bigcup_{\mathbf{b}\in\Gamma^{\#}} \mathcal{N}_{\mathbf{b}} = \bigcup_{\substack{\mathbf{b}\in\Gamma^{\#} \\ \nu=1,2}} \mathcal{N}_{\mathbf{b},\nu}
\end{aligned}$$

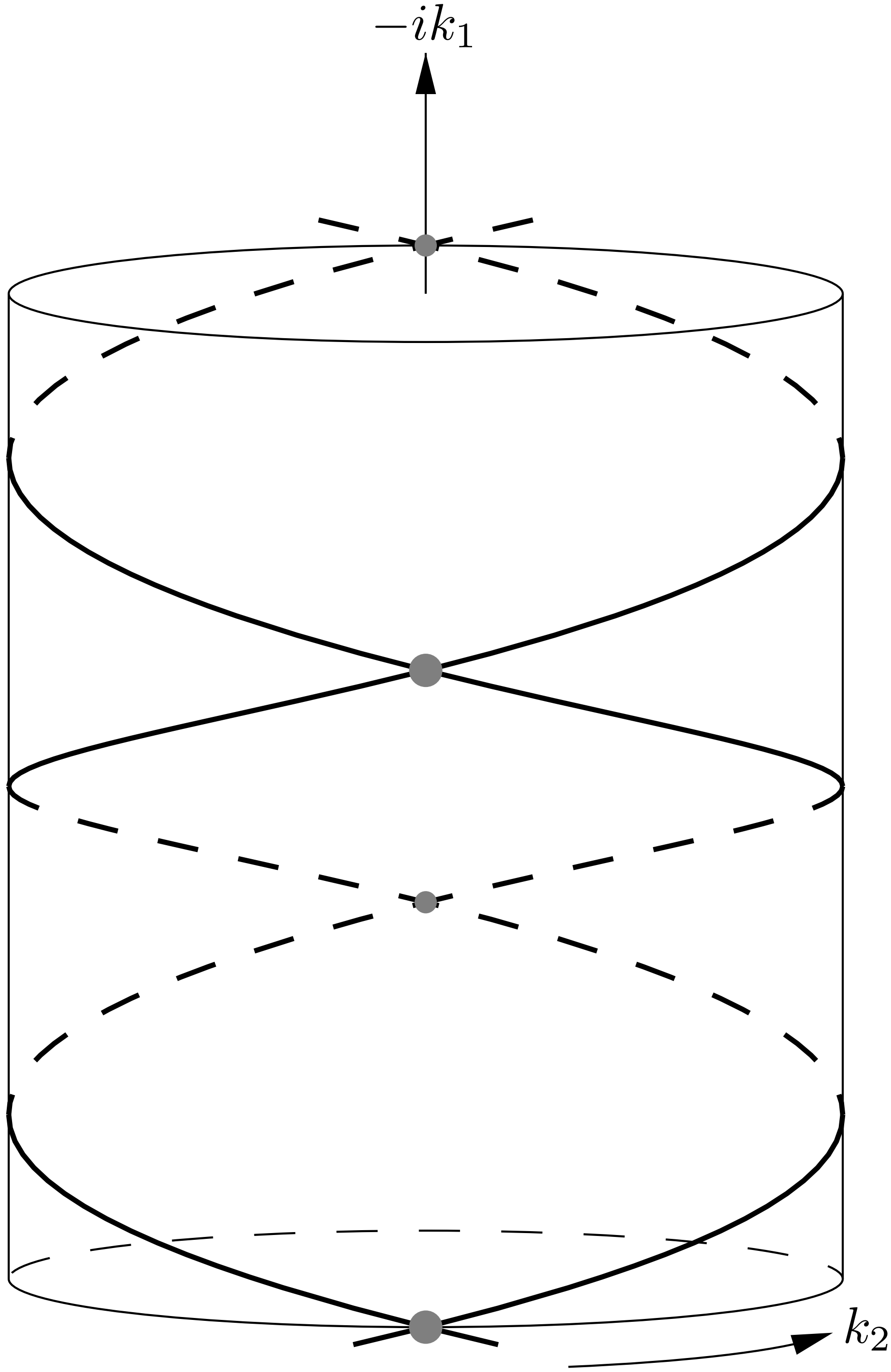
where

$$\mathcal{N}_{\mathbf{b}} = \left\{ \ \mathbf{k} \in \mathbb{C}^2 \ \middle| \ (k_1 + b_1)^2 + (k_2 + b_2)^2 = 0 \ \right\}$$

$$\mathcal{N}_{\mathbf{b},\nu} = \left\{ \ \mathbf{k} \in \mathbb{C}^2 \ \middle| \ (k_1 + b_1) + i(-1)^{\nu}(k_2 + b_2) = 0 \ \right\}$$




Free Fermi Curve: $\widehat{\mathcal{F}}(V=0)$




Free Fermi Curve: $\mathcal{F}(V = 0)$

**Theorem.** *Let $r > 1$. There exists an entire function $F$ on $\mathbb{C}^2 \times L^r(\mathbb{R}^2/\Gamma)$ such that*

$$\ker\Big(\big(i\nabla_{\mathbf{x}} - \mathbf{k}\big)^2 + V(\mathbf{x})\Big) \neq \{0\} \iff F(\mathbf{k}, V) \ = \ 0$$

**Idea of Proof.**

Write $\big(i\nabla_{\mathbf{x}} - \mathbf{k}\big)^2 + V(\mathbf{x}) = \mathbb{1} - \Delta + W(\mathbf{k}, \mathbf{x})$ .

$$\begin{aligned}
&\ker\Big(\big(i\nabla_{\mathbf{x}} - \mathbf{k}\big)^2 + V(\mathbf{x})\Big) \neq \{0\} \\
&\iff \ker\big(\mathbb{1} - \Delta + W(\mathbf{k}, \mathbf{x})\big) \neq \{0\} \\
&\iff \ker\big(\mathbb{1} + \tfrac{1}{\sqrt{\mathbb{1}-\Delta}} W(\mathbf{k}, \mathbf{x}) \tfrac{1}{\sqrt{\mathbb{1}-\Delta}}\big) \neq \{0\} \\
&\iff \det\big(\mathbb{1} + \tfrac{1}{\sqrt{\mathbb{1}-\Delta}} W(\mathbf{k}, \mathbf{x}) \tfrac{1}{\sqrt{\mathbb{1}-\Delta}}\big) = 0
\end{aligned}$$

Denote the eigenvalues of $\widehat{W} = \frac{1}{\sqrt{\mathbb{1}-\Delta}} W(\mathbf{k}, \mathbf{x}) \frac{1}{\sqrt{\mathbb{1}-\Delta}}$, $\lambda_i = \lambda_i(\mathbf{k}, V)$, $i = 1, 2, 3, \cdots$. Since $\sum_{\mathbf{n}\in\mathbb{Z}^2} \frac{1}{1+|\mathbf{n}|^{2r}} < \infty$

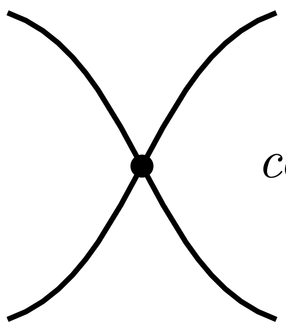
$$\sum_{i=1}^{\infty} |\lambda_i|^{2r} < \infty \Rightarrow \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i + \frac{1}{2}\lambda_i^2} \text{ converges}$$

Set

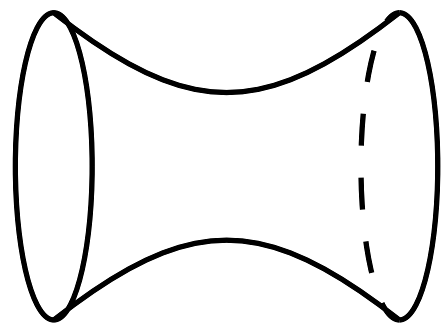
$$F(\mathbf{k}, V) = \lim_{\substack{\text{finite}\\ \text{rank}}} \det(\mathbb{1} + \widehat{W}) e^{-\operatorname{tr}\widehat{W} + \frac{1}{2}\operatorname{tr}\widehat{W}^2} = \det{}_3(\mathbb{1} + \widehat{W})$$

■

**"Theorem".** *Outside of a compact set $\mathcal{F}(V)$ is very close to $\mathcal{F}(0) = \widehat{\mathcal{F}}(0)/\Gamma^{\#}$ except that each*



*can open up to*



$z_1 z_2 = 0$ $\qquad$ $z_1 z_2 = t$

*Generically all the double points open up and then we have a Riemann surface (a one dimensional complex manifold).*

**Idea of Proof.** Fix any $\mathbf{k}$. This $\mathbf{k} \in \widehat{\mathcal{F}}(V)$ iff there is a nonzero $\psi$ obeying $\psi(\mathbf{x} + \boldsymbol{\gamma}) = \psi(\mathbf{x})\ \forall \boldsymbol{\gamma} \in \Gamma$ and

$$\Big[\big(i\boldsymbol{\nabla}_{\mathbf{x}} - \mathbf{k}\big)^2 + V(\mathbf{x})\Big]\psi = 0 \tag{1}$$

Expand $\psi(\mathbf{x}) = \sum_{\mathbf{c}\in\Gamma^{\#}} \tilde{\psi}_{\mathbf{c}} e^{i\mathbf{c}\cdot\mathbf{x}}$. Then (1) is equivalent to

$$(-\mathbf{b} - \mathbf{k})^2 \tilde{\psi}_{\mathbf{b}} + \sum_{\mathbf{c}\in\Gamma^{\#}} \tilde{V}(\mathbf{b} - \mathbf{c})\tilde{\psi}_{\mathbf{c}} = 0 \quad \forall \mathbf{b} \in \Gamma^{\#} \tag{2}$$

Recall that

$$\big(\mathbf{b}+\mathbf{k}\big)^2 = N_{\mathbf{b}}(\mathbf{k}) = N_{b,1}(\mathbf{k})N_{b,2}(\mathbf{k})$$

$$\text{where } N_{\mathbf{b},\nu}(\mathbf{k}) = (k_1+b_1) + i(-1)^\nu(k_2+b_2)$$

$N_{\mathbf{b}}(\mathbf{k})$ vanishes on $\mathcal{N}_{\mathbf{b}} = \big\{\ \mathbf{k}\in\mathbb{C}^2\ \big|\ N_{\mathbf{b}}(\mathbf{k})=0\ \big\}$. Fix $\epsilon$ small and define the ($\epsilon$-)tube about $\mathcal{N}_b$ by

$$\mathcal{T}_{\mathbf{b}} = \mathcal{T}_1(\mathbf{b})\cup\mathcal{T}_2(\mathbf{b})$$

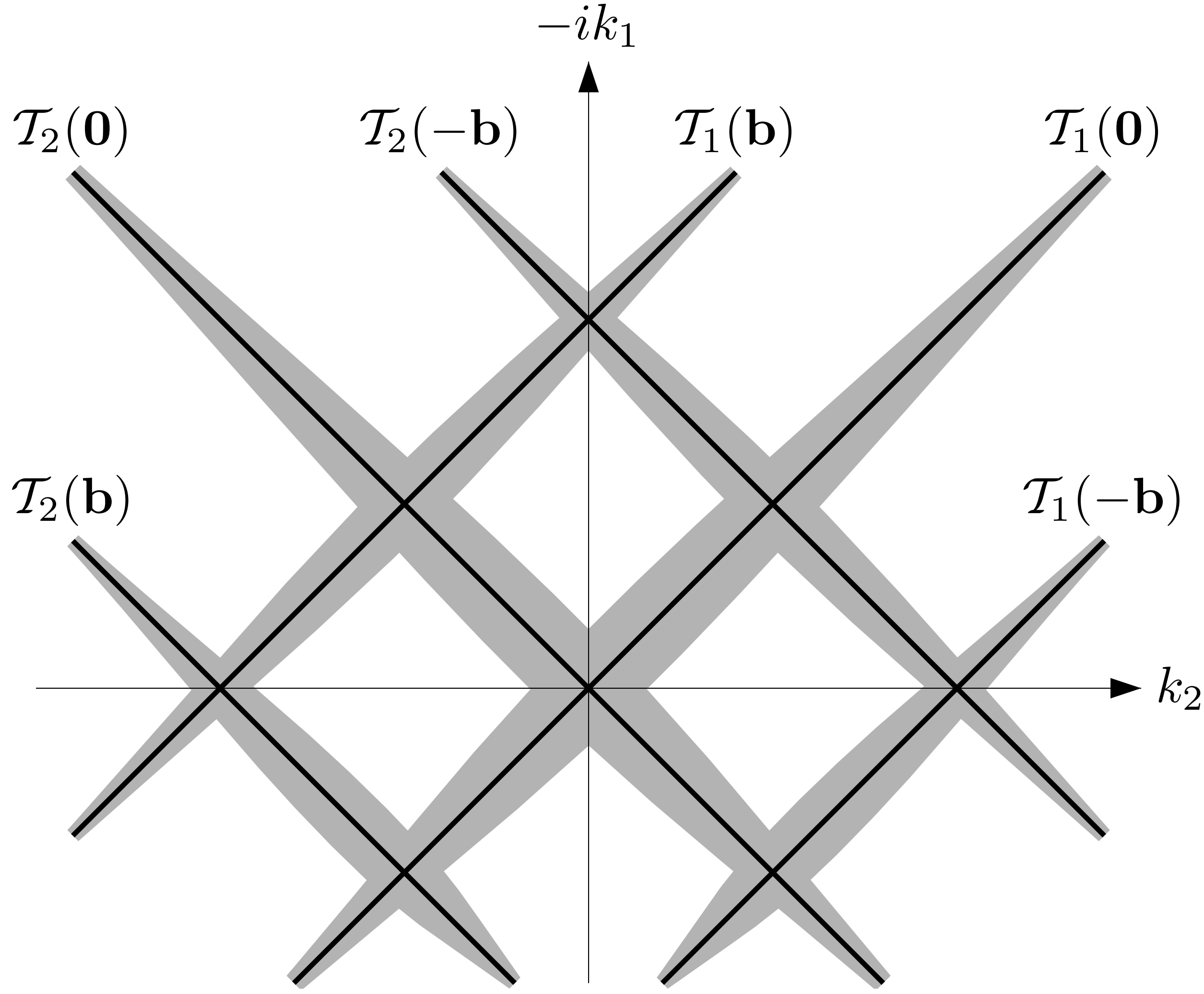
$$\mathcal{T}_\nu(\mathbf{b}) = \big\{\ \mathbf{k}\in\mathbb{C}^2\ \big|\ |N_{\mathbf{b},\nu}(\mathbf{k})| < \tfrac{\epsilon}{1+|\mathrm{Im}\,\mathbf{k}|^{1-\epsilon}}\ \big\}$$

Then

$$\mathbf{k}\notin\mathcal{T}_{\mathbf{b}} \quad\Longrightarrow\quad |N_{\mathbf{b}}(\mathbf{k})| \ge \frac{\epsilon|\mathrm{Im}\,\mathbf{k}|}{1+|\mathrm{Im}\,\mathbf{k}|^{1-\epsilon}}$$

and

- $\overline{\mathcal{T}}_{\mathbf{b}}\cap\overline{\mathcal{T}}_{\mathbf{b}'}$ is compact whenever $\mathbf{b}\neq\mathbf{b}'$
- $\overline{\mathcal{T}}_\nu(\mathbf{b})\cap\overline{\mathcal{T}}_\nu(\mathbf{b}') = \emptyset$ if $\mathbf{b}\neq\mathbf{b}'$
- $\overline{\mathcal{T}}_{\mathbf{b}}\cap\overline{\mathcal{T}}_{\mathbf{b}'}\cap\overline{\mathcal{T}}_{\mathbf{b}''} = \emptyset$ for all distinct $\mathbf{b},\ \mathbf{b}',\ \mathbf{b}''$ in $\Gamma^\sharp$


$-ik_1$
$\mathcal{T}_2(\mathbf{0})$
$\mathcal{T}_2(-\mathbf{b})$
$\mathcal{T}_1(\mathbf{b})$
$\mathcal{T}_1(\mathbf{0})$
$\mathcal{T}_2(\mathbf{b})$
$\mathcal{T}_1(-\mathbf{b})$
$k_2$

To study the part of $\hat{\mathcal{F}}(V)$ in the intersection of $\cup_{\mathbf{d}\in B}\mathcal{T}_{\mathbf{d}}$ and $\mathbb{C}^2 \setminus \cup_{\mathbf{b}\notin B}\mathcal{T}_{\mathbf{b}}$ for some finite subset $B$ of $\Gamma^\sharp$ it is natural to split (2)

$$N_{\mathbf{b}}(\mathbf{k})\tilde{\psi}_{\mathbf{b}} + \sum_{\mathbf{c}\in\Gamma^\#} \tilde{V}(\mathbf{b}-\mathbf{c})\tilde{\psi}_{\mathbf{c}} = 0 \quad \forall \mathbf{b}\in B \tag{$2_{\rm B}$}$$

$$N_{\mathbf{b}}(\mathbf{k})\tilde{\psi}_{\mathbf{b}} + \sum_{\mathbf{c}\in\Gamma^\#} \tilde{V}(\mathbf{b}-\mathbf{c})\tilde{\psi}_{\mathbf{c}} = 0 \quad \forall \mathbf{b}\in B' \tag{$2_{\rm B'}$}$$

where $B' = \Gamma^\# \setminus B$. Set $\tilde{\phi}_{\mathbf{b}} = \begin{cases} N_{\mathbf{b}}(\mathbf{k})\tilde{\psi}_{\mathbf{b}} & \text{if } \mathbf{b}\in B' \\ 0 & \text{if } \mathbf{b}\in B \end{cases}$. Then $(2_{\rm B})$ is

$$\tilde{\phi}_{\mathbf{b}} + \sum_{\mathbf{c}\in B'} \tfrac{\tilde{V}(\mathbf{b}-\mathbf{c})}{N_{\mathbf{c}}(\mathbf{k})}\tilde{\phi}_{\mathbf{c}} = -\sum_{\mathbf{c}\in B} \tilde{V}(\mathbf{b}-\mathbf{c})\tilde{\psi}_{\mathbf{c}} \quad \forall \mathbf{b}\in B'$$

Using

$$\begin{aligned} &\big\|\big[M_{\mathbf{b},\mathbf{c}}\big]_{\mathbf{b},\mathbf{c}\in B'}\big\| \\ &\qquad\qquad \le \max\Big\{\sup_{\mathbf{c}\in B'}\sum_{\mathbf{b}\in B'}\big|M_{\mathbf{b},\mathbf{c}}\big|, \sup_{\mathbf{b}\in B'}\sum_{\mathbf{c}\in B'}\big|M_{\mathbf{b},\mathbf{c}}\big|\Big\} \end{aligned}$$

one can easily see that, for $\mathbf{k}$ in the region under consideration, $\big\|\big[\tfrac{\tilde{V}(\mathbf{b}-\mathbf{c})}{N_{\mathbf{c}}(\mathbf{k})}\big]_{\mathbf{b},\mathbf{c}\in B'}\big\| \ll 1$.

Hence $\left[R_{\mathbf{b},\mathbf{c}}\right]_{\mathbf{b},\mathbf{c}\in B'}$, with $R_{\mathbf{b},\mathbf{c}} = \delta_{\mathbf{b},\mathbf{c}} + \frac{\tilde{V}(\mathbf{b}-\mathbf{c})}{N_{\mathbf{c}}(\mathbf{k})}$ has a bounded inverse and

$$\tilde{\phi}_{\mathbf{b}} = -\sum_{\substack{\mathbf{c}\in B \\ \mathbf{d}'\in B'}} R^{-1}_{\mathbf{b},\mathbf{d}'}\tilde{V}(\mathbf{d}'-\mathbf{c})\tilde{\psi}_{\mathbf{c}} \qquad \forall \mathbf{b}\in B'$$

$$\text{or } \tilde{\psi}_{\mathbf{c}'} = -\sum_{\substack{\mathbf{c}\in B \\ \mathbf{d}'\in B'}} \frac{1}{N_{\mathbf{c}'}(\mathbf{k})} R^{-1}_{\mathbf{c}',\mathbf{d}'}\tilde{V}(\mathbf{d}'-\mathbf{c})\tilde{\psi}_{\mathbf{c}} \quad \forall \mathbf{c}'\in B'$$

Substituting this into $(2_{\mathrm{B}})$ yields, for all $\mathbf{b}\in B$,

$$N_{\mathbf{b}}(\mathbf{k})\tilde{\psi}_{\mathbf{b}} + \sum_{\mathbf{c}\in B}\tilde{V}(\mathbf{b}-\mathbf{c})\tilde{\psi}_{\mathbf{c}} - \sum_{\substack{\mathbf{c}\in B \\ \mathbf{c}',\mathbf{d}'\in B'}} \frac{\tilde{V}(\mathbf{b}-\mathbf{c}')}{N_{\mathbf{c}'}(\mathbf{k})} R^{-1}_{\mathbf{c}',\mathbf{d}'}\tilde{V}(\mathbf{d}'-\mathbf{c})\tilde{\psi}_{\mathbf{c}} = 0$$

This has a nontrivial solution if and only if the $|B|\times|B|$ determinant

$$\det\left[N_{\mathbf{b}}(\mathbf{k})\delta_{\mathbf{b},\mathbf{c}} + \tilde{V}(\mathbf{b}-\mathbf{c}) - S^B_{\mathbf{b},\mathbf{c}}(\mathbf{k})\right]_{\mathbf{b},\mathbf{c}\in B} = 0$$

where

$$S^B_{\mathbf{b},\mathbf{c}}(\mathbf{k}) = \sum_{\mathbf{c}',\mathbf{d}'\in B'} \frac{\tilde{V}(\mathbf{b}-\mathbf{c}')}{N_{\mathbf{c}'}(\mathbf{k})} R^{-1}_{\mathbf{c}',\mathbf{d}'}\tilde{V}(\mathbf{d}'-\mathbf{c})$$

**Proposition 19.5,6** *Let* $\mathbf{k} \in \mathbb{C}^2$ *with* $|\mathrm{Im}\,\mathbf{k}| > \mathrm{const}$ .

*a) Let* $\mathbf{k} \in \mathbb{C}^2 \setminus \cup_{\mathbf{b}} T_{\mathbf{b}}$. *Then* $\mathbf{k} \notin \hat{\mathcal{F}}(V)$.

*b) Let* $\mathbf{k} \in T_{\mathbf{0}} \setminus \cup_{\mathbf{b}\neq\mathbf{0}} T_{\mathbf{b}}$. *Then* $\mathbf{k} \in \hat{\mathcal{F}}(V)$ *if and only if*

$$\mathbf{k}^2 = N_{\mathbf{0}}(\mathbf{k}) = \mathcal{A}(\mathbf{k})$$

*where* $\mathcal{A}(\mathbf{k}) = -\tilde{V}(\mathbf{0}) + S^{\{\mathbf{0}\}}_{\mathbf{0},\mathbf{0}}(\mathbf{k})$ *obeys*

$$\left|\mathcal{A}(\mathbf{k}) + \tilde{V}(\mathbf{0})\right| \le \tfrac{\mathrm{const}}{|\mathrm{Im}\,\mathbf{k}|^{2-\epsilon}}$$

$$\left|\tfrac{\partial^{n+m}}{\partial k_1^n \partial k_2^m}\mathcal{A}(\mathbf{k})\right| \le \tfrac{\mathrm{const}}{|\mathrm{Im}\,\mathbf{k}|} \qquad \text{if } m+n=1$$

*c) Let* $\mathbf{k} \in T_{\mathbf{0}} \cap T_{\mathbf{d}}$. *Then* $\mathbf{k} \in \hat{\mathcal{F}}(V)$ *if and only if*

$$\Big(N_{\mathbf{0}}(\mathbf{k}) + \tilde{V}(\mathbf{0}) - \mathcal{D}(\mathbf{k})_{1,1}\Big)\Big(N_{\mathbf{d}}(\mathbf{k}) + \tilde{V}(\mathbf{0}) - \mathcal{D}(\mathbf{k})_{2,2}\Big)$$
$$= \Big(\tilde{V}(-\mathbf{d}) - \mathcal{D}(\mathbf{k})_{1,2}\Big)\Big(\tilde{V}(\mathbf{d}) - \mathcal{D}(\mathbf{k})_{2,1}\Big)$$

*where*

$$\mathcal{D}(\mathbf{k})_{i,j} = S^{\{\mathbf{0},\mathbf{d}\}}_{\mathbf{d}_i,\mathbf{d}_j}(\mathbf{k}), \quad \mathbf{d}^{(1)} = \mathbf{0},\ \mathbf{d}^{(2)} = \mathbf{d}$$

*and obeys*

$$\left|\mathcal{D}(\mathbf{k})_{i,j}\right| \le \tfrac{\mathrm{const}}{|\mathrm{Im}\,\mathbf{k}|^{2-\epsilon}}$$

$$\left|\tfrac{\partial^{n+m}}{\partial k_1^n \partial k_2^m}\mathcal{D}(\mathbf{k})_{i,j}\right| \le \tfrac{\mathrm{const}}{|\mathrm{Im}\,\mathbf{k}|} \quad \text{if } m+n \ge 1$$

**Theorem 19.9** *Let* $V \in L^2(\mathbb{R}^2/\Gamma)$ *with* $\|\mathbf{b}\tilde{V}(\mathbf{b})\|_1 < \infty$. *Then* $\mathcal{F}(V)$ *is a reduced one dimensional complex analytic variety, which consists of at most two components. If* $\mathcal{F}(V)$ *is smooth, then it is irreducible.*

**Theorem 20.2** *Let* $q \in C^\infty(\mathbb{R}^2/\Gamma)$ *be such that* $\mathcal{F}(q)$ *is smooth. Then*

$$\mathcal{F}(q) = \mathcal{F}(q)^{\mathrm{com}} \cup \mathcal{F}(q)_1^{\mathrm{reg}} \cup \mathcal{F}(q)_2^{\mathrm{reg}} \cup \mathcal{F}(q)^{\mathrm{han}}$$

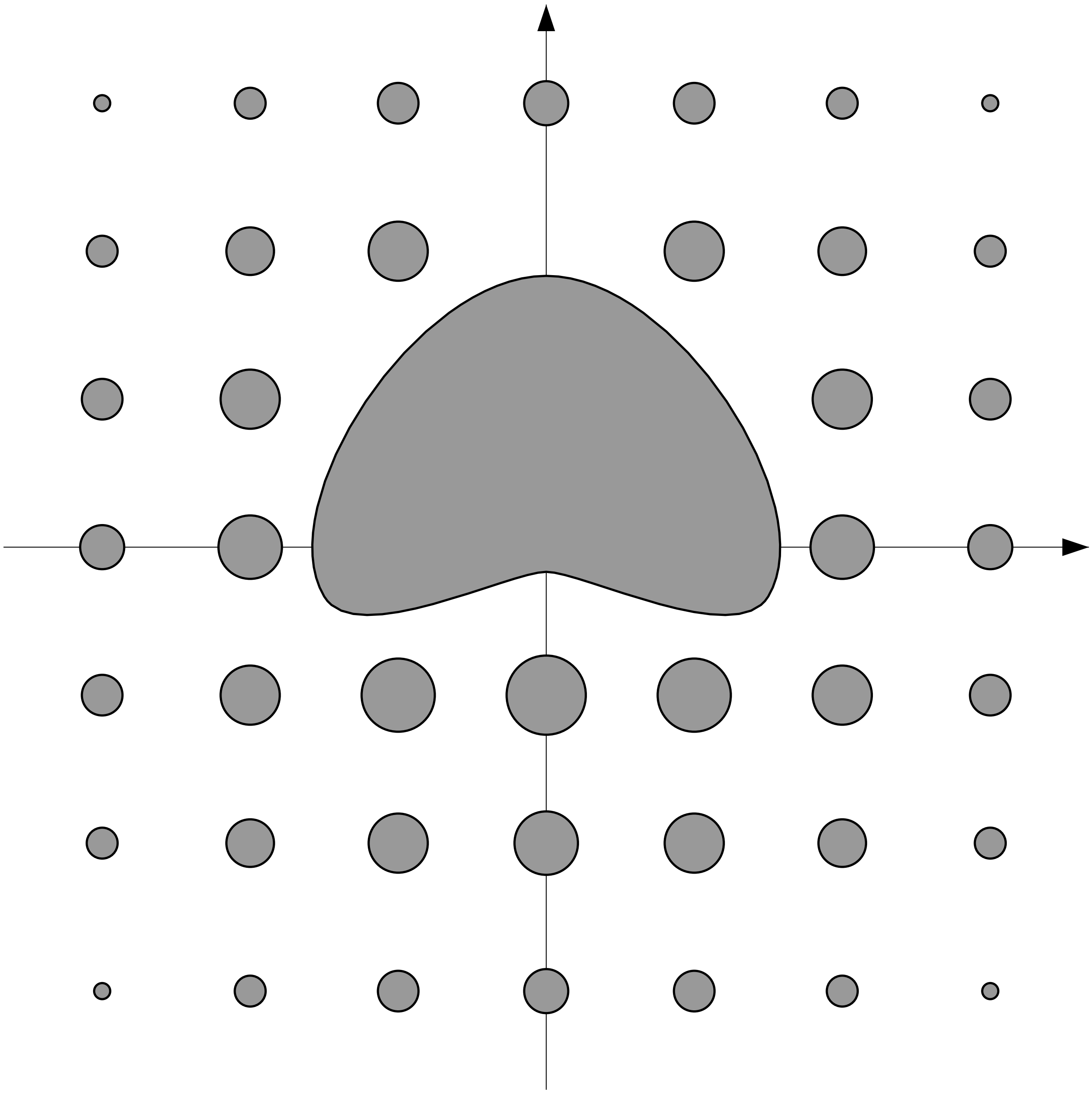
*where*

$$\begin{aligned}
\mathcal{F}(q)^{\mathrm{com}} &= \text{a compact Riemann surface with boundary} \\
\mathcal{F}(q)_1^{\mathrm{reg}} &\cong \mathbb{C} \setminus (\{\text{some disks}\} \cup \{\text{compact}\}) \\
\mathcal{F}(q)_2^{\mathrm{reg}} &\cong \mathbb{C} \setminus (\{\text{some disks}\} \cup \{\text{compact}\}) \\
\mathcal{F}(q)^{\mathrm{han}} &\cong \bigcup_{j \ge g=1} Y_j
\end{aligned}$$

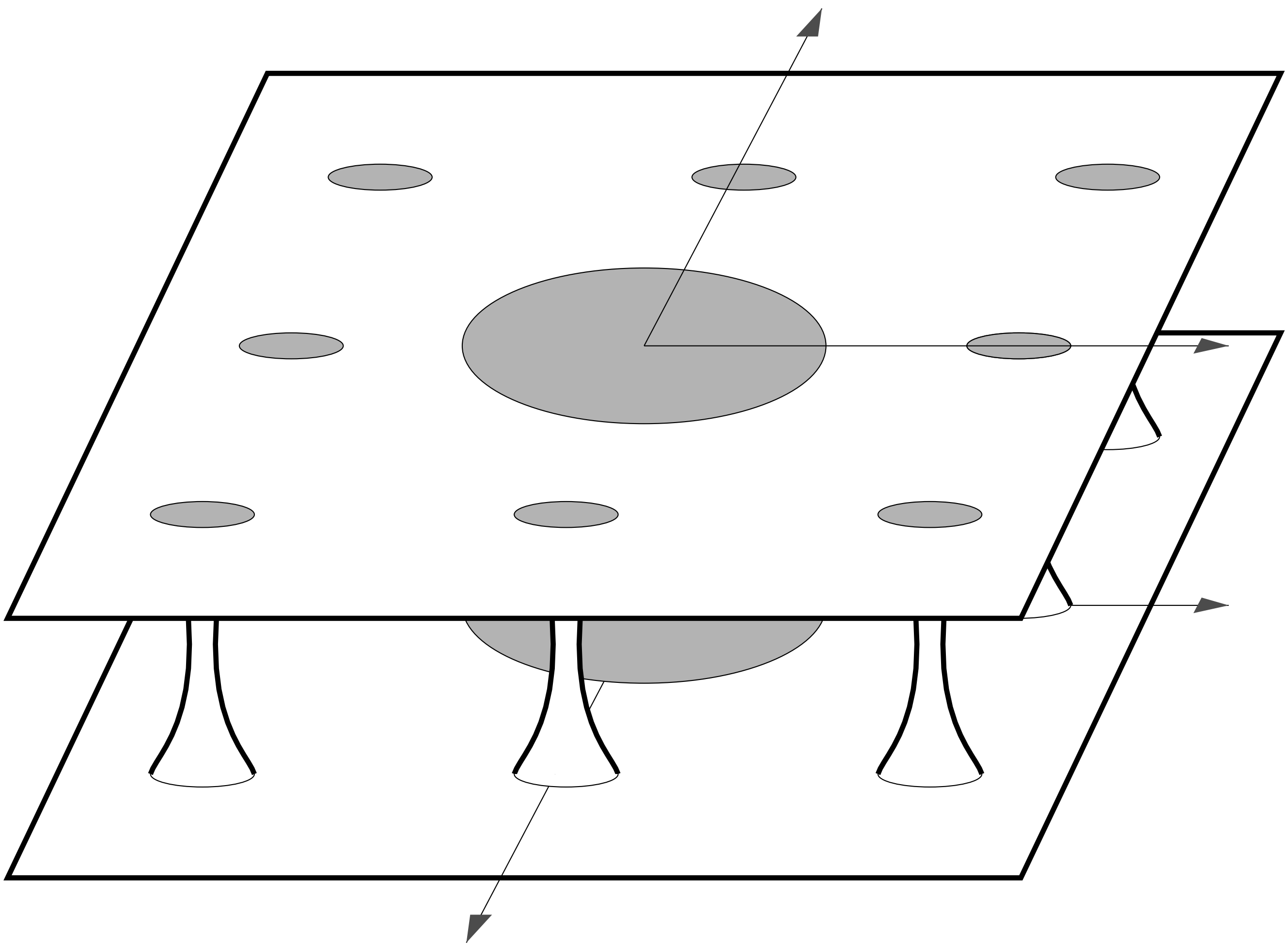
*Here* $g$ *is the genus of* $\mathcal{F}(q)^{\mathrm{com}}$, $Y_j$ *is biholomorphic to*

$$\left\{ \ (z_1, z_2) \in \mathbb{C}^2 \ \middle| \ z_1 z_2 = t_j, \ |z_1|, |z_2| \le 1 \ \right\}$$

*and, for all* $N$, $|t_j| \le \dfrac{C_N}{j^N}$ for all $j$ .



Planar Parts of the Fermi Curve



Interacting Fermi Curve: $\mathcal{F}(V)$