A Remark on Anisotropic Superconducting States

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Abstract

We show that, in three dimensions, there are no nontrivial, isotropic, unitary solutions of the gap equation for angular momentum greater than one, while in two dimensions they exist in all angular momentum sectors.

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Consider the many Fermion system in three dimensions characterized by the effective potential

$$\mathcal{G}(\psi^e, \bar{\psi}^e) = \log \frac{1}{Z} \int e^{-\lambda \mathcal{V}(\psi+\psi^e, \bar{\psi}+\bar{\psi}^e)} d\mu_C(\psi, \bar{\psi}),$$

$$\mathcal{V}(\psi, \bar{\psi}) = \frac{1}{2} \sum_{a_i \in \{\uparrow, \downarrow\}} \int \prod_{i=1}^{4} \frac{d^4 k_i}{(2\pi)^4} (2\pi)^4 \delta(k_1 + k_2 - k_3 - k_4) \delta_{\mathbf{a}_1, \mathbf{a}_3} \delta_{\mathbf{a}_2, \mathbf{a}_4}$$

$$\langle k_1, k_2 | V | k_3, k_4 \rangle \bar{\psi}(k_1, \mathbf{a}_1) \bar{\psi}(k_2, \mathbf{a}_2) \psi(k_4, \mathbf{a}_4) \psi(k_3, \mathbf{a}_3),$$

where $d\mu_C(\psi, \bar{\psi})$ is the fermionic Gaussian measure in the Grassmann variables

$$\left\{ \psi(\xi), \bar{\psi}(\xi) | \xi = (\tau, \mathbf{x}, \sigma), \tau \in \mathrm{I\!R}, \mathbf{x} \in \mathrm{I\!R}^3, \sigma \in \{\uparrow, \downarrow\} \right\}$$

with covariance

$$\begin{aligned} C(\xi_1, \xi_2) &= \langle \psi(\xi_1) \bar{\psi}(\xi_2) \rangle \\ &= \delta_{\sigma_1, \sigma_2} \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \frac{e^{i \langle k, \xi_1 - \xi_2 \rangle_-}}{i k_0 - e(\mathbf{k})} \\ \langle k, (\tau, \mathbf{x}) \rangle_- &= -k_0 \tau + \mathbf{k} \cdot \mathbf{x} \ , \ \mathbf{k} = (k_0, \mathbf{k}) \\ e(\mathbf{k}) &= \frac{\mathbf{k}^2}{2m} - \mu. \end{aligned}$$

and where the two-body interaction $\langle k_1, k_2 | V | k_3, k_4 \rangle$ is rotation invariant.That is

$$\langle Rk_1, Rk_2 | V | Rk_3, Rk_4 \rangle = \langle k_1, k_2 | V | k_3, k_4 \rangle$$

for any element R of $SO(3)$ acting on spatial components. The chemical potential μ in $e(\mathbf{k})$ determines the election density of the model.

The infrared behaviour of this model is determined (see [FT]) by a running coupling "constant" $F^{(h)}(t', s')$, $h \leq 0$ where at scale h the momentum k is restricted to a shell M^h away from the Fermi surface $e(\mathbf{k}) = 0$ and $t' = \left(0, \frac{\mathbf{t}}{|\mathbf{t}|} k_F\right)$ projects t onto the Fermi surface. Initially

$$F^{(0)}(t', s') = -\lambda \langle t', -t' | V | s', -s' \rangle.$$

The kernel $F^{(h)}(t', s')$ defines an operator on $L^2(k_F S^2)$.

By rotation invariance the operator $F^{(h)}$ commutes with the action of $SO(3)$. Therefore the eigenspaces of $F^{(h)}$ coincide with the $SO(3)$ irreducible invariant subspaces of $L^2(k_F S^2)$. Recall that the space H^n, obtained by restricting homogeneous harmonic polynomials of degree n to S^2, is a $2n+1$ dimensional $SO(3)$ irreducible invariant subspace of $L^2(k_F S^2)$ and that

$$L^2(k_F S^2) = \oplus_{n\geq 0} H^n.$$

It follows that

$$F^{(h)}(t', s') = \sum_{n\geq 0} \lambda_n^{(h)} \pi_n(t', s')$$

where π_n is the orthogonal projection onto H^n and $\lambda_n, n \geq 0$ is the spectrum of $F^{(h)}$. Here, $\pi_n(t', s') = (2n+1)k_F^{-2-n} P_n(\langle t', s'\rangle)$ where P_n is the Legendre polynomial of degree n.

It is widely believed that any (sufficiently weak) interaction $\langle k_1, k_2|V|k_3, k_4\rangle$ flows, after, say, h steps, to an effective interaction $F^{(h)}$ that is dominated by a single attractive angular momentum sector $\lambda_\ell^{(h)} > 0$ (see [KL]). The infrared behaviour is then likely to be determined by the corresponding BCS model with gap equation

$$\Delta(\mathbf{p}) = \frac{1}{2}\int_{|e(\mathbf{q})|\leq\epsilon} \frac{d^3\mathbf{q}}{(2\pi)^3}\ \lambda_\ell^{(h)} \pi_n(\mathbf{p}', \mathbf{q}')\Delta(\mathbf{q})\frac{1}{E(\mathbf{q})}\tanh\left(\frac{1}{2}\beta E(\mathbf{q})\right). \tag{1}$$

Here,

$$\Delta(\mathbf{p}) = \left(\Delta_{\sigma,\sigma'}(\mathbf{p})\right)_{\sigma,\sigma'\in\{\uparrow,\downarrow\}}$$

is a 2×2 matrix satisfying

$$\Delta(\mathbf{p}) = -\Delta(-\mathbf{p})^T$$

and

$$E(\mathbf{q})^2 = e(\mathbf{q})^2 + \Delta(\mathbf{q})^*\Delta(\mathbf{q}).$$

The expression $\frac{1}{E(\mathbf{q})}\tanh\left(\frac{1}{2}\beta E(\mathbf{q})\right)$ is unambiguously defined by expanding $\frac{1}{\sqrt{x}}\tanh\left(\frac{1}{2}\beta\sqrt{x}\right)$ as a power series in x. For a derivation of (1) see [AB], [BW].

Every solution of (1) is of the form

$$\Delta(\mathbf{p}) = \left(Y_{\sigma,\sigma'}(\mathbf{p})\right), \quad Y_{\sigma,\sigma'} \in H_\ell.$$

The simplest solutions are unitary and isotropic. A solution is unitary when

$$\Delta(\mathbf{p})^*\Delta(\mathbf{p}) = |d(\mathbf{p})|^2 I$$

and isotropic when $d(\mathbf{p})$ is a constant. In this case the quasiparticle dispersion relation $\left(e(\mathbf{q})^2+|d|^2\right)^{\frac{1}{2}}$ is isotropic and has a gap $|d|$ determined by

$$1=\frac{1}{2}\int_{|e(\mathbf{q})|\leq\epsilon}\frac{d^3\mathbf{q}}{(2\pi)^3}\,\lambda_\ell^{(h)}\left(e(\mathbf{q})^2+|d|^2\right)^{-\frac{1}{2}}\tanh\left[\frac{1}{2}\beta\left(e(\mathbf{q})^2+|d|^2\right)^{\frac{1}{2}}\right]\tag{2}$$

when $d\neq 0$. Intuitively, they have the best chance of being stable.

There are two important examples of isotropic, unitary solutions. For $\ell=0$ there is the BCS model

$$\Delta=\begin{bmatrix}0 & d\\ -d & 0\end{bmatrix}$$

for phononic superconductivity. Balian and Werthamer discovered, in the $\ell=1$ sector, the solution

$$\Delta=d\begin{bmatrix}-\mathbf{p}_1+i\mathbf{p}_2 & \mathbf{p}_3\\ \mathbf{p}_3 & \mathbf{p}_1+i\mathbf{p}_2\end{bmatrix},\quad \mathbf{p}_1^2+\mathbf{p}_2^2+\mathbf{p}_3^2=k_F^2$$

which describes the B phase of He^3.

Theorem *There are no nontrivial, isotropic, unitary solutions of* (1) *for* $\ell\geq 2$.

One therefore expects that solutions will have nodes for $\ell\geq 2$ making the flow harder to control. Such nodes are observed in the A phase of He^3 and in the $\ell=2$ theory of heavy fermionic superconductivity. Nodes also appear in the gap function for systems with cubic symmetry. See, for example, [VG].

The proof of Theorem 1 follows immediately from the

Lemma *Let* $f,g\in H_\ell$ *satisfy* $f\bar{f}+g\bar{g}=1$ *on* S^2. *Then,* $\ell=0,1$.

Proof Let P_ℓ, $\ell\geq 0$, be the homogeneous polynomials of degree ℓ on $\mathbb{R}^3$ with SO(3) invariant inner product

$$<f,g>:=f\left(\frac{\partial}{\partial k_1},\frac{\partial}{\partial k_2},\frac{\partial}{\partial k_3}\right)\bar{g}.$$

As usual H_ℓ is identified with H_ℓ^* by the SO(3) equivariant isomorphism

$$f\mapsto<\cdot,\bar{f}>.$$

We shall show that under the hypothesis of the lemma

$$U=f\otimes\bar{f}+\bar{f}\otimes f+g\otimes\bar{g}+\bar{g}\otimes g$$

is the (unique up to scalars) SO(3) invariant element of $H_\ell \otimes H_\ell$. It follows that the homomorphism

$$U \in H_\ell \otimes H_\ell \cong H_\ell \otimes H_\ell^* \cong \mathrm{Hom}\,(H_\ell, H_\ell)$$

commutes with SO(3) and is of rank at most four. Moreover, by Schur's Lemma, U is an isomorphism since H_ℓ is irreducible. Consequently, $2\ell + 1 \le 4$.

Consider the SO(3) equivariant multiplication map

$$H_\ell \otimes_s H_\ell \xrightarrow{M} P_{2\ell}$$

$$\sum c_j \phi_j \otimes \psi_j \longmapsto \sum_j c_j \phi_j \psi_j .$$

Observe that

$$\dim\, H_\ell \otimes_s H_\ell = 2\ell + 1 + \frac{(2\ell+1)(2\ell)}{2} = \binom{2\ell+2}{2} = \dim\, P_{2\ell}$$

and

$$MU = 2|k|^{2\ell}.$$

If M is surjective it is an isomorphism and U is invariant.

The projection of

$$M\left((k_1 + ik_2)^\ell \otimes_s (k_1 - ik_2)^\ell\right) = \left(k_1^2 + k_2^2\right)^\ell$$

onto the irreducible subspace $|k|^{2(\ell-m)} H_{2m}$ of $P_{2\ell}$ is nonzero because

$$\begin{aligned}
&\left\langle |k|^{2(\ell-m)} \left(k_1 + ik_3\right)^{2m}, \left(k_1^2 + k_2^2\right)^\ell \right\rangle \\
&= \left(\frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_3}\right)^{2m} \Delta^{\ell-m} \left(k_1^2 + k_2^2\right)^\ell \\
&= \prod_{j=0}^{\ell-m-1} 4(\ell-j)^2 \left(\frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_3}\right)^{2m} \left(k_1^2 + k_2^2\right)^m \\
&\neq 0.
\end{aligned}$$

Recall that every invariant subspace of $P_{2\ell}$ is of the form

$$\oplus_{j_i}\ |k|^{2j_i}\ H_{2(\ell-j_i)}$$

with $0 \le j_1 < j_2 \cdots < j_r \le \ell$ and in particular

$$P_{2\ell} = \oplus_{j=0}^{\ell}\ |k|^{2j}\ H_{2(\ell-j)}.$$

Finally the image of M is invariant and therefore all of $P_{2\ell}$. ∎

We observe that in two dimensions there are unitary isotropic solutions of the gap equation for every angular momentum. For example,

$$\Delta(\mathbf{p}) = d \begin{bmatrix} \cos \ell\theta & \sin \ell\theta \\ \sin \ell\theta & -\cos \ell\theta \end{bmatrix}$$

when ℓ is odd and

$$\Delta(\mathbf{p}) = d \begin{bmatrix} 0 & e^{i\ell\theta} \\ -e^{i\ell\theta} & 0 \end{bmatrix}$$

when ℓ is even. Here, $\mathbf{p} = |\mathbf{p}|(\cos\theta, \sin\theta)$.

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