Review of Riemann Surfaces

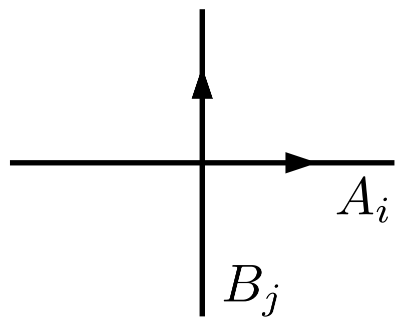
Let X be a Riemann surface (one complex dimensional manifold) of genus g. Then

(1) There exist curves $A_1, \cdots, A_g, B_1, \cdots, B_g$ with

$$A_i \times A_j = 0$$

$$A_i \times B_j = \delta_{i,j}$$

$$B_i \times B_j = 0$$



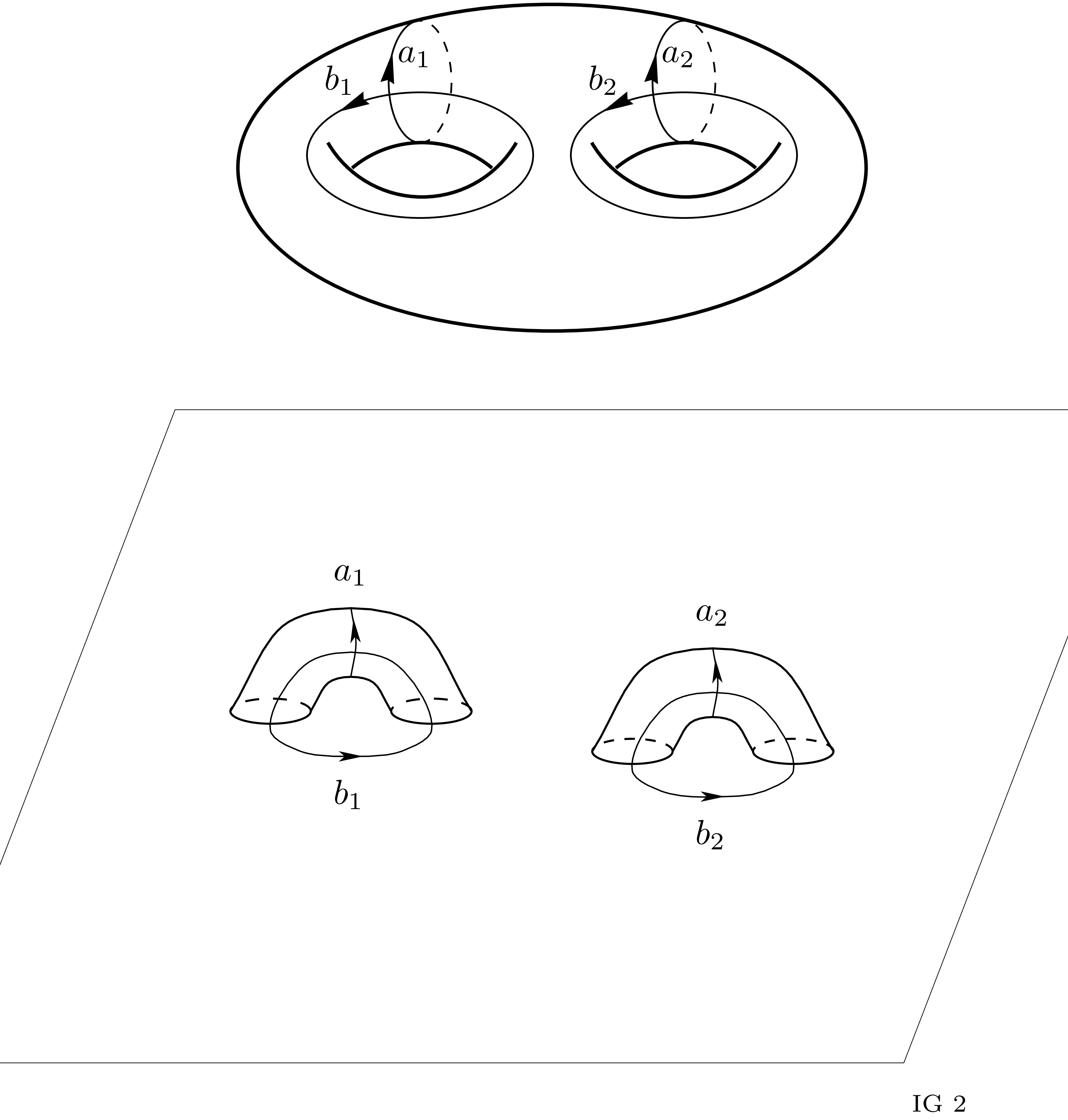
$A_i \times B_j = 1$

These curves are a basis for the homology of X. That is, if $C \times A_i = C \times B_i = 0$ for all i then $C \times D = 0$ for all curves D. [For smooth $\mathcal{F}(V)$: true but with $g = \infty$.]

(2) There is a basis $\omega_1, \cdots, \omega_g$ for the vector space of holomorphic one forms such that

$$\int_{A_i} \omega_j = \delta_{i,j}$$

[For smooth $\mathcal{F}(V)$: replace "vector space of" by "Hilbert space of square integrable"]



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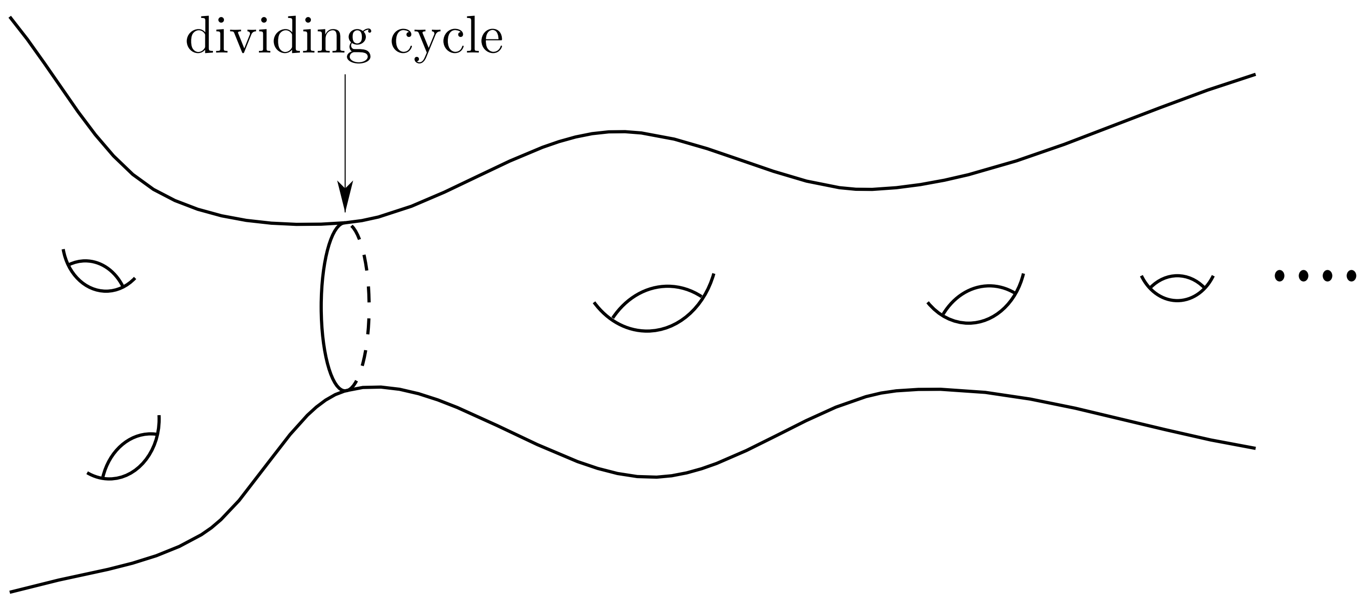

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(3) The Riemann period matrix

$$R_{i,j} = \int_{B_i} \omega_j$$

is symmetric ($R_{i,j} = R_{j,i}$).

Torelli Theorem: Let X and X' be Riemann surfaces. If $R_{i,j} = R'_{i,j}$ for all $1 \le i, j \le g$ then X and X' are biholomorphic.
[Infinite genus case [FKT2, Theorem 13.1]: X and X' have to obey axioms (all $\mathcal{F}(V)$'s do) restricting size and position of the handles.]

Proposition [FKT1, Proposition 4.4] *The Riemann period matrix obeys*

$$\sum_{i,j} n_i \,(\mathrm{Im}\, R_{i,j})\, n_j \ \ge\ \frac{1}{2\pi} \sum_j |\log t_j|\, n_j^2 \qquad \text{(RLB)}$$

[For smooth $\mathcal{F}(V)$: true]

Idea of Proof. The Riemann bilinear relations state that

$$\int_X \omega \wedge \eta \ =\ \sum_{i=1}^{\infty} \Big(\int_{A_i} \omega \int_{B_i} \eta - \int_{B_i} \omega \int_{A_i} \eta \Big)$$

(for all smooth, closed, square integrable one forms ω and η on X such that $\int_{A_i} \omega = \int_{A_i} \eta = 0$ for all but finitely many i, provided that there is an exhaustion function with finite charge on X.) Now use

$$\langle \vec{n}, (\mathrm{Im}\, R)\vec{n} \rangle = \Big\| \sum_{i\ge 1} n_i \omega_i \Big\|^2_{L^2(X)} \ge \sum_{j\ge 1} \Big\| \sum_{i\ge 1} n_i \omega_i \Big\|^2_{L^2(Y_j)}$$

and

Lemma [FKT1, Lemma 4.3] *Fix* $0 < t < 1$. *Let*

$$A \;=\; \Big\{\, (\sqrt{t}\, e^{i\theta}\,,\; \sqrt{t}\, e^{-i\theta}) \;\Big|\; 0 \le \theta \le 2\pi \,\Big\}$$

be the oriented waist on the model handle

$$H(t) \;=\; \Big\{ (z_1, z_2) \in \mathbb{C}^2 \;\Big|\; z_1 z_2 = t\,,\; |z_1|, |z_2| \le 1 \Big\}$$

For every holomorphic one form ω *on* $H(t)$,

$$\|\omega\|_2 \;\ge\; \sqrt{\tfrac{|\log t|}{2\pi}}\; \Big| \int_A \omega \Big|$$

Proof: Write $\omega \;=\; f(z_1)\, dz_1$. For any fixed r

$$\Big| \int_A \omega \Big|^2 \;=\; \Big| \int_0^{2\pi} 1 \cdot f(re^{i\theta})\; re^{i\theta} d\theta \Big|^2$$

$$\le\; 2\pi \int_0^{2\pi} |r f(re^{i\theta})|^2 d\theta$$

Hence

$$\|\omega\|_2^2 \;=\; \tfrac{1}{2} \int_{t \le |z_1| \le 1} |f(z_1)|^2 |dz_1 \wedge d\bar{z}_1|$$

$$=\; \int_t^1 \int_0^{2\pi} |r f(re^{i\theta})|^2 d\theta\, \tfrac{dr}{r}$$

$$\ge\; \tfrac{1}{2\pi} \Big| \int_A \omega \Big|^2 \int_t^1 \tfrac{dr}{r} \;=\; \tfrac{|\log t|}{2\pi} \Big| \int_A \omega \Big|^2$$

■

(4) The theta function, which is defined by

$$\theta(\vec{z}) = \sum_{\vec{n}\in\mathbb{Z}^g} e^{2\pi i<\vec{n},\vec{z}>} e^{\pi i<\vec{n},R\vec{n}>} : \mathbb{C}^g \to \mathbb{C}$$

obeys

$$\theta(\vec{z}+\vec{n}) = \theta(\vec{z}) \tag{θP1}$$

$$\theta(\vec{z}+\vec{R}_j) = e^{-2\pi i(z_j+R_{jj}/2)}\theta(\vec{z}) \tag{θP2}$$

for all $\vec{n}\in\mathbb{Z}^g$.

Theorem [FKT1, Theorem 4.6, Proposition 4.12] *Suppose that R obeys (RLB) with $t_j \in (0,1)$, $j \geq 1$ obeying $\sum_{j\geq 1} t_j^\beta < \infty$ for some $0 < \beta < \frac{1}{2}$. Then $\theta(\vec{z})$, with $\sum_{\vec{n}\in\mathbb{Z}^g}$ replaced by $\sum_{\vec{n}\in\mathbb{Z}^\infty\cap\ell^1}$, converges absolutely and uniformly on bounded subsets of the Banach space*

$$B = \left\{ \vec{z}\in\mathbb{C}^\infty \;\middle|\; \lim_{j\to\infty} \frac{z_j}{|\ln t_j|} = 0 \right\} \quad \|\vec{z}\|_B = \sup_{j\geq 1} \frac{z_j}{|\ln t_j|}$$

to an entire function that does not vanish identically. Furthermore, (θP1), (θP2) hold for all $\vec{n}\in\mathbb{Z}^\infty\cap B$ and all columns $\vec{R}_j$ of R with $\vec{R}_j\in B$.

(5) **Zeroes of the Theta Function.** One of Riemann's numerous classical results says for each fixed $\vec{e} \in \mathbb{C}^g$ and $x_0 \in X$,

$$\theta\left(\vec{e} + \int_{x_0}^{x} \vec{\omega}\right)$$

either vanishes identically or has exactly g roots. To show that, for any path joining x_1 to x_2 on X the infinite component vector

$$\int_{x_1}^{x_2} \vec{\omega} \;=\; \left(\int_{x_1}^{x_2} \omega_1\,,\, \int_{x_1}^{x_2} \omega_2\,,\, \cdots\right)$$

lies in the domain of definition of the theta function. depend on bounds on the frame $\omega_1, \omega_2, \cdots$.

Suppose that one end of the $j^{\rm th}$ handle is glued into the $\nu_1(j)^{\rm st}$ sheet near $s_1(j)$ and that the other end is glued into the $\nu_2(j)^{\rm nd}$ sheet near $s_2(j)$. When $\nu_1(j) = \nu_2(j)$, the pull back $w_j^\nu(z)dz = \Phi_\nu^* \omega_j$ of ω_j to the $\nu^{\rm th}$ sheet obeys [FKT2, Theorem 8.4]

$$\left| w_j^\nu(z) - \frac{1}{2\pi i}\frac{1}{z-s_1(j)} + \frac{1}{2\pi i}\frac{1}{z-s_2(j)} \right| \le \frac{\rm const}{1+|z^2|} \qquad \text{if } \nu \ne \nu_1(j)$$

$$\left| w_j^\nu(z) \right| \le \frac{\rm const}{1+|z^2|} \qquad \text{if } \nu = \nu_1(j)$$

On the other hand, when $\nu_1(j) \ne \nu_2(j)$,

$$\left| w_j^\nu(z) - \frac{1}{2\pi i}\frac{1}{z-s_1(j)} + \frac{1}{2\pi i}\frac{1}{z} \right| \le \frac{\rm const}{1+|z^2|} \quad \text{if } \nu = \nu_1(j)$$

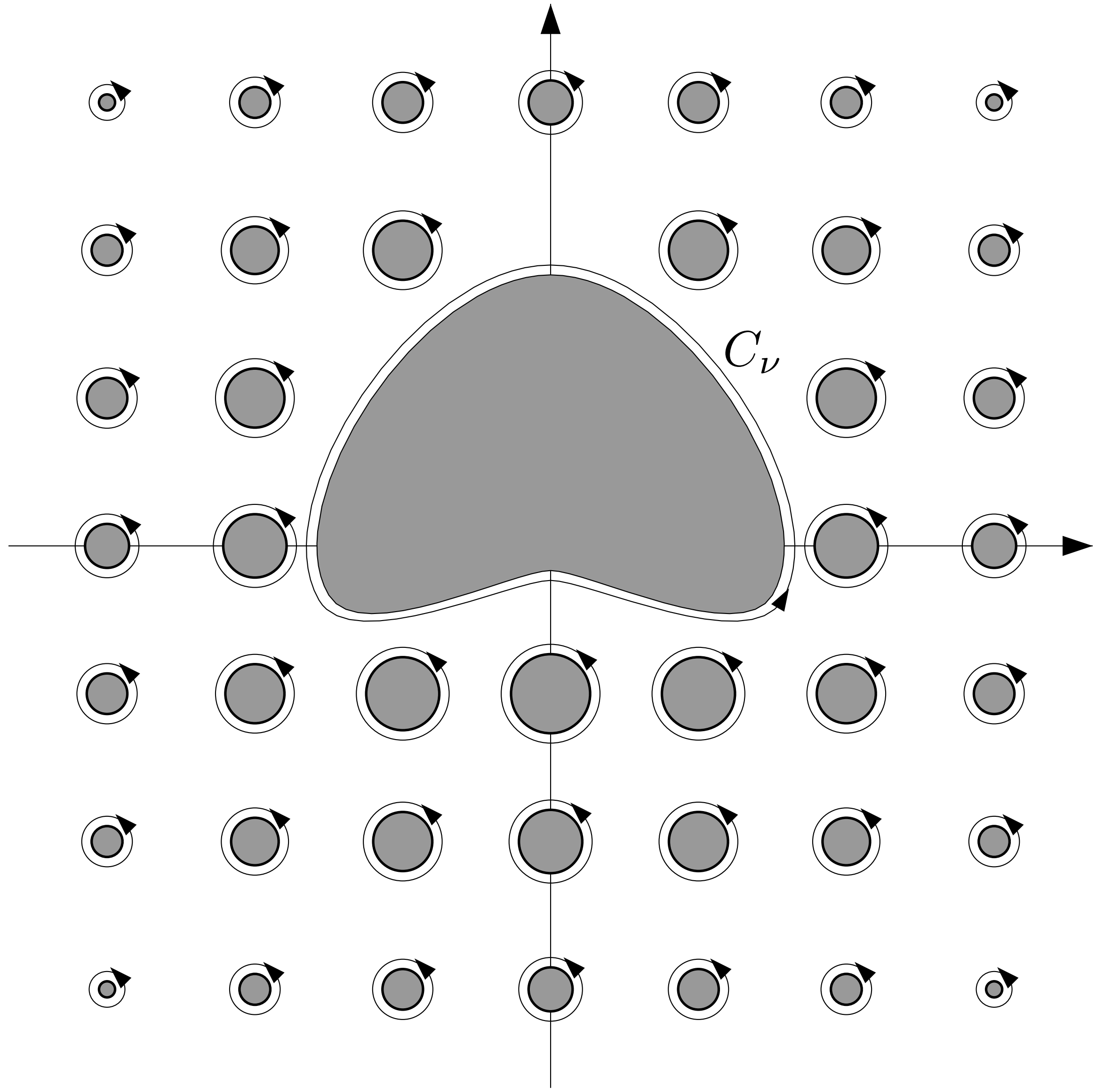
$$\left| w_j^\nu(z) - \frac{1}{2\pi i}\frac{1}{z} + \frac{1}{2\pi i}\frac{1}{z-s_2(j)} \right| \le \frac{\rm const}{1+|z^2|} \quad \text{if } \nu = \nu_2(j)$$

$$\left| w_j^\nu(z) \right| \le \frac{\rm const}{1+|z^2|} \qquad \text{if } \nu \ne \nu_i(j)$$

The const is independent of j. The pull backs of ω_j to Y_j obeys [FKT2, Proposition 8.16]

$$\left| \frac{\phi_j^* \omega_j(z)}{dz_1/z_1} - \frac{1}{2\pi i} \right| \le \frac{2}{5\pi}(|z_1| + |z_2|)$$

How to prove these bounds. The ν^{th} sheet is biholomorphic to a complex plane from which a compact neighbourhood of the origin and an infinite set of other small holes have been cut. Draw a contour C_ν around the first hole and circles $|z - s| = r(s),\ s \in S_\nu$ around the other small holes.



By the Cauchy integral formula

$$w_j^\nu(z) = w_{j,\mathrm{com}}^\nu(z) + \sum_{s\in S_\nu} w_{j,s}^\nu(z)$$

where

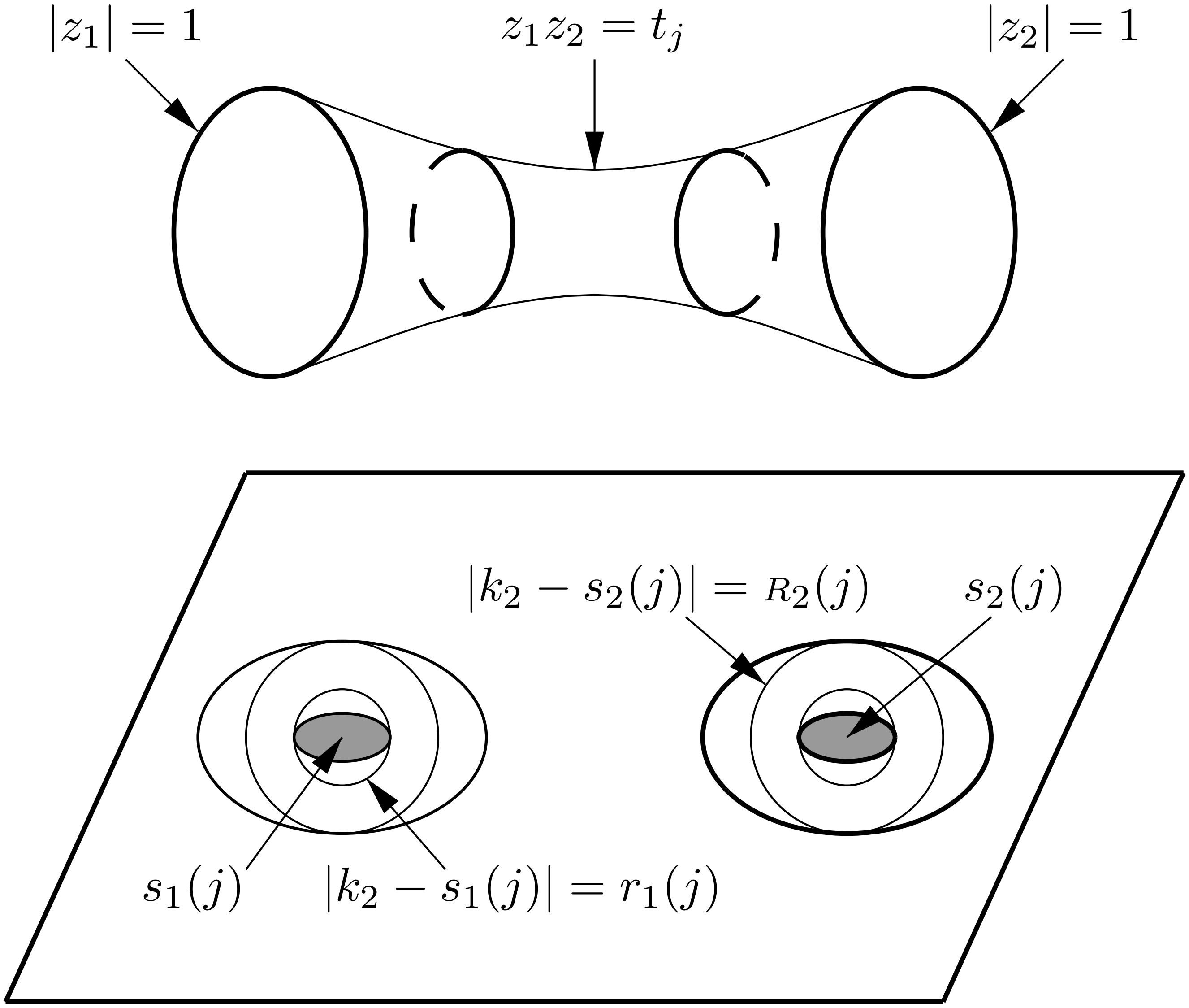
$$w_{j,s}^\nu(z) = -\frac{1}{2\pi i}\int_{|z-s|=r(s)} \frac{w_j^\nu(\zeta)}{\zeta - z} d\zeta$$

$$w_{j,\mathrm{com}}^\nu(z) = -\frac{1}{2\pi i}\int_{C_\nu} \frac{w_j^\nu(\zeta)}{\zeta - z} d\zeta$$

Step 1. By applying Cauchy-Schwarz to

$$w_{j,s}^\nu(z) = -\frac{1}{2\pi r(s) i}\int_{r(s)}^{2r(s)} \left[\int_{|z-s|=r} \frac{w_j^\nu(\zeta)}{\zeta - z} d\zeta\right] dr$$

one gets an upper bound on $w_{j,s}^\nu(z)$ for $|z-s| \geq 3r(s)$ in terms of the L^2 norm of ω_j restricted to the annulus $\Phi_\nu\big(\{r(s) \leq |z-s| \leq 2r(s)\big)$. To obtain a bound that decays like $\frac{1}{|z-s|^2}$ rather than $\frac{1}{|z-s|}$ one exploits the fact that $\int_{|z-s|=r} w_j^\nu(\zeta)\, d\zeta = 0$, unless the circle $|z-s| = r$ happens to be homologous to $\pm A_j$. If so, one works with $w_j^\nu(\zeta) \mp \frac{1}{2\pi i}\frac{1}{\zeta - s}$ instead of $w_j^\nu(\zeta)$. One also gets the analogous bound on $w_{j,\mathrm{com}}^\nu(z)$.



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Step 2. Consider the handle Y_i. One end is glued into sheet $\nu_1(i)$ near the point $s_1(i)$ and the other is glued into sheet $\nu_2(i)$ near $s_2(i)$. Denote by Y_i' the part of the handle Y_i bounded by $\Phi_{\nu_1(i)}\big(\{|z - s_1(i)| = R_1(i)\}\big)$ and $\Phi_{\nu_2(i)}\big(\{|z - s_2(i)| = R_2(i)\}\big)$. The radii $R_\mu(i)$ are chosen in (GH3,5) to be much larger than the corresponding radii $r_\mu(i)$. Using Stoke's Theorem and the holomorphicity of ω_j one obtains a bound on the L^2 norm of the restriction of ω_j to Y_i' in terms of the values of $w_j^{\nu_\mu(i)}$ on $\{|z - s_\mu(i)| = R_\mu(i)\}$, $\mu = 1, 2$.

Step 3. Substituting the first family of bounds into the second family gives a system of inequalities relating the L^2 norms of ω_j restricted to the $Y_i's$. This family of inequalities may be "solved" to get inequalities on the L^2 norms themselves.

Step 4. Bounds on the L^2 norms are turned into pointwise bounds on the sheets by the "Step 1" bounds above and on the handles by a similar method.

The above bounds on $\vec{\omega}$ imply that for any path joining x_1 to x_2 on X, the integral $\int_{x_1}^{x_2} \vec{\omega} \in B$ and remains in B in the limit as x_1 tends to infinity along a reasonable path. If X has m sheets we can choose m such paths each approaching infinity on a different sheet. Call the limits $\int_{\infty_\nu}^{x_2} \vec{\omega}$, $1 \leq \nu \leq m$. The precise statement that $\theta\left(\vec{e} + \int_{\infty_1}^{x} \vec{\omega}\right)$ has exactly "genus(X)" roots is

Theorem [FKT2, Theorem 9.11] *Let $\vec{e} \in B$ be such that $\theta(\vec{e}) \neq 0$ and $\theta\left(\vec{e} + \int_{\infty_1}^{\infty_\nu} \vec{\omega}\right) \neq 0$ for all $1 < \nu \leq m$. Then, there is a compact submanifold Y with boundary such that the multivalued, holomorphic function*

$$\theta\left(\vec{e} + \int_{\infty_1}^{x} \vec{\omega}\right)$$

has

(i) exactly genus(Y) *roots in Y*

(ii) exactly one root in each handle of X outside of Y

(iii) and no other roots.

Idea of Proof. That $\theta\left(\vec{e}+\int_{\infty_1}^{x}\vec{\omega}\right)$ has no zeroes near ∞_ν, except in handles, is a consequence of the facts that $\theta\left(\vec{e}+\int_{\infty_1}^{\infty_\nu}\vec{\omega}\right)\neq 0$ by hypothesis, that $\left\|\int_{x}^{\infty_\nu}\vec{\omega}\right\|_B$ is small for all sufficiently large x in the ν^{th} sheet and that θ is continuous in the norm of the Banach space B. The proof that there is exactly one zero in each, sufficiently far out, handle is based on the argument principle and the computation

$$\begin{aligned}
&\int_{A_j B_j A_j^{-1} B_j^{-1}} d\log\theta\left(\vec{e}+\int_{\infty_1}^{x}\vec{\omega}\right)\\
&=\int_{A_j} d\log\theta\left(\vec{e}+\int_{\infty_1}^{x}\vec{\omega}\right)+\int_{B_j} d\log\theta\left(\vec{e}+\vec{\imath}_j+\int_{\infty_1}^{x}\vec{\omega}\right)\\
&\quad-\int_{A_j} d\log\theta\left(\vec{e}+\vec{R}_j+\int_{\infty_1}^{x}\vec{\omega}\right)-\int_{B_j} d\log\theta\left(\vec{e}+\int_{\infty_1}^{x}\vec{\omega}\right)\\
&=2\pi i\int_{A_j} d\left(e_j+\int_{\infty_1}^{x}\omega_j+\tfrac{1}{2}R_{jj}\right)\\
&=2\pi i
\end{aligned}$$

where $\vec{\imath}_j$ is the vector whose k^{th} component is δ_{jk} and $\vec{R}_j$ is the j^{th} column of the Riemann period matrix. The periodicity properties of the theta function are used twice in this computation. We also used

$$\int_{B_j^{-1}} d \log \theta \left(\vec{e} + \vec{R}_j + \textstyle\int_{\infty_1}^{x} \vec{\omega} \right) = - \int_{B_j} d \log \theta \left(\vec{e} + \textstyle\int_{\infty_1}^{x} \vec{\omega} \right)$$

and an analogous formula for A_j^{-1}.

(6) **Riemann's Vanishing Theorem.** In preparation for Riemann's vanishing theorem, we introduce the notion of a divisor of degree "genus(X)" on the universal cover of X. This is done by fixing an auxiliary point $\hat{e} \in B$ with $\theta(\hat{e}) \neq 0$ and comparing sequences of points to the "genus(X)" many roots $\hat{x}_1, \hat{x}_2 \cdots$ of $\theta\left(\hat{e} + \int_{\infty_1}^{x} \vec{\omega}\right) = 0$. Precisely, let $\pi : \tilde{X} \to X$ be the universal cover of X and choose $\tilde{x}_j \in \pi^{-1}(\hat{x}_j)$. A sequence y_j, $j \geq 1$, on $\tilde{X}$ represents a divisor of degree "genus(X)" if eventually y_j lies in the same component of $\pi^{-1}(Y_j)$ as $\tilde{x}_j$ and the vector

$$\left(\int_{\tilde{x}_1}^{y_1} \omega_1 \, , \int_{\tilde{x}_2}^{y_2} \omega_2 \, , \cdots \right)$$

lies in B. The space $W^{(0)}$ of all these sequences is given the structure of a complex Banach manifold modeled on B. The quotient $S^{(0)}$ of $W^{(0)}$ by the group of all finite permutations is the manifold of divisors of degree "genus(X)". The construction is independent of the auxiliary point $\hat{e}$. We similarly construct Banach manifolds $S^{(-n)}$ of divisors of index n, that is, of degree "genus$(X)-n$", by deleting the first n components in a sequence $y_1, y_2, \cdots$ belonging to $W^{(0)}$.

Fix $\hat{e}$ as above. The right hand side of

$$(y_1\,,\, y_2\,, \cdots) \;\mapsto\; \hat{e} - \sum_{i \geq 1} \int_{\tilde{x}_i}^{y_i} \vec{\omega}$$

is invariant under permutations of the y_i's and induces the analog

$$f^{(0)} \;:\; S^{(0)} \;\longrightarrow\; B$$

of the Abel-Jacobi map. The map $f^{(0)}$ is holomorphic [FKT2, Proposition 10.1] and indeed is a biholomorphism between $f^{(0)\,-1}(B \setminus \Theta)$ and $B \setminus \Theta$ where

$$\Theta \;=\; \left\{ \vec{e} \in B \,\middle|\, \theta(\vec{e}) = 0 \right\}$$

is the theta divisor of X.

Similarly, the map

$$f^{(-1)} \;:\; S^{(-1)} \;\longrightarrow\; B$$

is induced by

$$(y_2\,,\, y_3\,,\cdots) \;\mapsto\; \hat{e} \,-\, \int_{\tilde{x}_1}^{\infty_1} \vec{\omega} \,-\, \sum_{i\geq 2} \int_{\tilde{x}_i}^{y_i} \vec{\omega}$$

The analogue of the Riemann vanishing theorem is

Theorem [FKT2, Theorem 10.4]

$$f^{(-1)}\left(S^{(-1)}\right) \;\subset\; \Theta$$

and

$$\left\{\; \vec{e} \in \Theta \;\middle|\; \theta\left(\vec{e} - \textstyle\int_{\infty_1}^{x} \vec{\omega}\right) \neq 0 \text{ for some}\, x \,\text{in}\, X \;\right\} \;\subset\; f^{(-1)}\left(S^{(-1)}\right)$$

In contrast to the case of compact Riemann surfaces, one can construct zeroes of the theta function that are not in the range of $f^{(-1)}$ by taking limits of $f^{(-1)}([y_1, y_2, \cdots])$ as some of the y_i's tend to infinity. The set

$$\left\{\; \vec{e} \in \Theta \;\middle|\; \theta\left(\vec{e} - \textstyle\int_{\infty_1}^{x} \vec{\omega}\right) = 0 \text{ for all}\, x \,\text{in}\, X \;\right\}$$

is stratified and studied in [FKT2, Theorem 11.1].

The proof of the Riemann Vanishing Theorem is based on the following result, which, in turn, is a residue computation.

Theorem 9.16 *Let* $\vec{e}, \vec{e}\,' \in B$ *be such that*

$$\theta(\vec{e} - \hat{e}_1 + \hat{e}_\nu) \neq 0,\ \theta(\vec{e}\,' - \hat{e}_1 + \hat{e}_\nu) \neq 0 \quad \text{for } \nu = 1, \cdots, m$$

Let $x_1, x_2, \cdots$ *and* $x_1', x_2', \cdots$ *be the zeroes of* $\theta\left(\vec{e} + \int_\infty^x \vec{\omega}\right)$ *and* $\theta\left(\vec{e}\,' + \int_\infty^x \vec{\omega}\right)$*, respectively, such that* $x_j, x_j' \in Y_j$ *for all sufficiently big* j*. Then there are paths* γ_j *joining* x_j *to* x_j' *such that* $\gamma_j \subset Y_j'$ *for all sufficiently large* j

$$\left(\int_{\gamma_1} \omega_1, \int_{\gamma_2} \omega_2, \cdots \right) \in B$$

and

$$\vec{e} - \vec{e}\,' = \sum_{j \geq 1} \int_{\gamma_j} \vec{\omega}$$

(7) **Torelli Theorem:** Let X and X' be Riemann surfaces. If $R_{i,j} = R'_{i,j}$ for all $1 \leq i, j \leq g$ then X and X' are biholomorphic.

[Infinite genus case: X and X' have to obey axioms (all $\mathcal{F}(V)$'s do) restricting size and position of the handles.]

Idea of Proof. The proof mimics the argument of Andreotti [An,GH] for the compact case. We look at how the tangent space $T_{\vec{e}}\Theta$ varies as $\vec{e}$ moves in directions $\vec{v} \in T_{\vec{e}}\Theta$. In particular, we look for directions $\vec{v}$ such that $T_{\vec{e}}\Theta$ is stationary, equivalently such that $\mathbb{C}\nabla\theta(\vec{e})$ is stationary. In other words, we investigate the ramification locus of the Gauss map on the theta divisor. Stationarity is given by the conditions

$$\nabla\theta(\vec{e}) \neq 0,\ \nabla\theta(\vec{e}) \cdot \vec{v} = 0,\ \frac{d}{d\lambda}\nabla\theta(\vec{e}+\lambda\vec{v})\Big|_{\lambda=0} \in \mathbb{C}\nabla\theta(\vec{e}) \tag{S}$$

For generic $\vec{e} = f^{(-1)}(y)$ we find, in [FKT2, Proposition 11.8], necessary and sufficient conditions that the set of $\vec{v}$'s satisfying (S) is of dimension 1 and determine precisely what the set is. The conditions are that the form $\omega_{\vec{e}}(z) = \sum_{k\geq 1} \nabla\theta(\vec{e})_k \omega_k(z)$ have a zero of precisely the right order, namely $\sharp\,\{i \mid \pi(y_i) = \pi(y_j)\}$, at each y_j $j \geq 2$ with one exception, say $y_j = x$. And that $\omega_{\vec{e}}(z)$ have one excess zero, in other words a zero of order

$\sharp\{i \mid \pi(y_i) = x\}+1$, at $z = x$. Then the set of stationary directions $\vec{v} \in T_{\vec{e}}\Theta$ is precisely $\mathbb{C}\vec{\omega}(x)$.

Note that the conditions (S) are stated purely in terms of the function θ. They do not involve the Riemann surface that gave rise to θ. On the other hand the statement "the set of stationary directions $\vec{v} \in T_{\vec{e}}\Theta$ is precisely $\mathbb{C}\vec{\omega}(x)$" does involve the Riemann surface and indeed assigns, in the nonhyperelliptic case, a unique point $x \in X$ to the given $\vec{e} \in \Theta$ [FKT1, Proposition 3.26]. In [FKT2, Proposition 11.10] we find, in the nonhyperelliptic case, a set $E \subset \Theta$ of such $\vec{e}$'s, that is dense in a subset of codimension 1 in Θ. Furthermore, for x in a dense subset of X, the set $\left\{ \vec{e} \in E \,\middle|\, \vec{e} \text{ is paired with } x \right\}$ is, roughly speaking, of codimension 2 in Θ. The pairing of points $\vec{e}$ in E with points $x \in X$ is the principal ingredient in the proof of the Torelli Theorem for the nonhyperelliptic case.

For hyperelliptic Riemann surfaces, the map $x \in X \mapsto \mathbb{C}\vec{\omega}(x)$ is of degree two. Except for a discrete set of points x, $\#\{\ x' \in X \ \big|\ \vec{\omega}(x') \parallel \vec{\omega}(x)\ \} = 2$. At the exceptional points, called Weierstrass points,

$$\#\{\ x' \in X \ \big|\ \vec{\omega}(x') \parallel \vec{\omega}(x)\ \} = 1$$

For each Weierstrass point $b \in X$, we find a set $H^{(b)}$ which is dense in a subset of codimension 1 in Θ with every point $e \in H^{(b)}$ paired, as above, with b.

Using these observations it is possible to recover the Riemann surface X from Θ, which in turn is completely determined by the period matrix of X.

Let

$$z = k_2$$

$$S = \text{ set of holes in } \mathcal{F}(V)^{\text{reg}}$$

$$r(s) = \text{ radius of hole } s$$

$$g = \text{ genus of } \mathcal{F}(V)^{\text{com}}$$

and for each $j > g$

$$\omega_j = w_j(z)dz \quad \text{ on } \mathcal{F}(V)^{\text{reg}}$$

$$s_1(j), s_2(j) = \text{ centres of holes to which } Y_j \text{ hooks}$$

By the Cauchy integral formula

$$w_j(z) = \sum_{s \in S} w_{j,s}(z)$$

where

$$w_{j,s}(z) = -\tfrac{1}{2\pi i} \int_{\partial s} \frac{w_j(\zeta)}{\zeta - z} d\zeta$$

is analytic in $\mathbb{C} \setminus s$.

Lemma. *If* $|z - s| > 3r(s)$ *for all* $s \in S$*, then for all* $s \neq s_1(j), s_2(j)$

$$|w_{j,s}(z)| \leq \frac{3r(s)}{|z-s|^2} \|\omega_j\|_{L^2(Y_s)}$$

For $s = s_\mu(j)$

$$\left| w_{j,s}(z) - \frac{(-1)^{\mu+1}}{2\pi i} \frac{1}{z - s_\mu(j)} \right|$$
$$\leq \frac{3r(s)}{|z-s|^2} \left\| \omega_j - (\phi_j^*) \left(\frac{1}{2\pi i} \frac{dz_1}{z_1} \right) \right\|_{L^2(Y_j)}$$