There Is No Two Dimensional Analogue of Lamé's Equation

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I Introduction

The Lamé equation is the best known of a class of one-dimensional, periodic Schrödinger equations for which all Bloch eigenvalues and multipliers can be explicitly parameterized by meromorphic functions defined on a compact Riemann surface. The purpose of this paper is to prove that there is no non-trivial two-dimensional analogue of this phenomenon. To make the last statement precise, we begin with a review of the basic properties of the Lamé equation.

Fix $\omega_1,\ \omega_2 > 0$. Let

$$\wp(z) \ = \ \frac{1}{z^2} \ + \sum_{\substack{\omega \in 2\omega_1\mathbb{Z} \oplus i2\omega_2\mathbb{Z} \\ \omega \neq 0}} \ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

be the Weierstrass function with primitive periods $2\omega_1$ and $i2\omega_2$. Then

$$2\wp\left(x + i\omega_2\right)$$

is a *real-valued*, real analytic, periodic function of x with primitive period $2\omega_1$. The Lamé equation is

$$-\frac{d^2}{dx^2}\psi \ + \ 2\wp\left(x + i\omega_2\right)\psi \ = \ \lambda\psi \qquad (I.1)$$

A solution $\psi(x,k)$ of (I.1) that satisfies

$$\psi(x + 2\omega_1, k) \ = \ e^{i2\omega_1 k}\psi(x,k) \qquad (I.2)$$

is called a Bloch solution. Recall that

$$2\wp(z) \ = \ -2\frac{d^2}{dz^2}\log\ \sigma(z) \qquad (I.3)$$

where

$$\sigma(z) \;=\; z \prod_{\substack{\omega\in 2\omega_1\mathbb{Z}\oplus i2\omega_2\mathbb{Z}\\ \omega\neq 0}} \left(1-\frac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{1}{2}\frac{z^2}{\omega^2}}$$

and

$$\begin{aligned} \zeta(z) &= \tfrac{d}{dz}\log\sigma(z) \\ &= \tfrac{1}{z} + \sum_{\substack{\omega\in 2\omega_1\mathbb{Z}\oplus i2\omega_2\mathbb{Z}\\ \omega\neq 0}} \tfrac{1}{z-\omega}+\tfrac{1}{\omega}+\tfrac{z}{\omega^2} \end{aligned}$$

There are constants η_1 and η_2 satisfying

$$\eta_1 i\omega_2 - \eta_2\omega_1 \;=\; \pi i$$

such that

$$\sigma(z+2\omega_1) \;=\; -\sigma(z)\; e^{\eta_1(z+\omega_1)}\;, \qquad \sigma(z+i2\omega_2) \;=\; -\sigma(z)\; e^{\eta_2(z+i\omega_2)}$$

and

$$\zeta(z+2\omega_1) \;=\; \zeta(z)+\eta_1\;, \qquad \zeta(z+i2\omega_2) \;=\; \zeta(z)+\eta_2$$

Now set

$$\begin{aligned} \lambda(z) &= -\wp(z) \\ k(z) &= -i\left(\zeta(z) - z\tfrac{\eta_1}{2\omega_1}\right) \\ \xi(z) &= e^{2\omega_1 ik} \;=\; e^{2\omega_1\zeta(z)-z\eta_1} \\ \psi(x,z) &= e^{\zeta(z)x}\tfrac{\sigma(z-x-i\omega_2)}{\sigma(x+i\omega_2)} \end{aligned}$$

By direct calculation

$$\psi(x+2\omega_1,z) \;=\; \xi(z)\;\psi(x,z)$$

and

$$-\frac{d^2}{dx^2}\psi(x,z) \;+\; 2\wp\left(x+i\omega_2\right)\psi(x,z) \;=\; \lambda(z)\psi(x,z) \qquad (I.4)$$

For (I.4), first observe that

$$\frac{d}{dx}\psi(x,z) \;=\; \left(\zeta(z) - \zeta(z - x - i\omega_2) - \zeta(x + i\omega_2)\right)\psi(x,z)$$

Then, differentiate again and apply the standard identities

$$\begin{aligned} \tfrac{1}{4}\left(\tfrac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2 &= \wp(u+v) + \wp(u) + \wp(v) \\ \tfrac{1}{2}\left(\tfrac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right) &= \zeta(u+v) - \zeta(u) - \zeta(v) \end{aligned}$$

Also,

$$\xi(z + 2\omega_1) \;=\; \xi(z)\ , \qquad \xi(z + i2\omega_2) \;=\; \xi(z)$$

Summarizing the discussion above, the energy λ and multiplier $\xi = e^{2\omega_1 ik}$

can be explicitly parameterized by meromorphic functions

$$\lambda(z) \;=\; -\wp(z)\ , \qquad \xi(z) \;=\; e^{2\omega_1\zeta(z) - z\eta_1}$$

on the elliptic curve $\mathbb{C}/2\omega_1\mathbb{Z} \oplus i2\omega_2\mathbb{Z}$ such that the boundary value problem (I.1), (I.2) has a solution if and only if $(\lambda, \xi) = (\lambda(z), \xi(z))$. The only if implication follows from the observation that for almost all z the functions $\psi(x,z)$ and $\psi(x,-z)$ are linearly independent solutions of (I.1) for $\lambda(z) = \lambda(-z)$. In particular, ξ is an algebraic function of λ .

We shall prove that there is no non-trivial two dimensional analogue of this phenomenon. Specifically, for any lattice $\Gamma = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$ and *real valued* function q in $L^2\left(\mathbb{R}^2/\Gamma\right)$ it is impossible to parameterize the energy λ

and multipliers $\xi_1,\ \xi_2$ by meromorphic functions $\lambda(p),\ \xi_1(p),\ \xi_2(p)$ defined on a *compact* complex surface $\mathcal{P}$ such that the boundary value problem

$$-\Delta\psi \ + \ q(x_1,x_2)\psi \ = \ \lambda\psi$$

$$\begin{aligned}\psi(x+\gamma_1) &= \xi_1\psi(x)\\ \psi(x+\gamma_2) &= \xi_2\psi(x)\end{aligned}$$

has a solution ψ if and only if $(\lambda,\xi_1,\xi_2) = (\lambda(p),\xi_1(p),\xi_2(p))$ for some $p\in\mathcal{P}$ unless q is essentially one-dimensional. That is,

$$q(x) \ = \ v(<\beta,x>)$$

where β is a primitive vector in the lattice $\Gamma^\sharp$ dual to Γ , or

$$q(x) \ = \ v_1(<\beta_1,x>) + v_2(<\beta_2,x>)$$

where $\beta_1,\ \beta_2$ are primitive, perpendicular vectors in $\Gamma^\sharp$. Here, $v(t)$, $v_1(t)$, $v_2(t)$ are one-dimensional, periodic, "finite gap" potentials. For example, the Lamé potential $2\wp(t+i\omega_2)$.

We now recall some necessary facts about one-dimensional potentials. See, for example, [Mc] and [MW]. Let v be a real valued function in $L^2\left(\mathbb{R}/T\mathbb{Z}\right)$. The associated one dimensional Bloch variety $B(v)$ is the set of all $(k,\lambda)\in\mathbb{C}\times\mathbb{C}$ such that there is a nontrivial function $\psi\in H^2_{\rm loc}(\mathbb{R}^1)$ satisfying

$$-\frac{d^2}{dx^2}\psi \ + \ v(x)\psi \ = \ \lambda\psi \qquad (I.5)$$

and

$$\psi(x+T) \ = \ e^{iTk}\psi(x) \qquad (I.6)$$

Set $D_k = d/dk + ik$. One can show (see,[KT]) that $B(v)$ is the set of all $(k,\lambda) \in \mathbb{C} \times \mathbb{C}$ obeying

$$(2\cos Tk - 2\cos\sqrt{\lambda})\ \det_2\left((-D_k^2 + v - \lambda)\cdot(-D_k^2 - \lambda)^{-1}\right) \;=\; 0$$

and that

$$(2\cos Tk - 2\cos\sqrt{\lambda})\ \det_2\left((-D_k^2 + v - \lambda)\cdot(-D_k^2 - \lambda)^{-1}\right)$$

is a complex analytic function on $\mathbb{C}\times\mathbb{C}$. Here, $\det_2$ is the second regularized determinant. It follows from representation above that $B(v)$ is a complex analytic subvariety of $\mathbb{C} \times \mathbb{C}$.

Denote by $y_1(x,\lambda)$ and $y_2(x,\lambda)$ the solutions of (I.5) satisfying the initial conditions

$$\begin{aligned} y_1(0,\lambda) &= y_2'(0,\lambda) \;=\; 1 \\ y_1'(0,\lambda) &= y_2(0,\lambda) \;=\; 0 \end{aligned}$$

and

$$\Delta(\lambda) \;=\; y_1(T,\lambda) + y_2'(T,\lambda)$$

Then, ([KT, p. 125])

$$(2\cos Tk - 2\cos\sqrt{\lambda})\ \det_2\left((-D_k^2 + v - \lambda)\cdot(-D_k^2 - \lambda)^{-1}\right) \;=\; 2\cos Tk - \Delta(\lambda)$$

so that

$$\begin{aligned} B(v) &= \{(k,\lambda) \in \mathbb{C}\times\mathbb{C} \mid 2\cos Tk - \Delta(\lambda) \;=\; 0\} \\ &= \left\{(k,\lambda) \in \mathbb{C}\times\mathbb{C} \mid e^{iTk} \text{ is a root of } \xi^2 - \Delta(\lambda)\xi + 1 \;=\; 0\right\} \end{aligned} \qquad (I.7)$$

The Bloch variety $B(v)$ is invariant under translation of k by elements of $\frac{2\pi}{T}\mathbb{Z}$. Consequently, the quotient

$$\begin{aligned} B(v)/\tfrac{2\pi}{T}\mathbb{Z} &= \{(\xi,\lambda)\in\mathbb{C}^*\times\mathbb{C} \mid \xi^2-\Delta(\lambda)\xi+1 \;=\; 0\} \\ &= C(v) \end{aligned}$$

is well defined.

The roots

$$\lambda_0 \;<\; \lambda_1 \le \lambda_2 < \;\lambda_3 \le \lambda_4 < \;\cdots$$

of $\Delta(\lambda) \;=\; \pm 2$ are all real and tend to $+\infty$. The smallest, λ_0 is a simple root of $\Delta(\lambda) \;=\; 2$. The next two, $\lambda_1,\ \lambda_2$ are roots of $\Delta(\lambda) \;=\; -2$ and so on. It follows that $B(v)/\frac{2\pi}{T}\mathbb{Z}$ is an irreducible transcendental hyperelliptic curve. Furthermore, the point (ξ,λ) is singular if and only if

$$\lambda_{2n-1} = \lambda = \lambda_{2n} \qquad\qquad \xi \;=\; (-1)^n$$

If $\lambda_{2n-1} = \lambda_{2n}$ for all but a finite number of subscripts n , the potential v is by definition “finite gap”. In this case the normalization $\mathcal{N}$ of $C(v) = B(v)/\frac{2\pi}{T}\mathbb{Z}$ is a compact Riemann surface with one point removed. We have

$$\begin{array}{ccccc} \mathcal{N} & \longrightarrow & C(v) & & (\xi,\lambda) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \longleftarrow & \mathbf{C} & & \lambda \end{array} \qquad (I.8)$$

In particular, the normalization map across the top parameterizes the energy λ and multiplier ξ by meromorphic functions on $\mathcal{N}$ just as for the Lamé potential. Conversely, if the normalization $\mathcal{N}$ of $B(v)/\frac{2\pi}{T}\mathbb{Z}$ is a compact Riemann surface with one point removed and (I.8) commutes, the potential v is finite gap. We remark that the set of finite gap potentials is dense in $L^2_{\text{real}}(\mathbb{R}/T\mathbb{Z})$. See [GT], [M].

The potentials of the last paragraph are referred to as finite gap since the complement of the continuous spectrum of the associated Schrödinger operator is a finite set of intervals. As we have explained, the finite gap condition is equivalent to the statement that the normalization of $B(v)/\frac{2\pi}{T}\mathbb{Z}$ is a compact Riemann surface with one point removed. In other words, the finite gap potentials are those with an algebraic Bloch structure.

To illustrate (I.8), observe that

$$\Delta(z) \;=\; 2\cos i\left(\zeta(z) - z\frac{\eta_1}{2\omega_1}\right)$$

for the Lamé potential $2\wp(x+i\omega_2)$. Then, the diagram

$$\begin{array}{ccccccc} z & \mathbb{C}/2\omega_1\mathbb{Z}\oplus i2\omega_2\mathbb{Z} \;-\; \{0\} & \longrightarrow & C(2\wp(\cdot+i\omega_2)) & (\xi,\lambda) \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \lambda(z) & \mathbb{P}^1 & \longleftarrow & \mathbf{C} & \lambda \end{array}$$

commutes. The map across the top is $z \longrightarrow (\xi(z),\lambda(z))$. Thus, the transcendental curve $B(2\wp(\cdot+i\omega_2))/\frac{2\pi}{T}\mathbb{Z}$ is covered by the complement of $\{0\}$ in the compact curve $\mathbb{C}/2\omega_1\mathbb{Z}\oplus i2\omega_2\mathbb{Z}$.

Finally, suppose v is a finite gap potential and $\mathcal{N}$ the normalization of $B(v)/\frac{2\pi}{T}\mathbb{Z}$. Let Θ be the Riemann theta function for $\mathcal{N}$. Then there are vectors Ω_1 and Ω_2 such that

$$v(x) \;=\; -2\frac{d^2}{dx^2}\log\,\Theta\left(x\Omega_1+\Omega_2\right) \qquad (I.9)$$

The representation (I.9) generalizes (I.3). Our discussion of one dimensional potentials is finished.

Let $\Gamma = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$ be a lattice of maximal rank in $\mathbb{R}^2$ and let q be a *real valued* function in $L^2\left(\mathbb{R}^2/\Gamma\right)$. The associated two dimensional Bloch variety $B(q)$ is the set of all $(k_1, k_2, \lambda) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ such that there is a nontrivial function $\psi \in H^2_{\rm loc}(\mathbb{R}^2)$ satisfying

$$-\Delta\psi \ + \ q(x_1, x_2)\psi \ = \ \lambda\psi$$

and

$$\begin{aligned} \psi(x+\gamma_1) &= e^{i<k,\gamma_1>}\psi(x) \\ \psi(x+\gamma_2) &= e^{i<k,\gamma_2>}\psi(x) \end{aligned}$$

where $k = (k_1, k_2)$. It is shown in [KT], by means of a regularized determinant, that $B(q)$ is a complex analytic hypersurface of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$. Define the projection π by

$$\begin{array}{cc} B(q) & (k_1, k_2, \lambda) \\ \downarrow & \downarrow \\ \mathbb{C} & \lambda \end{array}$$

and the Fermi curves by

$$F_\lambda \ = \ \pi^{-1}(\lambda)$$

Again, the lattice

$$\Gamma^\sharp \ = \ \left\{ b \in \mathbb{R}^2 \ | \ < \gamma, b > \in 2\pi\mathbb{Z} \text{ for all } \gamma \in \Gamma \right\}$$

dual to Γ acts by translation on $B(q)$. We can define the quotients

$$B(q)/\Gamma^\sharp \qquad C_\lambda \ = \ F_\lambda/\Gamma^\sharp$$

and the projection π'

$$\begin{array}{cc} B(q)/\Gamma^\sharp & (\xi_1, \xi_2, \lambda) \\ \downarrow & \downarrow \\ \mathbb{C} & \lambda \end{array}$$

It is proven in [KT, p.137] that $B(q)/\Gamma^\sharp$ is always irreducible.

By analogy with (I.8), we consider the class of *real valued* functions q in $L^2\left(\mathbb{R}^2/\Gamma\right)$ for which there is a compact complex analytic variety $\mathcal{P}$, a holomorphic projection map $\hat{\pi}$

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ \mathbb{P}^1 \end{array} \qquad (I.10)$$

a finite union $\mathcal{D}$ of curves on $\mathcal{P}$ and a finite, dominant holomorphic map (morphism, if $\mathcal{P}$ or $B(q)/\Gamma^\sharp$ is singular) Φ from $\mathcal{P}-\mathcal{D}$ to $B(q)/\Gamma^\sharp$ such that the diagram

$$\begin{array}{ccccc} & & \Phi & & \\ & \mathcal{P}-\mathcal{D} & \longrightarrow & B(q)/\Gamma^\sharp & \\ \hat{\pi} & \downarrow & & \downarrow & \pi' \\ & \mathbb{P}^1 & \longleftarrow & \mathbf{C} & \end{array} \qquad (I.11)$$

commutes. It is a direct consequence of (I.11) that for all λ the normalization of $C_\lambda = F_\lambda/\Gamma^\sharp$ is a compact Riemann surface with a finite set of points removed. In other words, after normalizing and closing the transcendental curves C_λ one obtains a holomorphic family of compact algebraic curves.

Suppose v_1 and v_2 are one-dimensional finite gap potentials with periods T_1 and T_2. It is easy to see by separating variables that the two-dimensional potential

$$q(x_1,x_2) \;=\; v_1(x_1)+v_2(x_2)$$

in $L^2\left(\mathbb{R}^2/T_1\mathbb{Z}\oplus T_2\mathbb{Z}\right)$ belongs to the class introduced above. In fact, the Bloch variety $B(v_1+v_2)/\Gamma^\sharp$ is the fiber product of $C(v_1)$ and $C(v_2)$. Also, by separation of variables, the potential

$$q(x_1,x_2) \;=\; v(<\alpha,x>)$$

belongs to this class. Here, v is finite gap and α is any vector in Γ .

For any $q\in L^2\left(\mathbb{R}^2/\Gamma\right)$ and $\gamma\in\Gamma$ set

$$q_\gamma(x) \;=\; \sum_{\substack{b\in\Gamma^\sharp\\ <b,\gamma>=0}} \hat{q}(b)\; e^{i<b,x>}$$

where

$$\hat{q}(b) \;=\; \frac{1}{|\mathbb{R}^2/\Gamma|}\int_{\mathbb{R}^2/\Gamma} q(x)\; e^{-i<b,x>}$$

We have the

Theorem *Let* q *be a real-valued function in* $L^2\left(\mathbb{R}^2/\Gamma\right)$ *. Suppose that there is a compact complex analytic variety* $\mathcal{P}$ *, a holomorphic projection map* $\hat{\pi}$

$$\begin{array}{c}\mathcal{P}\\ \downarrow\\ \mathbb{P}^1\end{array}$$

a finite union $\mathcal{D}$ *of curves on* $\mathcal{P}$ *and a finite, dominant holomorphic map*

Φ from $\mathcal{P}-\mathcal{D}$ to $B(q)/\Gamma^\sharp$ such that the diagram

$$\begin{array}{ccccc} & & \Phi & & \\ & \mathcal{P}-\mathcal{D} & \longrightarrow & B(q)/\Gamma^\sharp & \\ \hat{\pi} & \downarrow & & \downarrow & \pi' \\ & \mathbb{P}^1 & \longleftarrow & \mathbf{C} & \end{array}$$

commutes. Then

$$q \;=\; q_\gamma(x)$$

for some primitive $\gamma \in \Gamma$ and q_γ is finite gap, or

$$q \;=\; q_{\gamma_1}(x) + q_{\gamma_2}(x) - \hat{q}(0)$$

for perpendicular, primitive vectors $\gamma_1,\ \gamma_2\ \in \Gamma$ and $q_{\gamma_1},\ q_{\gamma_2}$ are finite gap. The converse also holds.

We actually prove a stronger result: Suppose that q is a real-valued function in $L^2\left(\mathbb{R}^2/\Gamma\right)$ such that for all $\lambda \in \mathbb{C}$ the normalization of the curve $C_\lambda \;=\; F_\lambda/\Gamma^\sharp$ is a compact Riemann surface $\mathcal{C}_\lambda$ with a discrete set of points removed and furthermore that the genus $g\left(\mathcal{C}_\lambda\right)$ is uniformly bounded in λ . Then, the conclusion of the Theorem holds.

In 1976 Dubrovin, Krichever and Novikov [DKN] constructed two-dimensional potentials such that for *one* energy λ the normalization of $C_\lambda \;=\; F_\lambda/\Gamma^\sharp$ is a compact Riemann surface with a discrete set of points removed (Also see, [NV]). Novikov has asked for regular potentials such that for all energies λ the normalization of $C_\lambda \;=\; F_\lambda/\Gamma^\sharp$ is a compact Riemann surface with a discrete set of points removed. It follows from our results that there are no nontrivial examples in $L^2\left(\mathbb{R}^2/\Gamma\right)$. We remark that in [CV] special potentials are constructed from simple Lie algebras that appear to have an algebraic Bloch structure. However, these potentials are either complex valued or have nonintegrable singularities.

II Outline of the Proof

If q is separable as a sum of finite gap potentials then, as we remarked above, one constructs a parametrization of $B(q)/\Gamma^\sharp$ using the normalizations of the spectral curves of the one-dimensional finite gap potentials. For proving the converse we can restrict ourselves to the case that the average $\hat{q}(0)$ of q is zero.

The assumption that there exists a parametrization of $B(q)/\Gamma^\sharp$ as in $(I.10)$, $(I.11)$ implies that for each $\lambda \in \mathbb{C}$ the normalization of C_λ is dominated by the normalization of $\hat{\pi}^{-1}(\lambda) - D$, which is a curve of finite and bounded genus. Therefore there is a constant B such that the rank of the homology of the normalization of C_λ is bounded above by B. In other words if $Sing(C_\lambda)$ denotes the set of singular points of C_λ and $H_1^{(r)}(C_\lambda, \mathbb{Z})$ denotes the image of the map

$$H_1\left(C_\lambda - Sing(C_\lambda), \mathbb{Z}\right) \rightarrow H_1(C_\lambda, \mathbb{Z})$$

induced by inclusion then

$$rank\ H_1^{(r)}(C_\lambda, \mathbb{Z}) \leq B \qquad \text{for all } \lambda \in \mathbb{C}$$

We use the directional compactification of $B(q)$ introduced in [KT] to construct elements of $H_1^{(r)}(C_\lambda, \mathbb{Z})$ for $\lambda \in \mathbb{R}$ close to $-\infty$. Recall that for each primitive vector $\gamma \in \Gamma$ there is a plane $E_\gamma := E_{\gamma,0}$ in the cradle constructed in [KT], Section 2 such that for any $\theta > 0$ the closure of the intersection of $B(q)$ with $\Sigma(\theta) := \left\{(k,\lambda) \in \mathbb{C}^2 \times \mathbb{C} \mid \arg(k_1^2 + k_2^2) \notin (-\theta, \theta)\right\}$ meets E_γ along a curve isomorphic to $B(q_\gamma)$, in short

$$\overline{B(q) \cap \Sigma(\theta)} \cap E_\gamma \cong B(q_\gamma) \ .$$

We identify $B(q_\gamma)$ with this subset of E_γ.

In addition it is shown in [KT] that near smooth points of $B(q_\gamma)$ the space $\overline{B(q) \cap \Sigma(\theta)}$ has a locally cone-like structure. More precisely we have

Proposition 1 *Let K' be a compact subset of $B(q_\gamma)$ consisting of smooth points of $B(q_\gamma)$ only. Then there is a map*

$$\psi : K' \times (-\varepsilon, \varepsilon) \times (-\theta, \theta) \longrightarrow \overline{B(q) \cap \Sigma(\theta)}$$

such that

(i) $\psi(s, 0, \varphi) = s$ *for all* $s \in K'$

$\psi(s, r, \varphi) \notin E_\gamma$ *for* $r \neq 0$

(ii) the restriction of ψ *to* $\{(s, r, \varphi) \in K' \times (-\varepsilon, \varepsilon) \times (-\theta, \theta) \mid r \neq 0\}$ *is a diffeomorphism onto its image*

(iii) the diagram

$$\begin{array}{ccc}
K' \times \{r \in \mathbb{R} \mid 0 < |r| < \varepsilon\} \times (-\theta, \theta) & \xrightarrow{\ \psi\ } & \psi\left(K' \times \{r \in \mathbb{R} \mid 0 < |r| < \varepsilon\} \times (-\theta, \theta)\right) \\
(s, r, \varphi) \mapsto r^2 e^{i\varphi} & \searrow \quad \mathbb{R} \quad \swarrow & -\frac{1}{\lambda} \leftarrow\!\shortmid (k, \lambda)
\end{array}$$

commutes

(iv) ψ is compatible with the action of $\Gamma^\sharp$, i.e.

if $s, s' \in K'$ and $b \in \Gamma^\sharp$ such that $b \cdot s = s'$

then $b \cdot \psi(s, r, \varphi) = \psi(s', r, \varphi)$ for all r, φ.

The differentiability of ψ and the situation near singular points will be investigated in Section III.

In any case, whenever K is a compact subset of the set of smooth points of the spectral curve $C(q_\gamma) = B(q_\gamma)/\left\{\gamma \in \Gamma^\sharp \mid \langle b, \gamma\rangle = 0\right\}$ then, by Proposition 1, for each $\lambda \in \mathbb{R}$ sufficiently close to $-\infty$ there is a subset $K_\lambda \subset C_\lambda$ diffeomorphic to K. If γ, γ' are linearly independent primitive vectors in Γ and K resp. K' are compact subsets of $C(q_\gamma) - Sing\,(C(q_\gamma))$ resp. $C(q_{\gamma'}) - Sing\,(C(q_{\gamma'}))$ then for $\lambda \in \mathbb{R}$ sufficiently close to $-\infty$ the sets K_λ and K'_λ are disjoint, since E_γ and $E_{\gamma'}$ were obtained by blowing up points in a cradle that lie in different $\Gamma^\sharp$-orbits.

Now assume that at least g gaps in the spectrum of the one-dimensional Schrödinger operator associated to q_γ are open. Then there exist cycles

$a_1, \ldots, a_g, b_1, \ldots, b_g$ in a compact subset K of $C(q_\gamma) - Sing\,(C(q_\gamma))$ whose intersection numbers fulfil $a_i \cdot a_j = b_i \cdot b_j = 0$ for $i, j = 1, \ldots, g$ and $a_i \cdot b_j = \delta_{ij}$. As K_λ is diffeomorphic to K there are cycles $a'_1, \ldots, a'_g, b'_1, \ldots, b'_g$ in K_λ with the same intersection properties. In particular $a'_1, \ldots, a'_g, b'_1, \ldots, b'_g$ represent independent elements of $H_1^{(r)}(C_\lambda, \mathbb{Z})$. So $2g \leq B$, and q_γ is a finite gap potential. If $\gamma_1, \ldots, \gamma_r$ are pairwise linearly independent primitive vectors in $\Gamma^\sharp$ such that for each $j = 1, \ldots, r$ at least one gap in the spectrum of the one dimensional Schrödinger operator associated to q_{γ_j} is open then the elements in $H_1^{(r)}(C_\lambda, \mathbb{Z})$ constructed as above are linearly independent. This shows that for all but finitely many primitive vectors $\gamma \in \Gamma$ the spectrum of the one dimensional operator associated to q_γ has no gaps. By Borg's theorem [B] this implies that $q_\gamma = 0$ for all but finitely many γ. Thus we have shown:

Lemma II.1 *Let $q \in L^2_{\mathbb{R}}(\mathbb{R}^2/\Gamma)$ with $\hat{q}(0) = 0$. If $(I.10), (I.11)$ hold then there are pairwise linearly independent primitive vectors $\gamma_1, \ldots, \gamma_r \in \Gamma$ such that*

$$q(x) = q_{\gamma_1}(x) + \ldots + q_{\gamma_r}(x) \ ,$$

and each q_{γ_1} is a finite gap potential.

Finite gap potentials are real analytic, so Lemma II.1 shows in particular that any L^2-potential for which $(I.10)$, $(I.11)$ holds is real analytic.

We now relax the condition that q be real and consider, possibly complex valued, potentials of the form $q(x) = q_{\gamma_1}(x) + \ldots + q_{\gamma_r}(x)$ where $\gamma_1, \ldots, \gamma_r$

are pairwise linearly independent primitive vectors of Γ and each q_{γ_i} is a finite gap potential. In this situation each of the spectral curves $C(q_\gamma)$ has infinitely many double points. We now want to construct cycles in $H_1^{(r)}(C_\lambda, \mathbb{Z})$ by opening up these double points. To make this precise, we use the following notation.

Definition A double point p of $C(q_\gamma)$ *opens up* if for one (and then every) point $p' \in B(q_\gamma)$ above p there is a neighbourhood U of p' in $\overline{\Sigma(\theta)}$, an integer $n \geq 1$ and a homeomorphism from a neighbourhood U' of 0 in

$$\{(x,y,t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} \mid xy = t^n,\ t \geq 0\}$$

to

$$\overline{\{(k,\lambda) \in B(q) \cap U \mid \lambda \in \mathbb{R},\ \lambda < 0\}} =: U''$$

which is a diffeomorphism on $\{(x,y,t) \in U' \mid t \neq 0\}$ such that the diagram

$$\begin{array}{ccccc} (x,y,t) & U' & \xrightarrow{\varphi} & U'' \supset U'' \setminus E_\gamma & (k,\lambda) \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ t & \mathbb{R} & \longrightarrow & \mathbb{R} & \lambda \\ & t & \longmapsto & -\frac{1}{t^2} & \end{array}$$

commutes.

Now assume that at least g double points of $C(q_\gamma)$, say $p_1, \ldots, p_g$ open up. Let V_i be a small neighbourhood of p_i in $C(q_\gamma)$, and K a compact subset of $C(q_\gamma)$ with smooth boundary such that $V_i \subset C(q_\gamma)$ and $K \setminus \{p_1, \ldots, p_g\}$ is connected. As above one constructs, for $\lambda \in \mathbb{R}$ sufficiently close to $-\infty$, two disjoint subsets $K_\lambda^{(i)}$ of F_λ and maps $\psi_i : K_\lambda^{(i)} \to K$ which are $\Gamma^\sharp$-compatible diffeomorphisms between $\psi_i^{-1}(K \setminus \{p_1, \ldots, p_g\})$ and $K \setminus \{p_1, \ldots, p_g\}$ and such that $\psi_i^{-1}(V_j)$ is diffeomorphic to a hyperbola. Then $K_\lambda^{(i)}$ is diffeomorphic to a Riemann surface of genus at least g with a certain number of holes. In particular there exist cycles $a'_1, \ldots, a'_g, b'_1, \ldots, b'_g$ on $K_\lambda^{(i)}$ with $a'_i \cdot a'_j = b'_i \cdot b'_j = 0$ for $i, j = 1, \ldots, g$, $a'_i \cdot b'_j = \delta_{ij}$. Again these cycles represent linearly independent elements of $H_1^{(r)}(C_\lambda, \mathbb{Z})$. So we have shown

Lemma II.2 *Let $q \in L^2_{\mathbb{C}}(\mathbb{R}^2/\Gamma)$ with $\hat{q}(0) = 0$. If $(I.10), (I.11)$ hold then for each primitive $\gamma \in \Gamma$ only finitely many double points of $C(q_\gamma)$ open up.*

In fact the argument shows that there are only finitely many γ for which a double point may open up; but we do not need this. The next step is to develop a criterion for double points to open up. If $\gamma \in \Gamma$ is a primitive vector such that $q_\gamma = 0$ then

$$B(q_\gamma) = \bigcup_{\substack{d\in\Gamma^\sharp \\ \langle d,\gamma\rangle = 0}} \bar{B}_d \cap E_\gamma$$

where B_d denotes the paraboloid $B_d = \Big\{(k,\lambda) \in \mathbb{C}^2 \times \mathbb{C} \mid (k+d)^2 - \lambda = 0\Big\}$, see [KT], Section 2. If $d \in \Gamma^\sharp \setminus \{0\}$ with $\langle d,\gamma\rangle = 0$ then $\bar{B}_0 \cap E_\gamma$ and $\bar{B}_d \cap E_\gamma$ intersect in one point, which we call p'_d. By p_d we denote its image in $C(q_\gamma)$. In Section IV we prove

Proposition 2 *Let $q \in L^2_{\mathbb{C}}(\mathbb{R}^2/\Gamma)$ such that $\sum_{b\in\Gamma^\sharp} |\hat{q}(b)|\,(1+|b|^{12}) < \infty$. Let γ be a primitive vector in Γ such that $q_\gamma = 0$, and let $d \in \Gamma^\sharp \setminus \{0\}$ with $\langle d,\gamma\rangle = 0$. Define*

$$\Delta_d := |\gamma|^2 \cdot \sum_{\substack{b,c\in\Gamma^\sharp \\ b+c=d}} \frac{\langle b,c\rangle}{\langle b,\gamma\rangle\,\langle c,\gamma\rangle} \hat{q}(b)\hat{q}(c)$$

If $\Delta_d\Delta_{-d}$ is non-zero then the double point p_d of $C(q_\gamma)$ opens up. When q is real $\Delta_{-d} = \bar{\Delta}_d$, so that the double point p_d opens up whenever $\Delta_d \neq 0$.

Under the assumption that q is a sum of finitely many one dimensional real valued finite gap potentials as in Lemma II.1 one can show that infinitely many of the numbers Δ_d are non-zero. Precisely, we show in Section V.

Proposition 3 *Assume that*

$$q(x) = q_{\gamma_1}(x) + \ldots + q_{\gamma_r}(x)$$

with $\gamma_1, \ldots, \gamma_r$ pairwise linearly independent primitive vectors in Γ, such that each q_{γ_j} is a non-constant real valued finite gap potential. Assume furthermore that $r \geq 3$ or $\langle \gamma_1, \gamma_2 \rangle \neq 0$ if $r = 2$. Then there is a primitive vector $\gamma \in \Gamma$ such that $q_\gamma = 0$, and infinitely many $d \in \Gamma^\sharp$ with $\langle d, \gamma \rangle = 0$ such that $\Delta_d \neq 0$.

Clearly the Theorem follows by putting Lemma II.1, Lemma II.2, Proposition 2 and Proposition 3 together.

III Smoothness of the Directional Compactification

Let γ be a primitive vector in Γ. We investigate the local behaviour of $\overline{B(q) \cap \Sigma(\theta)}$ near E_γ.

Proposition 4 *Suppose that $\sum\limits_{b \in \Gamma^\sharp} |\hat{q}(b)| \left(1 + |b|^{2k}\right) < \infty$. Let $S \subset E_\gamma$ be a compact subset with smooth boundary ∂S. Then for every $0 < \theta < \pi$ there is $\varepsilon > 0$, a C^k-parametrization*

$$\psi : S \times (-\varepsilon, \varepsilon) \times (-\theta, \theta) \rightarrow (\mathbb{C}^2 \times \mathbb{C}) \cup E_\gamma$$

and a C^k-map

$$F : S \times (-\varepsilon, \varepsilon) \times (-\theta, \theta) \rightarrow \mathbb{C}$$

such that

(i) $\psi(s, 0, \varphi) = s$ *for all* $s \in S$,

$\psi(s, r, \varphi) \notin E_\gamma$ *for* $r \neq 0$ *and*

the restriction of ψ *to* $\{(s, r, \varphi) \in S \times (-\varepsilon, \varepsilon) \times (-\theta, \theta) \mid r \neq 0\}$ *is a diffeomorphism onto its image.*

(ii) $\psi^{-1}\left(B(q) \cup B(q_\gamma)\right) = F^{-1}(0)$,

i.e. F *is a local equation for the Bloch variety*

(iii) The derivative of F *with respect to* s *has maximal rank at each point* $(s, 0, \varphi)$ *where* s *is a smooth point of* $B(q_\gamma)$

(iv) The diagram

$$\begin{array}{ccc}
S \times \{r \in \mathbb{R} \mid 0 < |r| < \varepsilon\} \times (-\theta, \theta) & \xrightarrow{\ \psi\ } & \psi\left(S \times \{r \in \mathbb{R} \mid 0 < |r| < \varepsilon\} \times (-\theta, \theta)\right) \\
(s, r, \varphi) & & (k, \lambda) \\
\searrow & & \swarrow \\
r^2 e^{i\varphi} \quad & \mathbb{R} & \quad -\frac{1}{\lambda}
\end{array}$$

commutes

(v) ψ *is compatible with the action of* $\Gamma^\sharp$, *i.e.*

if $s, s' \in S$ *and* $b \in \Gamma^\sharp$ *such that* $b \cdot s = s'$ *then*

$b \cdot \psi(s, r, \varphi) = \psi(s', r, \varphi)$ *for all* r, φ.

Proof: After rotating the lattice we may assume that $\gamma = (0, \gamma_2)$. Recall from [KT], p. 133 that there are coordinates (κ, u, v) in a neighbourhood of E_γ such that

$$k_1 = \kappa, \;\; k_2 = 1/v, \;\; \lambda = u + 1/v^2 \qquad (III.1)$$

and E_γ is given by $v = 0$. Let S be a compact subset of E_γ as in Proposition 4, and let G be a finite subset of $\Big\{ b \in \Gamma^\sharp \mid \langle b, \gamma \rangle = 0 \Big\}$ such that $(\kappa + c_1)^2 - u \neq 0$ for all $(\kappa, u) \in S$, $c = (c_1, 0) \notin G$. Let $H(\kappa, u, v) = (H_{b,c})_{b,c \in \Gamma^\sharp}$ be the matrix with entries

$$H_{b,c}(\kappa, u, v) = \begin{cases} \delta_{bc} + \frac{v \hat{q}(b-c)}{2c_2 + v\left((\kappa+c_1)^2 + c_2^2 - u\right)} & \text{if } c_2 \neq 0 \\ \delta_{bc} + \frac{\hat{q}(b-c)}{(\kappa+c_1)^2 - u} & \text{if } c_2 = 0,\ c \notin G \\ ((\kappa + c_1)^2 - u)\, \delta_{b\,c} + \hat{q}(b-c) & \text{if } c \in G \end{cases} \qquad (III.2)$$

In [KT], Section 2 it is shown that for every $0 < \theta < \pi$ there is an $\varepsilon > 0$ such that for all $(\kappa, u) \in S$ and

$$v \in \Sigma(\theta, \varepsilon) := \Big\{ z \in \mathbb{C} \mid |z| < \varepsilon, \arg(-z^2) \in (-\theta, \theta) \Big\}$$

the matrix $H(\kappa, u, v) - \mathbf{1}$ is Hilbert-Schmidt, depends continuously in the Hilbert Schmidt norm on $(\kappa, u, v) \in S \times \Sigma(\theta, \varepsilon)$, and that

$$\begin{aligned} &\overline{\{(\kappa, u, v) \in S \times \Sigma(\theta, \varepsilon) \mid v \neq 0\} \cap B(q)} \\ &\qquad = \{(\kappa, u, v) \in S \times \Sigma(\theta, \varepsilon) \mid \det{}_2 H(\kappa, u, v) = 0\} \end{aligned}$$

If G was chosen sufficiently big and ε sufficiently small then for all $(\kappa, u, v) \in S \times \Sigma(\theta, \varepsilon)$

$$\sum_{\substack{c \in \Gamma^\sharp, c_2 = 0 \\ c \notin G}} \frac{1}{\left|(\kappa + c_1)^2 - u\right|^2} + \sum_{\substack{c \in \Gamma^\sharp \\ c_2 \neq 0}} \frac{|v|^2}{\left|2c_2 + v((\kappa + c_1)^2 + c_2^2 - u)\right|^2} < \frac{1}{4||q||_2^2} \qquad (III.3)$$

so that the subblock

$$\mathbf{1} + W := (H_{bc})_{b,c \in \Gamma^\sharp \setminus G}$$

is invertible. In this case we block H in the form

$$H = \begin{array}{r} G\{ \\ \Gamma^\sharp \setminus G\{ \end{array} \overset{\begin{array}{cc} \overbrace{G} & \overbrace{\Gamma^\sharp \setminus G} \end{array}}{\begin{pmatrix} R & V \\ U & \mathbf{1} + W \end{pmatrix}} \tag{III.4}$$

Then

$$\det_2 H = (\det_2(\mathbf{1} + W)) \det M \tag{III.5}$$

where $M = M(\kappa, u, v)$ is the finite $G \times G$-matrix

$$M = R - V(\mathbf{1} + W)^{-1} U \ . \tag{III.6}$$

Proposition 4 will be a direct consequence of

Proposition 5 *Assume that* $\sum_{b \in \Gamma^\sharp} |\hat{q}(b)| \left(1 + |b|^{2k}\right) < \infty$. *Let*

$$\psi : S \times (-\varepsilon, \varepsilon) \times (-\theta, \theta) \to (\mathbb{C}^2 \times \mathbb{C}) \cup E_\gamma$$

be the map given by

$$(\kappa, u, r, \varphi) \to (\kappa, u, v(u, r, \varphi))$$

where $v(u, r, \varphi) := \frac{\sqrt{-1} r e^{i\varphi/2}}{\sqrt{1 + ur^2 e^{i\varphi}}}$. *If* ε *was chosen sufficiently small then* $M \circ \psi$ *is a* C^k*-differentiable map from* $S \times (-\varepsilon, \varepsilon) \times (-\theta, \theta)$ *to the space of* $(G \times G)$*-matrices.*

To see that Proposition 4 follows from Proposition 5 put $F(\kappa, u, r, \varphi) = \det M\,(\psi(\kappa, u, r, \varphi))$. Part (i) of Proposition 4 is obvious from the construction. Part (ii) follows from $(III.5)$ since $\det_2(\mathbf{1} + W)^{-1}$ is nowhere zero and

$\det_2 H = 0$ is an equation for $\overline{(B(q) \cap Im\,\psi)}$. Part (iii) follows from the fact that $\det H(\kappa, u, 0) = 0$ is a holomorphic reduced equation for $B(q_\gamma)$ on E_γ. Parts (iv) and (v) are obvious from the choice of ψ and $(III.1)$.

Proof of Proposition 5: Since ψ is a C^∞-map and for sufficiently small ε there is θ_0 with $\theta < \theta_0 < \pi$ such that the image of ψ is contained in $\left\{(\kappa, u, v) \in S \times \mathbb{C} \mid v = \sqrt{-1}re^{i\varphi} \text{ for some } r, \varphi \text{ with } |r| < 2\varepsilon, \varphi \in \left(-\frac{\theta_0}{2}, \frac{\theta_0}{2}\right)\right\}$ it suffices to show for each $\theta < \frac{\pi}{2}$ there is an $\varepsilon > 0$ such that $M(\kappa, u, \sqrt{-1}re^{i\varphi})$ is a C^k-differentiable matrix valued function of $(\kappa, u) \in S$, $r \in (-\varepsilon, \varepsilon)$, $\varphi \in (-\theta, \theta)$.

Three ingredients are used to control the matrix M and its derivatives. The first is the decay of the Fourier coefficients $\hat{q}(b - c)$ as $b - c$ gets large. For a general $A_{b,c} : \Gamma^\sharp \times \Gamma^\sharp \to \mathbb{C}$ we enforce decay between b and c through the norm

$$|||A||| = \max\left\{\sup_{b\in\Gamma^\sharp} \sum_{c\in\Gamma^\sharp} |A_{b,c}| \left[1 + |b - c|^2\right]^{k+1}, \sup_{c\in\Gamma^\sharp} \sum_{b\in\Gamma^\sharp} |A_{b,c}| \left[1 + |b - c|^2\right]^{k+1}\right\}$$

This norm obeys

$$\begin{aligned} |||AB||| &\leq |||A|||\ |||B||| \\ ||A|| &\leq |||A||| \end{aligned}$$

where $||A||$ is the operator norm of A viewed as the kernel of an operator on $\ell^2(\Gamma^\sharp)$. See [FKT, p. 261, 230]. We select an arbitrary positive constant Q and consider all potentials q for which

$$|||q||| \leq Q\ .$$

The second ingredient is the fact that the denominator

$$\left\{ \begin{array}{ll} 2c_2 + v\,((\kappa + c_1)^2 + c_2^2 - u) & \text{if } c_2 \neq 0 \\ (\kappa + c_1)^2 - u & \text{if } c_2 = 0,\, c \notin G \end{array} \right\}$$

that appears in V and W remains bounded away from zero. When $c_2 = 0$ this is part of the definition of G. When $c_2 \neq 0$, we prove it in

Lemma III.1 *There exist constants $E(S,\theta) > 0$ and $D(S,\theta) > 0$ such that*

$$c_2 \neq 0,\;\; (\kappa, u) \in S,\;\; |\varphi| \leq \theta < \frac{\pi}{2},\;\; |r| \leq E(S,\theta)$$

implies

$$\left|2c_2 + v\left((\kappa + c_1)^2 + c_2^2 - u\right)\right| \geq D(S,\theta)\;\;,$$

where

$$v = i\,r\,e^{i\varphi}\;.$$

Proof of Lemma III.1: We first evaluate the real and imaginary parts

$$\begin{aligned} R &= Re\,(-i\,e^{i\varphi}\,[2c_2 + v\,((\kappa + c_1)^2 + c_2^2 - u)]) \\ &= -2c_2 \sin\varphi + r\,[(c_1 + Re\kappa)^2 - (Im\,\kappa)^2 + c_2^2 - Re\,u] \\ I &= Im\,(-i\,e^{i\varphi}\,[2c_2 + v\,((\kappa + c_1)^2 + c_2^2 - u)]) \\ &= -2c_2 \cos\varphi + r\,[2(c_1 + Re\kappa)^2 Im\,\kappa - Im\,u]\;\;. \end{aligned}$$

There are two possiblities to be considered. Either

$$|r\,[2(c_1 + Re\kappa) Im\,\kappa - Im\,u]| \leq |c_2 \cos\varphi|\;\;,$$

in which case

$$|I| \geq |c_2 \cos\varphi| \geq \frac{2\pi}{|\gamma_2|} \cos\theta\;\;,$$

or

$$|r\,[2(c_1 + Re\kappa) Im\,\kappa - Im\,u]\,| \geq |c_2 \cos\varphi| \ .$$

In the latter case, since S is compact,

$$\begin{aligned} 2|r|\,|c_1 + Re\kappa|\,|Im\,\kappa| &\geq |c_2\cos\varphi| - |r|\,|Im\,u| \\ &\geq \tfrac{1}{2}|c_2|\cos\theta \end{aligned}$$

provided $E(S,\theta)$ is small enough. This implies, again by the compactness of S, that

$$|c_1 + Re\kappa| \geq D_1(S,\theta)\frac{|c_2|}{|r|} \ .$$

Hence, if $E(S,\theta)$ is small enough, the real part

$$\begin{aligned} |R| &\geq |r|\left[D_1^2\tfrac{c_2^2}{r^2} - c_2^2 - D_2^2\right] - 2|c_2| \\ &\geq |r| D_3 \tfrac{c_2^2}{r^2} - 2|c_2| \\ &\geq D_4 \tfrac{c_2^2}{|r|} \\ &\geq D_4 \left(\tfrac{2\pi}{|\gamma_2|}\right)^2 \tfrac{1}{E(S,\theta)} \quad . \end{aligned}$$

It suffices to choose

$$D(S,\theta) = \min\left(\frac{2\pi}{|\gamma_2|}\cos\theta,\ D_4\frac{4\pi^2}{\gamma_2^2}\frac{1}{E(S,\theta)}\right) \ .$$

□

The third and final ingredient used to control the derivatives of M is the observation that each derivative $\frac{\partial}{\partial v}$, $\frac{\partial}{\partial \kappa}$, $\frac{\partial}{\partial u}$ append to $H_{b,c}(\kappa,u,v)$ produces, at worst a “bad factor bounded by *const* $|c|^2$.” This may be seen by iteratively applying

$$\begin{aligned}
\frac{\partial}{\partial v}\frac{1}{2c_2+v\big((\kappa+c_1)^2+c_2^2-u\big)} &= -\frac{1}{\big[2c_2+v\big((\kappa+c_1)^2+c_2^2-u\big)\big]^2}\left[(\kappa+c_1)^2+c_2^2-u\right]\\
\frac{\partial}{\partial u}\frac{1}{2c_2+v\big((\kappa+c_1)^2+c_2^2-u\big)} &= -\frac{1}{\big[2c_2+v\big((\kappa+c_1)^2+c_2^2-u\big)\big]^2}[-v]\\
\frac{\partial}{\partial \kappa}\frac{1}{2c_2+v\big((\kappa+c_1)^2+c_2^2-u\big)} &= -\frac{1}{\big[2c_2+v\big((\kappa+c_1)^2+c_2^2-u\big)\big]^2}2v(\kappa+c_1)\\
\frac{\partial}{\partial u}\frac{1}{(\kappa+c_1)^2-u} &= \frac{1}{[(\kappa+c_1)^2-u]^2}\\
\frac{\partial}{\partial \kappa}\frac{1}{(\kappa+c_1)^2-u} &= -\frac{1}{[(\kappa+c_1)^2-u]^2}2(\kappa+c_1)
\end{aligned}$$

Define the multiplication operator

$$(\mathcal{M}f)(b)=\left(1+|b|^2\right)f(b)$$

on $\ell^2(\Gamma^\sharp)$. A precise version of the above is

Lemma III.2 *Let $(\kappa,u)\in S$, $|\varphi|\le\theta<\frac{\pi}{2}$ and $|r|\le E(S,\theta)$. Then there is a constant $C_1(S,\theta,G)$ such that is at least one of $b,c\notin G$*

$$\left|\frac{\partial^{n_1}}{\partial v^{n_1}}\frac{\partial^{n_2}}{\partial u^{n_2}}\frac{\partial^{n_3}}{\partial \kappa^{n_3}}\left(H_{bc}-\delta_{bc}\right)\right|\le(n_1+n_2+n_3)!C_1^{n_1+n_2+n_3+1}\left|\hat q(b-c)\right|\mathcal{M}_{cc}^{n_1+n_2+n_3}$$

Proof: The n_1,n_2,n_3 dependence of the above bound is of no interest to us. But we will prove it anyway. The proof is by induction on $n_1+n_2+n_3$ with the inductive hypothesis that $\frac{\partial^{n_1}}{\partial v^{n_1}}\frac{\partial^{n_2}}{\partial u^{n_2}}\frac{\partial^{n_3}}{\partial \kappa^{n_3}}\left(H_{bc}-\delta_{bc}\right)$ is a sum of at most $[2(n_1+n_2+n_3+1)]^{n_1+n_2+n_3}$ terms, each of which is a product of at most $2(n_1+n_2+n_3+1)$ factors. Each of the variables κ,u,v appears at most once in each factor. Each term is bounded by $const^{n_1+n_2+n_3+1}\left|\hat q(b-c)\right|\mathcal{M}_{cc}^{n_1+n_2+n_3}$. For example

$$\frac{v\hat q(b-c)}{2c_2+v\left((\kappa+c_1)^2+c_2^2-u\right)}=\left[\frac{\hat q(b-c)}{2c_2+v\left((\kappa+c_1)^2+c_2^2-u\right)}\right][v]$$

is viewed as a product of two factors.

Each action of a derivative on a product of N such factors produces N terms. By the derivative formulae just before the statement of the Lemma, each new term produced contains at most $N+2$ factors. For example $\frac{\partial}{\partial v}$ acting on the single factor $\frac{1}{2c_2+v\left((\kappa+c_1)^2+c_2^2-u\right)}$ produces a single term with the three factors

$$\left[\frac{1}{2c_2+v\left((\kappa+c_1)^2+c_2^2-u\right)}\right]\left[\frac{1}{2c_2+v\left((\kappa+c_1)^2+c_2^2-u\right)}\right]\left[-(\kappa+c_1)^2+c_2^2-u\right]$$

We start, at $n_1=n_2=n_3=0$, with one term containing at most two factors. So the parts of the inductive hypothesis dealing with the numbers of terms and factors is verified.

The bound on each term follows from the fact that there is always an explicit $\hat{q}(b-c)$, and

$$\begin{aligned}
|2c_2+v\left((\kappa+c_1)^2+c_2^2-u\right)| &\geq D(S,\theta) \\
|(\kappa+c_1)^2-u| &\geq const(G) \text{ for } c\notin G \\
|(\kappa+c_1)^2+c_2^2-u| &\leq const(S)\,[1+|c|^2] \\
2|\kappa+c_1| &\leq const(S)\,[1+|c|^2]
\end{aligned}$$

□

The "bad factors" of $[1+|c|^2]$ are controlled by using the decay of $\hat{q}(b-c)$ to move them to places where $|c|$ is small. The "bad factor moving lemma" is our next order of business. We consider, not only the lattice $\Gamma^\sharp$, but any $\Lambda\subset\Gamma^\sharp$. We denote by P_Λ the orthogonal projector onto $\ell^2(\Lambda)$. We introduce

Λ so as to enable us, later, to write $M = R - V(\mathbf{1}+W)^{-1}U$ as a limit $\lim\limits_{\Lambda\to\Gamma^\sharp} M_\Lambda$. In the Lemma we also prove the invertibility of $\mathbf{1}+W$. For this we need to insure that $\frac{1}{(\kappa+c_1)^2-u}$ is small for all $c\notin G$. Hence we impose the requirement on G that

$$(\kappa,u)\in S,\ c\notin G \Rightarrow \Big|(\kappa+c_1)^2-u\Big| \ge 2\cdot 4^{k+1}Q\ . \qquad (III.7)$$

Lemma III.3 *Let G be finite and obey (III.7). Let $\Lambda,\Lambda'\subset\Gamma^\sharp$. Let $(\kappa,u)\in S$, $|\varphi|\le\theta<\frac{\pi}{2}$ and $|r|\le\min\Big(E(S,\theta),\big[2\cdot 4^{k+1}Q\ D(S,\theta)\big]^{-1}\Big)$. Then there is a constant $C_2(S,\theta,G,Q,k)$ such that for all $|n|\le k+1$ and potentials q with $|||q|||\le Q$*

a) $||\mathcal{M}^n P_\Lambda \hat q(b-c) P_{\Lambda'}\mathcal{M}^{-n}|| \le 4^n Q$

b) $||\mathcal{M}^n P_\Lambda W\ P_{\Lambda'}\mathcal{M}^{-n}|| \le \frac{1}{2}$, $||\mathcal{M}^n P_\Lambda V\ P_{\Lambda'}\mathcal{M}^{-n}|| \le \frac{3}{2}$

c) $||\mathcal{M}^n P_\Lambda U||\ ,\ ||V\ P_\Lambda \mathcal{M}^n|| \le C_2$

d) $||\mathcal{M}^n (P_\Lambda + P_\Lambda W\ P_\Lambda)^{-1}\mathcal{M}^{-n}|| \le 2.$

The inverse here refers, of course, to the restriction to $\ell^2(\Lambda)$.

Proof: First note that for any operator on $\ell^2(\Gamma^\sharp)$ with kernel $A_{b,c}$

$$\begin{aligned} |(\mathcal{M}^n P_\Lambda A\ P_{\Lambda'}\mathcal{M}^{-n})| &\le (1+|b|^2)^n\,|A_{bc}|\,(1+|c|^2)^{-n} \\ &\le 4^n |A_{b,c}|\,(1+|c-b|^2)^{|n|} \end{aligned}$$

since

$$\begin{aligned} \tfrac{1+|b|^2}{1+|c|^2} &\le \tfrac{1+(|c|+|c-b|)^2}{1+|c|^2} \\ &\le \tfrac{1+4|c|^2+4|c-b|^2}{1+|c|^2} \\ &\le 4\,(1+|c-b|^2) \end{aligned}$$

and

$$\frac{1+|c|^2}{1+|b|^2} \le 4\left(1+|c-b|^2\right) \quad .$$

So, by the $L^1 - L^\infty$ operator bound

$$||B|| \le \left[\sup_{b\in\Gamma^\sharp}\sum_{c\in\Gamma^\sharp}|B_{b,c}|\right]^{1/2}\left[\sup_{c\in\Gamma^\sharp}\sum_{b\in\Gamma^\sharp}|B_{b,c}|\right]^{1/2} \quad ,$$

we have

$$\left|\left|\mathcal{M}^n P_\Lambda A\, P_{\Lambda'}\mathcal{M}^{-n}\right|\right| \le 4^n|||A||| \quad .$$

Part a) is now immediate.

Part b) follows from first,

$$\begin{aligned} \left|\tfrac{v\hat{q}(b-c)}{2c_2+v\left((\kappa+c_1)^2+c_2^2-u\right)}\right| &\le \tfrac{|v|}{D(S,\theta)}\,|\hat{q}(b-c)| \\ &\le \tfrac{1}{2}\tfrac{|\hat{q}(b-c)|}{4^{k+1}Q} \end{aligned}$$

for all $c_2 \ne 0$, and second

$$\left|\frac{\hat{q}(b-c)}{(\kappa+c_1)^2-u}\right| \le \frac{1}{2}\frac{|\hat{q}(b-c)|}{4^{k+1}Q}$$

for all $c_2 = 0$, $c \notin G$.

Part c) follows from part b) and the observation that, because G is finite, both $((\kappa+c_1)^2-u)$ and $\mathcal{M}^n_{cc}$ remain bounded as c runs over G.

Finally, to prove part d), expand

$$\mathcal{M}^{-n}(P_\Lambda + P_\Lambda W\ P_\Lambda)^{-1}\mathcal{M}^n = \sum_{j=0}^{\infty}(-1)^j P_\Lambda[\mathcal{M}^{-n}P_\Lambda W\ P_\Lambda\mathcal{M}^n]^j$$

and apply the bound on W from part b).

□

Completion of the proof of Proposition 5 Since R is a polynomial in (κ, u) we need only consider $V(\mathbf{1}+W)^{-1}U$. Let ∂^j denote any j^{th} order derivative with respect to (κ, u, v). Then any derivative of $V(\mathbf{1}+W)^{-1}U$ (of order at most k) is a sum of terms of the form

$$\begin{aligned}
&(\partial^{j_1}V)(\mathbf{1}+W)^{-1}(\partial^{j_2}V)(\mathbf{1}+W)^{-1}\cdots(\partial^{j_{p-1}}V)(\mathbf{1}+W)^{-1}(\partial^{j_p}U)\\
&=[\partial^{j_1}V\mathcal{M}^{-j_1}]\,[\mathcal{M}^{j_1}(\mathbf{1}+W)^{-1}\mathcal{M}^{-j_1}]\,[\mathcal{M}^{j_1}(\partial^{j_2}W)\mathcal{M}^{-j_1-j_2}]\\
&\quad[\mathcal{M}^{j_1+j_2}(\mathbf{1}+W)^{-1}\mathcal{M}^{-j_1-j_2}]\cdots[\mathcal{M}^{j_1+\cdots+j_{p-2}}(\partial^{j_{p-1}}W)\mathcal{M}^{-j_1\cdots-j_{p-1}}]\\
&\quad[\mathcal{M}^{j_1+\cdots+j_{p-1}}(\mathbf{1}+W)^{-1}\mathcal{M}^{-j_1\cdots-j_{p-1}}]\,[\mathcal{M}^{j_1+\cdots+j_{p-1}}\partial^{j_p}U\mathcal{M}^{-j_1-\cdots-j_p}]\\
&\quad[\mathcal{M}^{j_1+\cdots+j_p}P_G] \quad .
\end{aligned}$$

with $j_1 + \ldots + j_p \le k$.

The factors

$$[\partial^{j_1}V\mathcal{M}^{-j_i}]\,,\ \left[\mathcal{M}^{j_1+\cdots+j_{\alpha-1}}(\partial^{j_\alpha}W)\mathcal{M}^{-j_1\cdots-j_\alpha}\right]$$

and

$$[\mathcal{M}^{j_1+\cdots+j_{p-1}}\partial^{j_p}U\mathcal{M}^{-j_1\cdots-j_p}]$$

are all of bounded norm by Lemmas III.2 and III.3.a. The factors

$$\left[\mathcal{M}^{j_1+\cdots+j_\alpha}(\mathbf{1}+W)^{-1}\mathcal{M}^{-j_1\cdots-j_\alpha}\right]$$

are of bounded norm by Lemma III.3.d. The final factor $\mathcal{M}^{j_1+\cdots+j_p} P_G$ is of bounded norm by the finiteness of G.

$\square$

We now prove a Lemma that allows us to evaluate derivatives of M (or its determinant) as the limit of restrictions to finite subsets $\Lambda \subset \Gamma^\sharp$. More generally let Λ by any subset of $\Gamma^\sharp$ that contains G and define

$$M_\Lambda = R - V\ P_\Lambda(\mathbf{1} + P_\Lambda W\ P_\Lambda)^{-1} P_\Lambda U$$

Lemma III.4 *Let G be finite and obey (III.7) and let $\theta < \frac{\pi}{2}$. Then there is a constant $C_3(S, \theta, G, Q, k)$ such that for all $(\kappa, u) \in S$, $|\varphi| \le \theta$, $|r| \le \min\left(E(S,\theta), \left[2 \cdot 4^{k+1} Q\ D(S,\theta)\right]^{-1}\right)$, $n_1 + n_2 + n_3 \le k$ and potentials q with $|||q||| \le Q$*

$$\left\| \frac{\partial^{n_1}}{\partial v^{n_1}} \frac{\partial^{n_2}}{\partial u^{n_2}} \frac{\partial^{n_3}}{\partial \kappa^{n_3}} [M - M_\Lambda] \right\| \le \frac{C_3}{dist(0, \Gamma^\sharp \setminus \Lambda)^2} \quad .$$

Remark The decay rate depends primarily on the smoothness of q. If q is C^∞ the difference will decrease to zero faster than any inverse power of $dist(0, \Gamma^\sharp \setminus \Lambda)$.

PROOF: As in the proof of Proposition 5, $\partial^j[M - M_\Lambda]$ is a sum of terms of the form

$$
\begin{aligned}
&(\partial^{j_1}V)(\mathbf{1}+W)^{-1}(\partial^{j_2}W)(\mathbf{1}+W)^{-1}\cdots(\partial^{j_{p-1}}W)(\mathbf{1}+W)^{-1}(\partial^{j_p}U)\\
&-(\partial^{j_1}V)P_\Lambda(\mathbf{1}+P_\Lambda W\,P_\Lambda)^{-1}(\partial^{j_2}P_\Lambda W\,P_\Lambda)(\mathbf{1}+P_\Lambda W\,P_\Lambda)^{-1}\\
&\qquad\qquad\cdots(\partial^{j_{p-1}}P_\Lambda W\,P_\Lambda)(1+P_\Lambda W\,P_\Lambda)^{-1}(\partial^{j_P}P_\Lambda U)\ .
\end{aligned}
$$

Now apply

$$
A_1A_2\cdots A_m - B_1B_2\cdots B_m = \sum_{\alpha=1}^{m} B_1\cdots B_{\alpha-1}(A_\alpha - B_\alpha)A_{\alpha+1}\cdots A_m
$$

with

$$
\begin{aligned}
A_1 &= \partial^{j_1}V & B_1 &= \partial^{j_1}V\,P_\Lambda\\
A_2 &= (\mathbf{1}+W)^{-1} & B_2 &= (\mathbf{1}+P_\Lambda W\,P_\Lambda)^{-1}
\end{aligned}
$$

and so on and then apply

$$
\begin{aligned}
\partial^{j_1}V - \partial^{j_1}V\,P_\Lambda &= \partial^{j_1}V\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}\\
\partial^{j_p}U - \partial^{j_p}P_\Lambda U &= \mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}\partial^{j_P}U\\
\partial^{j_\alpha}W - \partial^{j_\alpha}P_\Lambda W\,P_\Lambda &= \mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}\partial^{j_\alpha}W + P_\Lambda\partial^{j_\alpha}W\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathbf{1}+W)^{-1} - (\mathbf{1}+P_\Lambda W\,P_\Lambda)^{-1}\\
&= -(\mathbf{1}+P_\Lambda W\,P_\Lambda)^{-1}(W - P_\Lambda W\,P_\Lambda)(\mathbf{1}+W)^{-1}\\
&= -(\mathbf{1}+P_\Lambda W\,P_\Lambda)^{-1}\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}W(\mathbf{1}+W)^{-1}\\
&\quad - (\mathbf{1}+P_\Lambda W\,P_\Lambda)^{-1}P_\Lambda W\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}(\mathbf{1}+W)^{-1}\ .
\end{aligned}
$$

The result is a finite sum of terms of much the same form as

$$
(\partial^{j_1}V)(\mathbf{1}+W)^{-1}(\partial^{j_2}W)(\mathbf{1}+W)^{-1}\cdots(\partial^{j_{P-1}}W)(\mathbf{1}+W)^{-1}(\partial^{j_P}U)
$$

but with same extra P_Λ's tossed in, possibly with an extra $W(\mathbf{1}+W)^{-1}$ tossed in and definitely with one $\mathcal{M}^{-1}(\mathbf{1}-\mathbf{P}_{\boldsymbol{\Lambda}})\mathcal{M}$. The additional P_Λ's and

$W(\mathbf{1}+W)^{-1}$ are unimportant. On the other hand the $\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}$ is crucial because

$$\left|\left|\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\right|\right| = \max_{b\in\Gamma^\sharp\setminus\Lambda} \frac{1}{1+|b|^2} \le \left[dist(0,\Gamma^\sharp\setminus\Lambda)\right]^{-2} .$$

We may now continue just as in the proof of Proposition 5. The remaining $\mathcal{M}$ from $\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}$ just causes the replacement of some of the exponents $j_1+\cdots+j_\alpha$ (those to the right of the $\mathcal{M}^{-1}(\mathbf{1}-P_\Lambda)\mathcal{M}$) by $j_1+\cdots+j_\alpha+1$. Since $j_1+\ldots+j_\alpha+1\le k+1$ we may still apply Lemma III.3.

□

As an almost immediate consequence of Lemma III.4, we have the analyticity of M and its derivatives in the potential q.

Lemma III.5 *Let $0<\theta<\frac{\pi}{2}$ and $k\ge 0$. Let G be finite and obey (III.7). If $n_1+n_2+n_3\le k$, $|\varphi|\le\theta$ and $|r|\le\min\left(E(S,\theta),\left[2\cdot 4^{k+1}QD(S,\theta)\right]^{-1}\right)$ then*

$$\frac{\partial^{n_1}}{\partial v^{n_1}}\frac{\partial^{n_2}}{\partial u^{n_2}}\frac{\partial^{n_3}}{\partial \kappa^{n_3}}M(v,u,\kappa,q)$$

is analyic in q on the domain $|||q|||<Q$.

Proof: For any finite Λ the finite matrices $VP_\Lambda, P_\Lambda U, R$ and $P_\Lambda W P_\Lambda$ and their derivatives with respect to (κ,u,v) are polynomials (of degree 1) in q and hence trivially analytic. Furthermore $||P_\Lambda W\ P_\Lambda||=\frac{1}{2}$ so $(\mathbf{1}+P_\Lambda W\ P_\Lambda)^{-1}$ and consequently $\frac{\partial^{n_1}}{\partial v^{n_1}}\frac{\partial^{n_2}}{\partial u^{n_2}}\frac{\partial^{n_3}}{\partial \kappa^{n_3}}M_\Lambda$ are analytic too. By Lemma III.4 the latter converge uniformly on $|||q||| < Q$ to $\frac{\partial^{n_1}}{\partial v^{n_1}} \frac{\partial^{n_2}}{\partial u^{n_2}} \frac{\partial^{n_3}}{\partial \kappa^{n_3}} M$. The Lemma now follows by Weierstrass' theorem.

□

IV Opening up double points

In this Section we prove Proposition 2. Again we assume that $\gamma = (0, \gamma_2)$ and use the coordinates κ, u, v described in (*III*.1). Then for $b \in \Gamma^\sharp$ with $\langle b, \gamma \rangle = 0$

$$\bar{B}_b \cap E_\gamma = \left\{ (\kappa, u, 0) \in E_\gamma \mid (\kappa + b_1)^2 - u = 0 \right\}$$

so that the point of intersection between $\bar{B}_0 \cap E_\gamma$ and $\bar{B}_d \cap E_\gamma$ in (κ, u, v) coordinates is

$$p'_d = (-d_1/2, d_1^2/4, 0) \ .$$

In a neighbourhood U of p'_d we can use

$$x := \kappa^2 - u, \ \ y := (\kappa + d_1)^2 - u \text{ and } v \tag{IV.1}$$

as local coordinates on $(\mathbb{C}^2 \times \mathbb{C}) \cup E_\gamma$.

We shall work with two matrices $H_{b,c}(x, y, v)$ of the form (*III*.2). The first, denoted $H^s_{b,c}$, (with s standing for small) has $G = G^s = \{0, d\}$. It is

$$H^s_{b,c} = \begin{cases} x\delta_{b,c} + \hat{q}(b-c)N(c) & \text{if } c = 0 \\ y\delta_{b,c} + \hat{q}(b-c)N(c) & \text{if } c = d \\ \delta_{b,c} + \hat{q}(b-c)N(c) & \text{if } c \neq 0, d \end{cases} \qquad (IV.2)$$

where

$$N(c) := \begin{cases} 1 & \text{for } c = 0, d \\ \frac{1}{(\kappa+c_1)^2 - u} & \text{for } c \in \Gamma^\sharp \setminus \{0, d\} \text{ with } c_2 = c \\ \frac{v}{2c_2 + v\left((\kappa+c_1)^2 + c_2^2 - u\right)} & \text{for } c \in \Gamma^\sharp \text{ with } c_2 \neq 0. \end{cases} \qquad (IV.3)$$

When the potential q is sufficiently small the 2×2 matrix

$$M^s = R^s - V^s(\mathbf{1} + W^s)^{-1}U^s\,, \qquad (IV.4)$$

where R^s, U^s, V^s and W^s are defined by the blocking $(III.4)$ of H^s with $G = G^s$, is well-defined. We shall do most of our computations using M^s.

For the second H, we first select a bounded open subset $S \subset E_\gamma$ containing p'_d, and a positive real number Q. Then we choose a finite subset $G^\ell \subset \{b \in \Gamma^\sharp \mid b_2 = 0\}$ that obeys $(III.7)$ with $k = 5$.

The matrix

$$M^\ell = R^\ell - V^\ell(\mathbf{1} + W^\ell)^{-1}U^\ell \qquad (IV.5)$$

arising from the blocking $(III.4)$ of H^ℓ with $G = G^\ell$ is defined for all q with $|||q||| \leq Q$. Furthermore, since M^ℓ and its derivatives with respect to (κ, u, v) are analytic in q we can get all the information we need about them from M^s. The precise relationship between $\det M^\ell$ and $\det M^s$ is, for $\hat{q}$ sufficiently small,

$$\begin{aligned}
\det M^\ell &= \lim_{\substack{\Lambda\to\Gamma^\sharp \\ \Lambda \text{ finite}}} \frac{\det H^\ell_\Lambda}{\det(\mathbf{1}+W^\ell_\Lambda)} \\
&= \prod_{c\in G^\ell\setminus\{0,d\}} N(c)^{-1} \lim_{\Lambda\to\Gamma^\sharp} \frac{\det H^s_\Lambda}{\det(\mathbf{1}+W^\ell_\Lambda)} \\
&= \prod_{c\in G^\ell\setminus\{0,d\}} N(c)^{-1} \lim_{\Lambda\to\Gamma^\sharp} \frac{\det(\mathbf{1}+W^s_\Lambda)}{\det(\mathbf{1}+W^\ell_\Lambda)} \frac{\det H^s_\Lambda}{\det(\mathbf{1}+W^s_\Lambda)} \\
&= \prod_{c\in G^\ell\setminus\{0,d\}} N(c)^{-1} \det\left(\tilde{R}-\tilde{V}(\mathbf{1}+W^\ell)^{-1}\tilde{U}\right)\det M^s
\end{aligned} \tag{IV.6}$$

where $\tilde{R}, \tilde{V}$ and $\tilde{U}$ are defined by blocking $\mathbf{1}+W^s = (H^s)_{b,c\in\Gamma^\sharp} \setminus \{0,d\}$ as

$$\mathbf{1}+W^s = \begin{array}{r} \\ G\setminus\{0,d\} \\ \Gamma^\sharp\setminus G \end{array} \begin{array}{c} \overbrace{}^{G\setminus\{0,d\}} \quad \overbrace{}^{\Gamma^\sharp\setminus G} \\ \left[\begin{array}{cc} \tilde{R} & \tilde{V} \\ \tilde{U} & \mathbf{1}+W^\ell \end{array}\right] \end{array} \tag{IV.7}$$

The blocks $\tilde{R}, \tilde{U}, \tilde{V}$ are restrictions of R^ℓ, U^ℓ, V^ℓ with R^ℓ and U^ℓ multiplied by the finite matrix $[N(c)^{-1}\delta_{b,c}]_{b,c\in G\setminus\{0,d\}}$. Hence, by Lemmas III.3 and III.4, $\tilde{R}-\tilde{V}(\mathbf{1}+W^\ell)^{-1}\tilde{U}$ is well-defined for all $|||q||| \le Q$ and its determinant is $\lim\limits_{\Lambda\to\Gamma^\sharp} \frac{\det(\mathbf{1}+W^s_\Lambda)}{\det(\mathbf{1}+W^\ell_\Lambda)}$.

We want to determine the first few terms in the Taylor series of $\det M^\ell$ along imaginary v-directions. For this purpose we introduce a grading in the formal power series ring $\mathbb{C}\,[[x,y,v]]$ by giving x and y weight two and v weight one. Let I_r be the ideal in $\mathbb{C}\,[[\kappa,y,v]]$ consisting of all elements of weight at least r, and put

$$R := \mathbb{C}\,[[\kappa,y,v]]\,/I_5 \ .$$

Below we shall prove

Proposition 6 *Assume that* $q_\gamma \equiv 0$ *and* $\sum_{b\in\Gamma^\sharp} (1+|b|^{12})\,|\hat q(b)|^2$ *is sufficiently small. Then*

(i) $\det M^s = (x-T_1)(y-T_2) - \frac{v^4}{16}\Delta_d\Delta_{-d}$ *in* R
with $T_i \in I_2$

(ii) $\det\left(\tilde R - \tilde V(\mathbf{1}+W^\ell)^{-1}\tilde U\right) = 1$ *mod* I_2

(iii) $\prod_{c\in G^\ell\setminus\{0.d\}} N(c)^{-1} = \prod_{c\in G^\ell\setminus\{0.d\}} \frac{1}{c_1(c_1-d_1)}$ *mod* I_2

We first show how Proposition 6 implies Proposition 2. Proposition 6 shows that, in R, for small q,

$$\det M^\ell = \prod_{c\in G^\ell\setminus\{0,d\}} \frac{1}{c_1(c_1-d_1)}\left[(x-T_1)(y-T_2) - \frac{v^4}{16}\Delta_d\Delta_{-d}\right]$$

so that the Taylor polynomial of $\det M^\ell$ for $\arg(-v^2)\in(-\theta,\theta)$ is, up to a nonzero multiplicative constant, equal to $(x-T_1)(y-T_2) - \frac{v^4}{16}\Delta_d\Delta_{-d}$ mod I_5. By analyticity in q (Lemma III.5) this is the case for all $|||q|||<Q$. We make the change of variables

$$x' = x - T_1\ ,\ \ y' = y - T_2\ ,\ \ w = \frac{v}{\sqrt{1+uv^2}}\ \ .$$

Then

$$\lambda = w^{-2}$$

and by Proposition 5

$$\left|\det M^\ell(x',y',w) - \left(x'y' - \frac{1}{16}\Delta_d\Delta_{-d}w^4\right)\right| = o\left(|x'|^2 + |y'|^2 + |w|^4\right) \quad . \tag{IV.8}$$

If $\Delta_d\Delta_{-d} \neq 0$ then the double point p_d opens up by the topological Lemma stated in the appendix.

Before we begin with the proof of Proposition 6 we note that, since $x = \kappa^2 - u$, $y = \kappa^2 - u + 2d_1\kappa + d_1^2$

$$(\kappa + c_1)^2 - u = \frac{d_1 - c_1}{d_1}x + \frac{c_1}{d_1}y + c_1(c_1 - d_1) \tag{IV.9}$$

and

$$N(c) = \begin{cases} 1 & c = 0, d \\ \frac{d_1}{c_1d_1(c_1-d_1)+(d_1-c_1)x+c_1y} & c_2 = 0,\ c \neq 0, d \\ \frac{vd_1}{2c_2d_1+v\left[(d_1-c_1)x+c_1y+c_1d_1(c_1-d_1)+c_2^2d_1\right]} & c_2 \neq 0 \end{cases} \tag{IV.10}$$

$$= \begin{cases} 1 & c = 0, d \\ \frac{1}{c_1(c_1-d_1)} \mod I_2 & c_2 = 0,\ c \neq 0, d \\ \frac{v}{2c_2+v\left[c_1(c_1-d_1)+c_2^2\right]} \mod I_4 & c_2 \neq 0 \\ \qquad = \frac{v}{2c_2} \mod I_2 & \end{cases}$$

so that

$$b, c \in \Gamma^\sharp \text{ with } b_2 = -c_2 \neq 0 \Rightarrow N(b) + N(c) \in I_2 \tag{IV.11}$$

$$\begin{aligned} b \in \Gamma^\sharp,\ b_2 \neq 0 \Rightarrow N(b) + N(d-b) &= -v^2\frac{b_1(b_1-d_1)+b_2^2}{2b_2^2} \mod I_4 \\ &= \frac{v^2}{2}\frac{\langle b, d-b\rangle}{b_2^2} \mod I_4 \end{aligned} \tag{IV.12}$$

We now give the

Proof of Proposition 6: Part (iii) is an immediate consequence of $(IV.10)$. Since $q_\gamma = 0$ all the Fourier coefficients $\hat{q}(b-c)$ with $b_2 = c_2$ are zero. So

$$\begin{aligned} \det M^s &= \det\left\{\left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right] - S^s\right\} \\ &= (x - S^s_{0,0})(y - S^s_{d,d}) - S^s_{0,d}S^s_{d,0} \end{aligned}$$

and

$$\det\left(\tilde{R} - \tilde{V}(\mathbf{1} + W^\ell)^{-1}\tilde{U}\right) = \det(\mathbf{1} - \tilde{S})$$

where $S^s = V^s(\mathbf{1} + W^s)^{-1}U^s$ and $\tilde{S} = \tilde{V}(\mathbf{1} + W^\ell)^{-1}\tilde{U}$.

Expanding in a geometric series in W (which by Lemma III.3 converges)

$$S_{b,c} = \sum_{n=0}^{\infty}(-1)^n \sum_{b^{(1)},\ldots,b^{(n+1)}} V_{b,b^{(1)}}W_{b^{(1)},b^{(2)}}W_{b^{(2)},b^{(3)}}\cdots W_{b^{(n)},b^{(n+1)}}U_{b^{(n+1)},c}$$

with $b^{(1)},\ldots,b^{(n+1)}$ summed over $\Gamma^\sharp \setminus \{0,d\}$ in the case of S^s and $\Gamma^\sharp \setminus G^\ell$ in the case of $\tilde{S}$. Note that $S_{b,c}$ always has $b,c \in G^\ell$ and hence $b_2 = c_2 = 0$ and that

$$\begin{aligned} V_{b,c} = W_{b,c} = U_{b,c} \quad &\text{if} \quad b_2 = c_2 \\ W_{b,c}, V_{b,c} \in I_1 \quad &\text{if} \quad c_2 \neq 0 \ . \end{aligned}$$

Thus $S_{b,c} \in I_1$ and

$$\begin{aligned} S_{b,c} \;=\; & \sum_{\substack{b^{(1)} \\ b^{(1)}_2 \neq 0}} V_{b,b^{(1)}}U_{b^{(1)},c} - \sum_{\substack{b^{(1)},b^{(2)} \\ b^{(1)}_2\neq 0, b^{(2)}_2 \neq 0}} V_{b,b^{(1)}}W_{b^{(1)},b^{(2)}}U_{b^{(2)},c} \\ &+ \sum_{\substack{b^{(1)},b^{(2)},b^{(3)} \\ b^{(1)}_2\neq 0, b^{(3)}_2\neq 0 \\ b^{(2)}_2 = 0}} V_{b,b^{(1)}}W_{b^{(1)},b^{(2)}}W_{b^{(2)},b^{(3)}}U_{b^{(3)},c} \mod I_3 \ . \end{aligned} \qquad (IV.13)$$

In fact for any b, c with $b_2 = c_2 = 0$ the first sum is

$$\sum_{\substack{b^{(1)} \\ b_2^{(1)} \neq 0}} \hat{q}(b - b^{(1)})\hat{q}\left(b^{(1)} - c)N(c)N(b^{(1)}\right)$$
$$= \frac{1}{2} \sum_{\substack{b^{(1)} \\ b_2^{(1)} \neq 0}} \hat{q}(b - b^{(1)})\hat{q}(b^{(1)} - c)N(c)\left[N(b^{(1)}) + N(b + c - b^{(1)})\right]$$

which is in I_2 by $(IV.11)$. This proves part ii of the Proposition as well as the claims that $T_1 = S^s_{0,0} \in I_2$ and $T_2 = S^s_{d,d} \in I_2$. It also proves that the third term in $(IV.13)$ is in I_3.

It remains only to calculate $S^s_{0,d}$ and $S^s_{d,0}$. In $\mathbb{C}\left[[x, y, v]\right]/I_3$

$$\begin{aligned}
S^s_{0,d} &= \tfrac{1}{2} \sum_b \hat{q}(-b)\hat{q}(b - d)\left[N(b) + N(d - b)\right] \\
&\quad - \sum_{b^{(1)}, b^{(2)}} \hat{q}(-b^{(1)})N(b^{(1)})\hat{q}(b^{(1)} - b^{(2)})N(b^{(2)})\hat{q}(b^{(2)} - d) \\
&= \tfrac{v^2}{4} \sum_b \tfrac{\hat{q}(-b)\langle b, d-b\rangle \hat{q}(b-d)}{b_2^2} \\
&\quad - \tfrac{1}{4} \sum_{\substack{b^{(1)}, b^{(2)}, b^{(3)} \\ s.t.\ b^{(1)} + b^{(2)} + b^{(3)} = d}} \hat{q}(-b^{(1)})\hat{q}(-b^{(2)})\hat{q}(-b^{(3)}) \tfrac{v^2}{b_2^{(1)}(b_2^{(1)} + b_2^{(2)})} \quad \text{by } (IV.10), (IV.12) \\
&= -\tfrac{v^2}{4}\Delta_{-d}
\end{aligned}$$

since, for $b^{(1)} + b^{(2)} + b^{(3)} = d$, $b_2^{(1)} \neq 0$, $b_2^{(2)} \neq 0$, $b_2^{(3)} \neq 0$

$$\begin{aligned}
&\tfrac{1}{b_2^{(1)}(b_2^{(1)} + b_2^{(2)})} + \tfrac{1}{b_2^{(1)}(b_2^{(1)} + b_2^{(3)})} + \tfrac{1}{b_2^{(2)}(b_2^{(1)} + b_2^{(2)})} \\
&+ \tfrac{1}{b_2^{(2)}(b_2^{(2)} + b_2^{(3)})} + \tfrac{1}{b_2^{(3)}(b_2^{(1)} + b_2^{(3)})} + \tfrac{1}{b_2^{(3)}(b_2^{(2)} + b_2^{(3)})} \\
&= \tfrac{1}{b_2^{(1)} b_2^{(2)}} + \tfrac{1}{b_2^{(1)} b_2^{(3)}} + \tfrac{1}{b_2^{(2)} b_2^{(3)}} \\
&= \tfrac{b_2^{(1)} + b_2^{(2)} + b_2^{(3)}}{b_2^{(1)} b_2^{(2)} b_2^{(3)}} \; = \; 0
\end{aligned}$$

Similarly $S^s_{d,0} = -\frac{v^2}{4}\Delta_d$ so that

$$\det M^s = (x - S^s_{0,0})(y - S^s_{d,d}) - \frac{v^4}{16}\Delta_d \Delta_{-d} \mod I_5 \quad .$$

$\square$

V Sums of finite gap potentials

We first note a property of the asymptotic expansion of the Fourier coefficients of finite gap potentials which we will use later.

Lemma V.1 *Let $V \in L^2_{\mathbb{R}}(\mathbb{R}/2\pi\mathbb{Z})$ be a potential such that the spectrum of $-\frac{d^2}{dz^2} + V(z)$ has only finitely many gaps. Then there are complex numbers ζ_j, $j = 1, \ldots, m$ with $|\zeta_j| = 1$, positive integers k_j, and $y > O$ such that*

$$\hat{V}(n) = -|n| e^{-|n|y} \sum_{j=1}^{m} k_j \zeta_j^n + O(e^{-|n|y'})$$

for some $y' > y$.

Proof: It is well known that there is an entire function $\theta(z)$ on $\mathbb{C}$ such that

$$V(z) = -2 \frac{d^2}{dz^2} \log \theta(z) + const.$$

(see I.9). Therefore at each pole a of $V(z)$

$$V(z) = \frac{2m_a}{(z-a)^2} + O(1)$$

where m_a is the order of the zero of $\theta(z)$ at a. Let $a_1, \ldots, a_r$ be the zeroes with maximal negative imaginary part $-y < 0$ in $\{z \in \mathbb{C} \mid 0 \leq Re\, z < 2\pi\}$. Put

$k_j := 2m_{a_j}$. Choose $y' > y$ such that all poles a of $V(z)$ with $-y' < Im\, a \leq 0$ fulfil $Im\, a = -y$. Then by the residue theorem for $n \geq 0$

$$\begin{aligned}
\hat{V}(n) &= \tfrac{1}{2\pi}\int_0^{2\pi} V(x - i\,y')e^{-i(x-i\,y')n}dx - i\sum_{j=1}^{m} res_{z=a_j}\left(V(z)e^{-i\,n\,z}\right) \\
&= -\sum_{j=1}^{m} k_j n\, e^{-i\,n\,a_j} + O(e^{-n\,y'}) \\
&= -n\, e^{-n\,y} \cdot \sum_{j=1}^{m} k_j \zeta_j^n + O(e^{-n\,y'})
\end{aligned}$$

with $\zeta_j := e^{-i\,Re\,a_j}$. For $n < 0$, $\hat{V}(n) = \overline{\hat{V}(-n)}$.

□

For sums like in Lemma V.1 we need the following

Lemma V.2 *Let $\zeta_1, \ldots, \zeta_r$ be pairwise different complex numbers of absolute value one, and let $k_j \in \mathbb{C} \setminus \{0\}$. Put*

$$F(n) := \sum_{j=1}^{r} k_j \zeta_j^n \ .$$

Then

$$\limsup_{n\to\infty} |F(n)| > 0 \quad .$$

PROOF: Write $\zeta_j = e^{2\pi i\theta_j}$ with $\theta_j \in \mathbb{R}$. Choose real numbers $\varphi_0 = 1$, $\varphi_1, \ldots, \varphi_\ell$ which are linearly independent over $\mathbb{Q}$ such that each θ_j lies in the $\mathbb{Q}$-vectorspace spanned by $\varphi_0, \ldots, \varphi_\ell$. Then there exist $N \in \mathbb{N}$ and integers a_{ij} such that

$$\theta_j = \frac{1}{N} \sum_{m=0}^{\ell} a_{mj}\varphi_m$$

so

$$\begin{aligned} F(n) &= \sum_{j=1}^{r} k_j \left(e^{2\pi i\ a_{0j}/N}\right)^n \prod_{m=1}^{\ell} e^{2\pi i\ (n\varphi_m/N)\ a_{mj}} \\ &= \sum_{a=(a_1,\ldots,a_\ell)\in\mathbb{Z}^\ell} A_a(n) \cdot \prod_{m=1}^{\ell} e^{2\pi i\ (n\varphi_m/N)\ a_m} \end{aligned}$$

where each $A_a(n)$ is of the form

$$A_a(n) = \sum_{k=1}^{N} A_{a,k} e^{2\pi i\ kn/N} \quad ,$$

and not all the $A_{a,k}$ are zero. However, all but finitely many $A_a(n)$ are zero. Each $A_a(n)$ is periodic with period N. Using the Vandermonde determinant

$$\det \left(e^{2\pi i\ kn/N}\right)_{k,n=1,\ldots,N}$$

one sees that $A_a(n)$ is not identically zero whenever $A_{a,k} \neq 0$ for some k. Choose $n_0 \in \mathbb{N}$ such that $A_a(n_0) \neq 0$ for some $a \in \mathbb{Z}^\ell$. Put

$$f(z_1, \ldots, z_\ell) := \sum_{a\in\mathbb{Z}^\ell} \left(A_a(n_0) \cdot \prod_{m=1}^{\ell} e^{2\pi i\ (n_0\varphi_m/N)\ a_m} \right) z_1^{a_1} \cdots z_\ell^{a_\ell} \quad .$$

Then

$$F(n_0 + nN) = f(e^{2\pi i\ n\varphi_1}, \ldots, e^{2\pi i\ n\varphi_\ell}) \quad .$$

The polynomial $f(z_1, \ldots, z_\ell)$ does not vanish identically on the torus

$$T := \left\{ (z_1, \ldots, z_\ell) \in \mathbb{C}^d \,|\, |z_m| = 1 \quad , \quad m = 1, \ldots, d \right\} .$$

The sequence $(e^{2\pi i \, n\varphi_1}, \ldots, e^{2\pi i \, n\varphi_\ell})$ is ergodic on this torus. This shows that $\limsup_{n\to\infty} |F(n_0 + n\,N)| \neq 0$.

We now begin with the *proof of Proposition 3.* By Lemma V.1 there is for each $j = 1, \ldots, r$ a positive number y_j such that for $b \in \Gamma^\sharp$ with $\langle b, \gamma_j \rangle = 0$

$$|\hat{q}(b)| = O\left(|b| e^{-y_j |b|} \right) \quad ,$$

but for no $\varepsilon > 0$ $|\hat{q}(b)| = O(e^{-(y_j+\varepsilon)|b|})$. Let β_j be a vector of length $1/y_j$ perpendicular to γ_j. Again by Lemma V.1 there are for each $j = 1, \ldots, r$ strictly positive real numbers k_{ij}, and real number φ_{ij}, $i = 1, \ldots, m_j$ such that

$$\hat{q}(\lambda\beta_j) = -|\lambda| e^{-|\lambda|} \cdot \sum_{i=1}^{m_j} k_{ij} e^{i\lambda\varphi_{ij}} + O(e^{-|\lambda|}) \tag{V.1}$$

whenever $\lambda\beta_j \in \Gamma^\sharp$.

Now fix any $\gamma \in \Gamma$, $d \in \Gamma^\sharp$ such that $\langle \gamma, d \rangle = 0$ and d is not parallel to any β_j. Then, for each $1 \leq i < j \leq r$, d has a unique representation

$$d = \lambda_{ij}\beta_i + \mu_{ij}\beta_j$$

and

$$\Delta_{nd} = 2|\gamma|^2 \sum_{\substack{i<j \\ n\lambda_{ij}\beta_i \in \Gamma^\sharp \\ n\mu_{ij}\beta_j \in \Gamma^\sharp}} \frac{\langle \beta_i, \beta_j \rangle}{\langle \beta_i, \gamma \rangle \langle \gamma, \beta_j \rangle} \hat{q}(n\lambda_{ij}\beta_i) \hat{q}(n\mu_{ij}\beta_j) \ .$$

Note that the decay rate as $n \to \infty$ of

$$\begin{aligned} \hat{q}(n\lambda_{ij}\beta_i)\hat{q}(n\mu_{ij}\beta_j) &= n^2|\lambda_{ij}|\,|\mu_{ij}|e^{-n(|\lambda_{ij}|+|\mu_{ij}|)} \\ &\quad\times\left(\sum_{\alpha=1}^{m_i} k_{\alpha i}e^{i\lambda\varphi_{\alpha i}n}\right)\left(\sum_{\beta} k_{\beta j}e^{i\lambda\varphi_{\beta j}n}\right)+O(e^{-n(|\lambda_{ij}|+|\mu_{ij}|)}) \end{aligned}$$

is controlled by $|\lambda_{ij}|+|\mu_{ij}|$, which has the following geometric interpretation. First choose two signs $\sigma_i, \sigma_j \in \{+, -\}$ so that d is in the cone generated by $\sigma_i\beta_i$ and $\sigma_j\beta_j$. Then

$$d = |\lambda_{ij}|(\sigma_i\beta_i) + |\mu_{ij}|(\sigma_j\beta_j)$$

and $|\lambda_{ij}|+|\mu_{ij}|$ is the factor by which one must scale d in order that the head of $(|\lambda_{ij}| + |\mu_{ij}|)^{-1}\,d$ lie on the line through $\sigma_i\beta_i$ and $\sigma_j\beta_j$.

If we choose d appropriately we may identify precisely which i, j give the smallest decay rate. This is done using the following Lemma. Put

$$B := \{\pm\beta_j \mid j = 1 \dots r\} \quad .$$

Lemma V.3 *Under the hypotheses of Proposition 3 there exist points*

$$\beta^{(1)}, \dots, \beta^{(\ell)} \in B \, , \, \ell \geq 2$$

with the following properties.

(i) $\beta^{(1)}, \dots, \beta^{(\ell)}$ *lie on a line* g *in* $\mathbb{R}^2$*, and* $\beta^{(i)}$ *lies strictly between* $\beta^{(i-1)}$ *and* $\beta^{(i+1)}$ *on* g

(ii) $g \cap B = \{\beta_1, \dots, \beta_\ell\}$

(iii) $\left\langle \beta^{(1)}, \beta^{(i)} \right\rangle \geq 0$ *for* $i = 2, \dots, \ell$ *and if* $\left\langle \beta^{(1)}, \beta^{(\ell)} \right\rangle = 0$ *then* $\ell \geq 3$*.*

(iv) All points of $B - \{\beta^{(1)}, \dots, \beta^{(\ell)}\}$ *lie in the halfplane of* $\mathbb{R}^2 - g$ *containing 0, with the possible exception of one point* β*, which then fulfils* $\left\langle \beta, \beta^{(1)} \right\rangle = 0$*. In this case the lines* $\operatorname{span}(\beta, \beta')$*,* $\beta' \in B - \{\beta^1\}$ *all intersect* g *strictly on the same side of* $\beta^{(1)}$ *as* $\beta^{(2)}$*.*

PROOF: Let D be the boundary of the convex hull of B. Since $r \geq 2$ it is a convex polygon which is symmetric with respect to the origin. If e is an edge of D which forms an angle less than $\frac{\pi}{2}$ with 0 we can take for $\beta^{(1)}, \dots, \beta^{(\ell)}$ the points of $e \cap B$. If there is no such edge, all the edges of D form the angle $\frac{\pi}{2}$ with the origin, i.e. D is a diamond. If one of the edges e of the diamond contains at least 3 points of B we can again take $\{\beta^{(1)}, \ldots, \beta^{(\ell)}\} = e \cap D$. Otherwise there are $\beta^{(1)}$, $\beta \in B$ such that $D \cap B = \{\pm\beta^{(1)}, \pm\beta\}$ and $\left\langle \beta, \beta^{(1)} \right\rangle = 0$. Let D' be the convex hull of $B - \{\pm\beta\}$. By assumption it is again a convex polygon which is symmetric around 0. Therefore it contains an edge e, with $\beta^{(1)}$ as vertex, which forms an angle less than $\frac{\pi}{2}$ with 0. We then take for $\beta^{(1)}, \ldots, \beta^{(\ell)}$ the points of $e \cap B$.

$\square$

Without loss of generality we may assume that $\beta^{(i)} = \beta_i$ for $i = 1, \ldots, \ell$. If all points of $B \setminus \{\beta_1, \ldots, \beta_\ell\}$ lie on the same side of g let C be the interior of the cone $\mathbb{R}_+\beta_1 + \mathbb{R}_+\beta_2$. If there exists $\beta \in B$ which is separated from 0 by g let $\hat{\beta}$ be the point of $\{span(\beta, \beta') \cap g \mid \beta' \in B - \{\beta_1\}\}$ closest to β_1. It lies in the segment between β_1 and β_2. In this case let C be the interior of the cone spanned by β_1 and $\hat{\beta}$. By construction we have

Lemma V.4 *Let $d \in C$. For $1 \leq i < j \leq r$ write*

$$d = \lambda_{ij}\beta_i + \mu_{ij}\beta_j \ .$$

Then for $j = 2, \ldots, \ell$ one has $\lambda_{ij} > 0$, $\mu_{ij} > 0$ and

$$\lambda_{12} + \mu_{12} = \lambda_{13} + \mu_{13} = \cdots = \lambda_{1\ell} + \mu_{1\ell} \ .$$

We call this number r_d. Furthermore if $(i, j) \notin \{(1, 2), \ldots, (1, \ell)\}$ then

$$|\lambda_{ij}| + |\mu_{ij}| > r_d \ , \quad \textit{unless possibly } i = 1,\ \beta_j = \beta$$

Proof: First consider $i = 1$, $j \in \{2, \ldots, \ell\}$. Since $d \in C \subset \mathbb{R}_+\beta_1 + \mathbb{R}_+\beta_j$ we have $\lambda_{ij}, \mu_{i,j} > 0$. Since $span(\beta_1, \beta_2) = span(\beta_1, \beta_j)$ we have $\lambda_{12} + \mu_{12} = \lambda_{1j} + \mu_{1j}$ for all $2 \leq j \leq \ell$.

Next consider any $i < j$ except $i = 1$, $j \in \{2, \ldots, \ell\}$. Also exclude $\beta_i = \beta$ and $\beta_j = \beta$ in the event that there exists a $\beta \in B$ separated from 0 by g. Choose σ_i, $\sigma_j \in \{\pm 1\}$ so that $\sigma_i\lambda_{ij} > 0$, $\sigma_j\mu_{ij} > 0$. Then the line segment $\alpha\sigma_i\beta_i + (1 - \alpha)\sigma_j\beta_j$, $0 \leq \alpha \leq 1$, is contained in the convex hull of $B \setminus \{\pm\beta\}$ but is not contained in the edge of this hull with end points β_1 and β_ℓ. Thus as ζ increases, starting with $\zeta = 0$, ζd must hit the line segment joining $\sigma_i\beta_i$ and $\sigma_j\beta_j$ before it hits the line segment joining β_1 and β_ℓ (where it leaves the convex hull). Consequently $|\lambda_{ij}| + |\mu_{ij}| > \lambda_{12} + \mu_{12}$.

Finally consider β_i, $i \neq 1$ and $\beta_j = \beta$. By Lemma V.3.iv and the definition of $\hat{\beta}$, the line $span(\sigma_i\beta_i, \sigma_j\beta_j)$ separates the origin from the line segment joining β_1 and $\hat{\beta}$. Therefore $|\lambda_{ij}| + |\mu_{ij}| > \lambda_{12} + \mu_{12}$.

$\square$

For $j = 1, \ldots, \ell$ the sublattice

$$\Gamma_j^\sharp = (\mathbb{R}\beta_1 \cap \Gamma^\sharp) \oplus (\mathbb{R}\beta_j \cap \Gamma^\sharp)$$

is of finite index in $\Gamma^\sharp$. Choose

$$d \in C \cap \bigcap_{j=1}^{\ell} \Gamma_j^\sharp \ ,$$

and let $\gamma \in \Gamma$ be a primitive vector perpendicular to d. Define λ_{ij}, μ_{ij} as in Lemma V.4. By $(V.1)$ and Lemma V.4 there is an $\varepsilon > 0$ such that

$$\Delta_{n\,d} \;=\; |\gamma|^2 \sum_{j=1}^{\ell} \frac{\langle \beta_1, \beta_j \rangle}{\langle \beta_1, \gamma \rangle \, \langle \beta_j, \gamma \rangle} \hat{q}(n\lambda_{1j}\beta_1)\hat{q}(n\mu_{1j}\beta_j) + O\left(e^{-n(r_d+\varepsilon)}\right)$$

$$(V.2)$$

$$= \quad |\gamma|^2 n^2 e^{-nr_d} F(n) + O(ne^{-nr_d})$$

where

$$F(n) = \sum_{j=1}^{\ell} \frac{\langle \beta_1, \beta_j \rangle \, \lambda_{1j}\mu_{1j}}{\langle \beta_1, \gamma \rangle \, \langle \beta_j, \gamma \rangle} \left(\sum_{i=1}^{m_1} k_{i1} (e^{i\lambda_{1j}\varphi_{i1}})^n \right) \cdot \left(\sum_{i=1}^{m_j} k_{ij} (e^{i\mu_{1j}\varphi_{ij}})^n \right)$$

Since $\frac{\langle \beta_1, \beta_j \rangle \lambda_{1j}\mu_{1j}}{\langle \beta_1, \gamma \rangle \langle \beta_j, \gamma \rangle} \leq 0$, and it is equal to zero only if $j = \ell$, $\langle \beta_1, \beta_\ell \rangle = 0$, and $k_{ij} > 0$, the function $F(n)$ is of the form of Lemma V.2. So

$$\limsup_{n \to \infty} |F(n)| > 0$$

and Proposition 3 follows from $(V.2)$

VI Appendix

Here we prove a topological statement used in Section IV.

Lemma VI.1 *Let U be a neighbourhood of the origin in $\mathbb{C}^2 \times \mathbb{R}$ and let $g : U \to \mathbb{C}$ be a continuous function whose restriction to $\{(z, w) \in U \mid w = 0\}$ and to $\{(z, w) \in U \mid w \neq 0\}$ are analytic such that for some $n > 0$*

$$\lim_{(z,w) \to 0} \frac{g(z_1, z_2, w) - (z_1 z_2 - w^n)}{|z_1|^2 + |z_2|^2 + |w|^n} = 0 \ .$$

Then there is a homeomorphism $\psi : U' \to U''$ between neighbourhoods of 0 in $\mathbb{C}^2 \times \mathbb{R}$ with the following properties *(i)* *ψ commutes with the projection $\pi : U \to \mathbb{R}$, $(z, w) \mapsto w$, i.e. the diagram*

$$\begin{array}{ccc} U' & \xrightarrow{\ \psi\ } & U'' \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathbb{R} & \end{array}$$

is commutative.

(ii) *$\psi(z, 0) = (z, 0)$ for all $(z, 0) \in U'$*

(iii) *The restriction of ψ to $\{(z, w) \in U' \mid w \neq 0\}$ is a diffeomorphism*

(iv) *ψ maps $\{(z, w) \in U' \mid g(z_1, z_2, w) = 0\}$ onto*

$$\{(z_1, z_2, w) \in U'' \mid z_1 z_2 - w^n = 0\}$$

In particular, for small $\varepsilon > 0$ and $w_0 \neq 0$ the set

$$\{(z, w_0) \in U \mid g(z, w_0) = 0, |z| < \varepsilon\}$$

is diffeomorphic to a cylinder.

Proof: By the Morse Lemma we may assume that $g(z_1, z_2, 0) = z_1 z_2$. Write

$$\begin{aligned} g_0(z_1, z_2, w) &= z_1 z_2 - w^n \\ g(z_1, z_2, w) &= g_0(z_1, z_2, w) + h(z_1, z_2, w) \end{aligned}$$

Then $h(z_1, z_2, 0) = 0$ and

$$\lim_{(z,w)\to 0} \frac{h(z_1, z_2, w)}{|z_1|^2 + z_2|^2 + |w|^n} = 0$$

The multiplicative group $\mathbb{R}^* := \{\tau \in \mathbb{R} \mid \tau \neq 0\}$ acts on $\mathbb{C}^2 \times \mathbb{R}$ by

$$\tau \cdot (z_1, z_2, w) = (\tau^n z_1, \tau^n z_2, \tau^2 w).$$

This action preserves

$$X_0 := \Big\{(z_1, z_2, w) \in \mathbb{C}^2 \times \mathbb{R} \mid g_0(z_1, z_2, w) = 0\Big\} \ .$$

Since the set of $\mathbb{R}^*$-orbits on $\mathbb{C}^2 \times \mathbb{R} - \{0\}$ is compact there is an $\mathbb{R}^*$-invariant open neighbourhood T of $X_0 - \{0\}$ in $\mathbb{C}^2 \times \mathbb{R} \setminus \{0\}$, a finite covering T_i of T by $\mathbb{R}^*$-invariant open sets and C^∞-projections

$$\pi_i : T_i \rightarrow U_i := T_i \cap X_0$$

whose fibres $\pi_i^{-1}(z, w)$ over $(z, w) \in X_0$ are complex submanifolds of $\mathbb{C}^2 \times \{w\}$ isomorphic to $\{\zeta \in \mathbb{C} \mid \ |\zeta| < 1\}$. By the assumption on h and the compactness of $(\mathbb{C}^2 \times \mathbb{R} \setminus T)/\mathbb{R}^*$ we may, after possibly shrinking U, assume that

$$|h(z_1, z_2, w)| < |g_0(z_1, z_2, w)|$$

for all $(z_1, z_2, w) \in U - T$. So, by Rouché's theorem, for each $(z, w) \in U_i$ and each $t \in [0, 1]$ the function $g_0(z, w) + t\, h(z, w)$ has a unique zero in $\pi_i^{-1}(z, w)$. Therefore there are a neighbourhood U' of 0 in $\mathbb{C}^2 \times \mathbb{R}$ and $\mathbb{R}^*$-invariant vectorfields V_i on $T_i \cap U'$ such that integration for time $t \in [0, 1]$ maps $U_i \cap U'$ to the intersections of U' with

$$X_t := \{(z, w) \in U' \mid g_0(z, w) + t\, h(z, w) = 0\} \ .$$

V_i can be chosen to be zero in $\{(z, w) \in T_i \mid w = 0\}$ and on ∂T_i and to be C^∞ on $\{(z, w) \in T_i \mid w \neq 0\}$. Using a partition of unity we get a neighbourhood U' of 0 in $\mathbb{C}^2$ and a $\mathbb{R}^*$-invariant vectorfield V on $\mathring{T} \cap U'$ such that integration for time t maps $X_0 \cap U'$ to $X_t \cap T'$ and such that $V = 0$ on $\{(z,w) \in T \mid w = 0\} \cup \partial T$, and $V\Big|_{\{(z,w)\in T \mid w\neq 0\}}$ is C^∞. V can be prolonged by 0 to $U' \setminus T$, and integration of V gives the desired map ψ.

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