Evaluation of Fermion Loops by Higher Residues

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§I Introduction

Let

$$e(\mathbf{k}) \;=\; \tfrac{1}{2m}\,|\mathbf{k}|^2 \;-\; \mu \quad , \quad \mathbf{k} \in \mathbb{R}^2$$

be the dispersion relation for a two dimensional electron gas with chemical potential $\mu > 0$. By definition, the amplitude of the $(n+1)$ Fermion loop with external momenta $q_i = (q_{i\,0}, \mathbf{q}_i) \in \mathbb{R} \times \mathbb{R}^2$, $i = 1, 2, \cdots, n+1$, is

$$I(q_1, \cdots, q_{n+1}) \;=\; \tfrac{1}{(2\pi)^3} \int_{\mathbb{R}^2} dk_1\, dk_2 \int_{-\infty}^{+\infty} \frac{dk_0}{\prod\limits_{i=1}^{n+1} \big(\imath(k-q_i)_0 - e(\mathbf{k}-\mathbf{q}_i)\big)}$$

Recall that the amplitude of an arbitrary diagram contributing to the formal power series expansion of the corresponding many fermion Green's functions in powers of the coupling constant is obtained by integrating products of Fermion loops and interactions.

It is not hard to evaluate the 2-loop (the "polarization bubble") $I(q_1, q_2)$ when $(q_1 - q_2)_0 \neq 0$ and $\mathbf{q}_1 - \mathbf{q}_2 \neq 0$. One finds (see, for example, [FKST], Proposition II.1)

$$I(q_1, q_2) \;=\; -\tfrac{m}{2\pi} + \tfrac{m k_F}{2\pi\,|\mathbf{q}_1 - \mathbf{q}_2|}\, \Re\left(\alpha - \frac{1}{\alpha}\right)$$

where α is the root outside the unit circle of

$$z^2 - \tfrac{1}{k_F}\left(|\mathbf{q}_1 - \mathbf{q}_2| - 2m\imath\, \tfrac{(q_1-q_2)_0}{|\mathbf{q}_1 - \mathbf{q}_2|}\right) z + 1 = 0$$

and $k_F = \sqrt{2m\mu}$. The zero frequency limit has a particularly interesting behaviour. Namely,

$$I\big((0\,,\mathbf{q}_1), (0\,,\mathbf{q}_2)\big) \;=\; \begin{cases} -\,\tfrac{m}{2\pi}\;, & 0 \le |\mathbf{q}_1 - \mathbf{q}_2| \le 2k_F \\ -\,\tfrac{m}{2\pi}\left(1 - \tfrac{1}{|\mathbf{q}_1-\mathbf{q}_2|}\sqrt{|\mathbf{q}_1-\mathbf{q}_2|^2 - 4k_F^2}\right)\;, & |\mathbf{q}_1 - \mathbf{q}_2| \ge 2k_F \end{cases}$$

In particular, the value of the 2-loop is $-\frac{m}{2\pi}$ whenever the disks of radius k_F centered at $\mathbf{q}_1$ and $\mathbf{q}_2$ overlap.

In this paper, we evaluate the $(n+1)$ Fermion loop explicitly for all $n \geq 2$. Our result (see, Theorem III.1 and the Remarks after it) is

$$I(q_1, \cdots, q_{n+1}) \;=\; \tfrac{m^n}{2\pi\imath} \sum_{\substack{i,j=1 \\ i\neq j}}^{n+1} \int_{w_{ij}} \varphi_{ij}(z)\; dz$$

where φ_{ij} is an explicitly computable rational function of one complex variable and w_{ij} is an explicitly given curve in $\mathbb{C}$. For $n = 2$, that is the 3-loop, we have written out the function φ_{ij} and the path w_{ij} in all detail.

Again, the zero frequency limit

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) = \lim_{\substack{q_i \to (0, \mathbf{p}_i) \\ q_{i\,0} \neq q_{j\,0}}} I(q_1, \cdots, q_{n+1})$$

is interesting. The third author (see, [Si]) evaluated $J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ numerically and observed that

$$J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = 0$$

whenever the disks of radius k_F *centered at* $\mathbf{p}_1$, $\mathbf{p}_2$ *and* $\mathbf{p}_3$ *have a point in common.* We have evaluated (Theorem III.2) $J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1})$. In particular (Corollary III.3 (ii)), for all $n \geq 2$,

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) = 0$$

when all $n+1$ disks with radius k_F around the points $\mathbf{p}_1, \cdots . \mathbf{p}_{n+1}$ have at least one point in common. * Also (Corollary III.3 (iii)),

$$J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \frac{m^2}{2 \times \text{area of the triangle with vertices } \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3}$$

when the three disks with radius k_F around the points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ have no point in common but any two of the disks intersect.

To evaluate $I(q_1, \cdots, q_{n+1})$, we perform (see, the beginning of §III) the integral over k_0 to obtain

$$I(q_1, \cdots, q_{n+1}) = \frac{m^n}{(2\pi)^2} \sum_{i=1}^{n+1} \int_{|\mathbf{k} - \mathbf{q}_i| < k_F} \frac{dk_1 \, dk_2}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n+1} f_{ij}(\mathbf{k})}$$

where

$$f_{ij}(\mathbf{k}) = (\mathbf{q}_j - \mathbf{q}_i) \cdot \mathbf{k} + \tfrac{1}{2} \left(\mathbf{q}_i^2 - \mathbf{q}_j^2 \right) + \imath \, m \, (q_{i\,0} - q_{j\,0})$$

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* The Appendix contains an elementary proof of this fact for the special case that $n = 2$ and the triangle with vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is acute.

Next, it is observed that each summand

$$\int_{|\mathbf{k}-\mathbf{q}_i|<k_F} \frac{dk_1 dk_2}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n+1} f_{ij}(\mathbf{k})}$$

is of the generic form

$$\int_{\mathrm{D}_\rho(\mathbf{q})} \frac{dx_1 \wedge dx_2}{\prod\limits_{j=1}^{n} (a_{j1}x_1 + a_{j2}x_2 - b_j)}$$

where the coefficients a_{j1}, a_{j2} , $j = 1, \cdots, n$, are real and b_j is complex. Here, $\mathrm{D}_\rho(\mathbf{q})$ is the disk of radius $\rho > 0$ centered at $\mathbf{q}$. In §II, we replace the disk by a homologous cycle in the complex projective plane $\mathbb{P}^2$ where higher residues can be applied.

It is a great pleasure to thank Andrea Cavalli and Peter Wagner for many helpful suggestions. Peter Wagner also found a different proof of Theorem III.2, using methods similar to those of [OW]. He has generously allowed us to incorporate some of his improvements in the statement of Theorem III.2 itself.

§II The Integral of a Particular Rational Function over a Disk

Let $n \geq 2$ and fix a real matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{pmatrix}$$

such that no two by two subdeterminant vanishes. Fix $b \in \mathbb{C}^n$ such that $\Im b_j \neq 0$, for all $j = 1, \cdots, n$. For each $\mathbf{q} \in \mathbb{R}^2$ and $\rho \geq 0$ let

$$\mathrm{D}_\rho(\mathbf{q}) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, (x_1 - \mathbf{q}_1)^2 + (x_2 - \mathbf{q}_2)^2 \leq \rho^2 \right\}$$

be the closed disk of radius ρ with the standard orientation, centered at $\mathbf{q}$. We shall compute

$$\int_{\mathrm{D}_\rho(\mathbf{q})} \frac{dx_1 \wedge dx_2}{\prod_{j=1}^{n} (a_{j1}x_1 + a_{j2}x_2 - b_j)}$$

To formulate the result, let L_j, $j = 1 \cdots, n$, be the line in the complex projective plane $\mathbb{P}^2$ given by

$$L_j = \left\{ [z_0, z_1, z_2] \in \mathbb{P}^2 \,\middle|\, a_{j1}z_1 + a_{j2}z_2 = b_j z_0 \right\}$$

and let

$$L_\infty = \left\{ [z_0, z_1, z_2] \in \mathbb{P}^2 \,\middle|\, z_0 = 0 \right\}$$

We assume that the intersections

$$L_\alpha \cap L_\beta = \left\{ d^{\alpha,\beta} \right\}$$

for $\alpha, \beta = 1, \cdots, n, \infty$ with $\alpha \neq \beta$ are pairwise different and do not lie on the projective quadric

$$Q_s = \left\{ [z_0, z_1, z_2] \in \mathbb{P}^2 \,\middle|\, (z_1 - \mathbf{q}_1 z_0)^2 + (z_2 - \mathbf{q}_2 z_0)^2 = s^2 z_0^2 \right\}$$

for either $s = 0$ or $s = \rho$.

If $n \geq 3$, let η_j, $j = 1, \cdots, n$, be the unique rational one form on the projective line L_j that has a simple pole at the point $d^{j,k}$ for all $k = 1, \cdots, n$, $k \neq j$, with residue

$$\frac{1}{\det \begin{pmatrix} a_{j1} & a_{j2} \\ a_{k1} & a_{k2} \end{pmatrix}} \prod_{\ell \neq j,k} \left(\frac{1}{d_0^{j,k}} \left(a_{\ell 1} d_1^{j,k} + a_{\ell 2} d_2^{j,k} - b_\ell d_0^{j,k} \right) \right)^{-1}$$

no other poles , and a zero of order $n-3$ at $d^{j,\infty}$. We will see below that the sum of these residues is zero and for this reason the form η_j does in fact exist. If $n=2$, let η_j , $j=1,2$, be the unique rational one form on the projective line L_j that has a simple pole at the point $d^{j,k}$ for $k \neq j$, with residue

$$\frac{1}{\det\begin{pmatrix} a_{j1} & a_{j2} \\ a_{k1} & a_{k2} \end{pmatrix}}$$

a simple pole at $d^{j,\infty}$, and no other poles.

For all $s \in \mathbb{C}$, set

$$\begin{aligned} Q_s^+ &= \left\{ [z_0, z_1, z_2] \in Q_s \;\middle|\; \Im \tfrac{z_2 - \mathbf{q}_2}{z_1 - \mathbf{q}_1} > 0 \right\} \\ Q_s^- &= \left\{ [z_0, z_1, z_2] \in Q_s \;\middle|\; \Im \tfrac{z_2 - \mathbf{q}_2}{z_1 - \mathbf{q}_1} < 0 \right\} \end{aligned}$$

We will show, see Lemma II.2, that for all $s \geq 0$ and all $j = 1, \cdots, n$, the intersection $Q_s^+ \cap L_j$ consists of exactly one point, say, $c_j^+(s)$ and the intersection $Q_s^- \cap L_j$ consists of exactly one point, say, $c_j^-(s)$.

Theorem II.1 *Suppose that for each $j = 1, \cdots, n$, the paths $c_j^{\pm}(s)$, $0 \leq s \leq \rho$, do not pass through any of the double points $L_j \cap L_k$, $j,k = 1, \cdots, n$, $j \neq k$. Then,*

$$\begin{aligned} \int_{D_\rho(\mathbf{q})} \frac{dx_1 \wedge dx_2}{\prod_{j=1}^{n} (a_{j1}x_1 + a_{j2}x_2 - b_j)} &= -2\pi\imath \sum_{j=1}^{n} \int_{\left\{ c_j^+(s) \,\middle|\, 0 \leq s \leq \rho \right\}} \eta_j \\ &= +2\pi\imath \sum_{j=1}^{n} \int_{\left\{ c_j^-(s) \,\middle|\, 0 \leq s \leq \rho \right\}} \eta_j \end{aligned}$$

Remark. Observe that for any path $s(\sigma)$, $0 \leq \sigma \leq 1$, in $\mathbb{C}$ joining 0 to ρ and lying sufficiently close to the interval $[0,\rho]$, the intersection $Q_{s(\sigma)}^{\pm} \cap L_j$ consists of exactly one point $c_j^{\pm}(s(\sigma))$. Our hypotheses imply that $c_j^{\pm}(0)$ and $c_j^{\pm}(\rho)$ are not double points $L_j \cap L_k$. It is therefore possible to choose a path $s(\sigma)$, $0 \leq \sigma \leq 1$, arbitrarily close to $[0,\rho]$ such that

$c_j^{\pm}(s(\sigma))$ is not a double point for all $j = 1, \cdots, n$ and $0 \le \sigma \le 1$. With this choice

$$\int_{D_\rho(\mathbf{q})} \frac{dx_1 \wedge dx_2}{\prod_{j=1}^{n} (a_{j1}x_1 + a_{j2}x_2 - b_j)} = 2\pi\imath \sum_{j=1}^{n} \int_{\left\{c_j^+(s(\sigma)) \,\middle|\, 0 \le \sigma \le 1\right\}} \eta_j$$

$$= -2\pi\imath \sum_{j=1}^{n} \int_{\left\{c_j^-(s(\sigma)) \,\middle|\, 0 \le \sigma \le 1\right\}} \eta_j$$

Two lemmas are required for the proof of Theorem II.1. Observe that by translation invariance, we may assume that $\mathbf{q} = 0$.

Lemma II.2 *For all $s \ge 0$ and all $\alpha = 1, \cdots, n, \infty$, the intersection $Q_s^+ \cap L_\alpha$ consists of exactly one point, say, $c_\alpha^+(s)$ and the intersection $Q_s^- \cap L_\alpha$ consists of exactly one point, say, $c_\alpha^-(s)$. Furthermore, $c_\infty^+(s)$ and $c_\infty^-(s)$ do not depend on s.*

Proof: First, fix $1 \le j \le n$. We can rotate (z_1, z_2) around zero to make $a_{j2} = 0$. The intersection $Q_s \cap L_j$ is determined by the equation

$$z_1^2 + z_2^2 = s^2 \left(\tfrac{a_{j1}}{b_j}\right)^2 z_1^2$$

Thus,

$$\left(\tfrac{z_2}{z_1}\right)^2 = s^2 \left(\tfrac{a_{j1}}{b_j}\right)^2 - 1$$

It follows that there is exactly one root with positive imaginary part and one with negative imaginary part since a_{j1} is real and $\Im b_j \ne 0$. By inspection,

$$Q_s \cap L_\infty = \Big\{ [0, 1, \imath], [0, 1, -\imath] \Big\}$$

■

Embed $\mathbb{C}^2$ in $\mathbb{P}^2$ by the standard map

$$(z_1, z_2) \in \mathbb{C}^2 \longrightarrow [z_0, z_1, z_2] \in \mathbb{P}^2$$

Let ω be the meromorphic two form on $\mathbb{P}^2$ given by

$$\omega \;=\; \frac{dz_1 \wedge dz_2}{\prod\limits_{j=1}^{n} (a_{j1}z_1 + a_{j2}z_2 - b_j)}$$

If $n \geq 3$, then ω has simple poles on the lines $L_1 \cdots, L_n$ and is holomorphic on the difference $\mathbb{P}^2 \setminus \big(L_1 \cup L_2 \cup \cdots \cup L_n\big)$. If $n = 2$, then ω has simple poles on $L_1\, L_2\, L_\infty$, and is holomorphic on $\mathbb{P}^2 \setminus \big(L_1 \cup L_2 \cup L_\infty\big)$.

Regard the union of lines

$$C \;=\; L_1 \cup L_2 \cup \cdots \cup L_n \cup L_\infty$$

as a singular algebraic curve of degree $n+1$ in $\mathbb{P}^2$. Then, the Poincare' residue of ω along the curve C (see, for example [GH], p.147),

$$\eta \;=\; \mathrm{res}_C\, \omega$$

is a Rosenlicht differential on C (see [S], Ch. IV.9).

Fix, $j = 1, \cdots, n$. For each $k \neq j$,

$$\omega \;=\; \frac{1}{\det\begin{pmatrix} a_{j1} & a_{j2} \\ a_{k1} & a_{k2}\end{pmatrix} \prod\limits_{\ell \neq j,k}^{n} (a_{\ell 1}z_1 + a_{\ell 2}z_2 - b_\ell)} \; \frac{d(a_{j1}z_1 + a_{j2}z_2 - b_j)}{a_{j1}z_1 + a_{j2}z_2 - b_j} \wedge \frac{d(a_{k1}z_1 + a_{k2}z_2 - b_k)}{a_{k1}z_1 + a_{k2}z_2 - b_k}$$

It follows from this representation that the restriction of η to the projective line L_j has a simple pole at the point $d^{j,k}$ for all $k = 1, \cdots, n$, $k \neq j$, with residue

$$\frac{1}{\det\begin{pmatrix} a_{j1} & a_{j2} \\ a_{k1} & a_{k2}\end{pmatrix}} \prod_{\ell \neq j,k} \left(\frac{1}{d_0^{j,k}} \big(a_{\ell 1} d_1^{j,k} + a_{\ell 2} d_2^{j,k} - b_\ell d_0^{j,k} \big) \right)^{-1}$$

Furthermore, it is holomorphic and does not vanish on $L_j \setminus \big\{ d^{j,\alpha} \,\big|\, \alpha = 1, \cdots, n, \infty\,,\; j \neq \alpha \big\}$. Consequently, $\eta\big|_{L_j}$ has a zero of order $n-3$ at $d^{j,\infty}$, when $n \geq 3$, and a simple pole when $n = 2$. It follows from the preceding remarks that the restriction $\eta\big|_{L_j}$, has all the properties of the rational one form η_j introduced above. In particular, η_j exists and

$$\eta_j \;=\; \eta\big|_{L_j}$$

Set

$$\eta_\infty = \eta\big|_{L_\infty}$$

For $n \geq 3$, the restriction $\eta_\infty = 0$, since ω is regular along L_∞. If $n = 2$, then η_∞ has simple poles at the points $d^{1,\infty}$ and $d^{2,\infty}$, and no other poles.

We need the following residue formula.

Lemma II.3 *Fix* $\alpha = 1, \cdots, n, \infty$. *Let* p_1 *and* p_2 *be any two points on*

$$L_\alpha \setminus \{\, d^{\alpha,\beta} \,\big|\, \beta = 1, \cdots, n, \infty\,,\ \beta \neq \alpha \,\}$$

Suppose S_1 *and* S_2 *are algebraic curves in* $\mathbb{P}^2$ *such that* L_α *meets* S_1 *transversally at* p_1 *and* S_2 *transversally at* p_2. *Let*

$$\pi : T_\alpha \longrightarrow L_\alpha \setminus \{\, d^{\alpha,\beta} \,\big|\, \beta = 1, \cdots, n, \infty\,,\ \beta \neq \alpha \,\}$$

be a tubular neighborhood of $L_\alpha \setminus \{\, d^{\alpha,\beta} \,\big|\, \beta = 1, \cdots, n, \infty\,,\ \beta \neq \alpha \,\}$ *such that*

$$S_i \cap T_\alpha = \pi^{-1}(p_i)$$

for $i = 1, 2$. *Finally, let* γ *be a path on* $L_\alpha \setminus \{\, d^{\alpha,\beta} \,\big|\, \beta = 1, \cdots, n, \infty\,,\ \beta \neq \alpha \,\}$ *joining* p_1 *to* p_2 *and set*

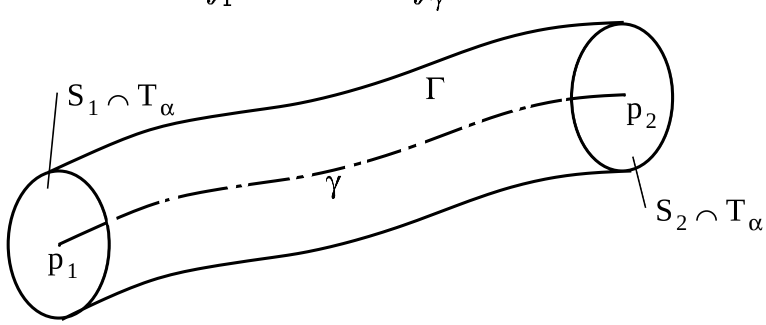
$$\Gamma = \pi^{-1}(\gamma) \cap \partial T$$

The "cylinder" Γ *is oriented so that*

$$\partial\Gamma = \partial\big(S_2 \cap T_\alpha\big) - \partial\big(S_1 \cap T_\alpha\big)$$

Here, $S_i \cap T_\alpha$, $i = 1, 2$, *is given the standard orientation of an open subset of the Riemann surface* S_i. *Then,*

$$\int_\Gamma \omega = -2\pi\imath \int_\gamma \eta_\alpha$$



Proof: The proof is similar to that of [Seb]. ■

Remark. By convention, the boundary of an oriented surface is oriented such that the ordered pair of the outward pointing normal and an oriented tangent vector gives the orientation of the surface.

Proof of Theorem II.1 : By construction, for all $\alpha = 1, \cdots, n, \infty$, the projective line L_α intersects the quadric $\mathcal{Q}_s$ transversally at the points $c_\alpha^\pm(s)$. For $0 \le s \le \rho$ let

$$\pi_{\alpha,s},\colon T_{\alpha,s} \longrightarrow L_\alpha \setminus \{ d^{\alpha,\beta} \,|\, \beta = 1, \cdots, n, \infty, \beta \neq \alpha \}$$

be tubular neighborhoods of $L_\alpha \setminus \{ d^{\alpha,\beta} \,|\, \beta = 1, \cdots, n, \infty, \beta \neq \alpha \}$ such that

$$\mathcal{Q}_t^\pm \cap T_{\alpha,s} \;=\; \pi_{\alpha,s}^{-1}\big(c_\alpha^\pm(t)\big)$$

for $t = 0, s$, and

$$\Delta_{\alpha,0}^\pm \;=\; \mathcal{Q}_0^\pm \cap T_{\alpha,s}$$

is independent of s. Set

$$\Delta_{\alpha,s}^\pm \;=\; \mathcal{Q}_s^\pm \cap T_{\alpha,s}$$

By convention, the disks $\Delta_{\alpha,t}^\pm$, $t = 0, s$, are oriented as subsets of the Riemann surface $\mathcal{Q}_t^\pm$. Also, set

$$\Gamma_{\alpha,s}^\pm \;=\; \pi_{\alpha,s}^{-1}\left(\{\, c_\alpha^\pm(t) \;\big|\; 0 \le t \le s \,\}\right) \cap \partial T_{\alpha,s}$$

The "cylinder" $\Gamma_{\alpha,s}^\pm$ is oriented so that

$$\partial\Gamma_{\alpha,s}^\pm \;=\; \partial\Delta_{\alpha,s}^\pm - \partial\Delta_{\alpha,0}^\pm$$

By Lemma II.3,

$$\int_{\Gamma_{\alpha,s}^\pm} \omega \;=\; -\,2\pi\imath \int_{\{\, c_\alpha^\pm(t) \;|\; 0 \le t \le s \,\}} \eta_\alpha$$

Now, for each $0 \le s \le \rho$, set

$$Z_s^\pm \;=\; \pm D_s(0) - \left(\mathcal{P}_s^\pm - \mathcal{P}_0^\pm + \sum_{j=1}^{n} \Gamma_{j,s}^\pm + \Gamma_{\infty,s}^\pm\right)$$

where, for $t = 0, s$

$$\mathcal{P}_t^{\pm} \;=\; \overline{\mathcal{Q}_t^{\pm} \setminus \Big(\bigcup_{j=1}^{n} \Delta_{j,t}^{\pm} \cup \Delta_{\infty,t}^{\pm} \Big)}$$

We have for $t = 0, s$

$$\partial \mathcal{P}_t^{\pm} \;=\; \partial \mathcal{Q}_t^{\pm} \;-\; \sum_{j=1}^{n} \partial \Delta_{j,t}^{\pm} \;-\; \partial \Delta_{\infty,t}^{\pm}$$

and

$$\partial \mathcal{Q}_s^{\pm} \;=\; \pm \partial D_s(0) \qquad , \qquad \partial \overline{\mathcal{Q}_0^{\pm}} \;=\; \emptyset$$

Therefore,

$$\begin{aligned} \partial Z_s^{\pm} \;&=\; \pm\, \partial D_s(0) - \Big(\partial \mathcal{Q}_s^{\pm} \;-\; \sum_{j=1}^{n} \partial \Delta_{j,s}^{\pm} \;-\; \partial \Delta_{\infty,s}^{\pm} \Big) + \Big(\partial \mathcal{Q}_0^{\pm} \;-\; \sum_{j=1}^{n} \partial \Delta_{j,0}^{\pm} \;-\; \partial \Delta_{\infty,0}^{\pm} \Big) \\ &\quad - \sum_{j=1}^{n} \Big(\partial \Delta_{j,s}^{\pm} - \partial \Delta_{j,0}^{\pm} \Big) \;-\; \Big(\partial \Delta_{\infty,s}^{\pm} - \partial \Delta_{\infty,0}^{\pm} \Big) \\ &= 0 \end{aligned}$$

In other words, $Z_s^{\pm}$ is a 2-cycle in $\mathbb{P}^2 \setminus C$.

Observe that the homology class $[Z_s^{\pm}] \in H_2(\mathbb{P}^2 \setminus C, \mathbb{Z})$ represented by $Z_s^{\pm}$ is 0 since, by construction, $Z_s^{\pm}$ depends continuously on s and $[Z_0^{\pm}] = 0$. The integral of the holomorphic two form ω over the one dimensional complex manifolds $\mathcal{P}_t^{\pm}$, $t = 0, s$ vanishes. It follows that

$$\begin{aligned} 0 \;=\; \int_{Z_s^{\pm}} \omega \;&=\; \pm \int_{D_s(0)} \omega \;-\; \sum_{j=1}^{n} \int_{\Gamma_{j,s}^{\pm}} \omega \;-\; \int_{\Gamma_{\infty,s}^{\pm}} \\ &=\; \pm \int_{D_s(0)} \omega \;+\; 2\pi\imath \sum_{j=1}^{n} \int_{\left\{ c_j^{\pm}(s) \,\middle|\, 0 \le s \le \rho \right\}} \eta_j \;+\; 2\pi\imath \int_{\left\{ c_\infty^{\pm}(s) \,\middle|\, 0 \le s \le \rho \right\}} \eta_\infty \end{aligned}$$

Since, $c_\infty^{+}(s)$ and $c_\infty^{-}(s)$ do not depend on s,

$$\int_{\left\{ c_\infty^{\pm}(s) \,\middle|\, 0 \le s \le \rho \right\}} \eta_\infty \;=\; 0$$

Therefore

$$\int_{\mathrm{D}_\rho(\mathbf{q})} \omega \;=\; \mp\; 2\pi\imath \sum_{j=1}^{n} \int_{\left\{ c_j^{\pm}(s) \,\middle|\, 0 \le s \le \rho \right\}} \eta_j$$

■

Remark. For all $0 \leq s \leq \rho$, the divisor of the rational function

$$\frac{z_1^2 + z_2^2 - s^2 z_0^2}{z_1^2 + z_2^2}$$

on the curve C is

$$\sum_{j=1}^{n} \left(c_j^+(s) - c_j^+(0)\right) + \left(c_j^-(s) - c_j^-(0)\right)$$

It therefore also follows from Abel's theorem for the singular curve C and the Rosenlicht differential η that

$$\sum_{j=1}^{n} \int_{\{ c_j^+(s) \,|\, 0 \leq s \leq \rho \}} \eta_j = - \sum_{j=1}^{n} \int_{\{ c_j^-(s) \,|\, 0 \leq s \leq \rho \}} \eta_j$$

§III The Evaluation of Fermion Loops

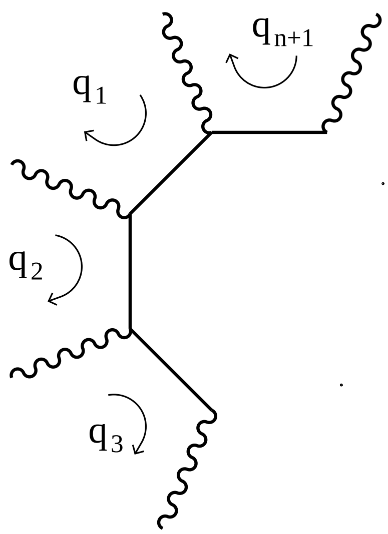
Let $q_i = (q_{i\,0}, \mathbf{q}_i)$, $i = 1, 2, \cdots, n+1$, be vectors in $\mathbb{R} \times \mathbb{R}^2$. We assume that no three of the points $\mathbf{q}_1, \cdots, \mathbf{q}_{n+1}$ lie on a line. The amplitude of the (n+1)-loop with momenta $q_1, q_2, \cdots, q_{n+1}$ is

$$I(q_1, \cdots, q_{n+1}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} dk_1\, dk_2 \int_{-\infty}^{+\infty} \frac{dk_0}{\prod_{i=1}^{n+1} \big(\imath(k-q_i)_0 - e(\mathbf{k}-\mathbf{q}_i)\big)}$$

where the dispersion relation $e(\mathbf{k})$ is given by

$$e(\mathbf{k}) = \frac{1}{2m}\left(\mathbf{k}^2 - k_F^2\right)$$

Here, $k_F = \sqrt{2\mathbf{m}\mu}$ is the Fermi momentum.



If $e(\mathbf{k} - \mathbf{q}_i) \neq 0$ for $i = 1, \cdots, n+1$, and $n \geq 1$ then, by the residue theorem (closing the contour in the upper half plane),

$$\int_{-\infty}^{+\infty} \frac{dk_0}{\prod_{i=1}^{n+1} \big(\imath(k_0 - q_{i\,0}) - e(\mathbf{k}-\mathbf{q}_i)\big)} = 2\pi\, m^n \sum_{i=1}^{n+1} \frac{\chi\big(e(\mathbf{k}-\mathbf{q}_i)<0\big)}{\prod_{\substack{j=1\\ j\neq i}}^{n+1} f_{ij}(\mathbf{k})}$$

where

$$\begin{aligned} f_{ij}(\mathbf{k}) &= m\Big(e(\mathbf{k}-\mathbf{q}_i) - e(\mathbf{k}-\mathbf{q}_j) + \imath(q_{i\,0} - q_{j\,0})\Big) \\ &= (\mathbf{q}_j - \mathbf{q}_i)\cdot\mathbf{k} + \tfrac{1}{2}\left(\mathbf{q}_i^2 - \mathbf{q}_j^2\right) + \imath m\,(q_{i\,0} - q_{j\,0}) \end{aligned}$$

Observe that $f_{ij}(\mathbf{k})$, $i,j = 1, \cdots n+1$, is an affine linear function of $\mathbf{k}$. Substituting, we obtain

$$I(q_1, \cdots, q_{n+1}) \;=\; \tfrac{m^n}{(2\pi)^2} \sum_{i=1}^{n+1} \int_{|\mathbf{k}-\mathbf{q}_i|<k_F} \frac{dk_1 dk_2}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n+1} f_{ij}(\mathbf{k})}$$

It follows from this representation that $I(q_1, \cdots, q_{n+1})$ is a continuous function on the set

$$\left\{ \left((q_{1\,0}, \mathbf{q}_1), \cdots, (q_{n+1\,0}, \mathbf{q}_{n+1})\right) \;\in\; \left(\mathbb{R} \times \mathbb{R}^2\right)^{n+1} \;\middle|\; q_{i\,0} \neq q_{j\,0} \text{ for } i \neq j \right\}$$

Observe that Theorem II.1 can be used to compute each of the summands

$$\int_{|\mathbf{k}-\mathbf{q}_i|<k_F} \frac{dk_1 dk_2}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n+1} f_{ij}(\mathbf{k})}$$

For each pair $1 \le i \ne j \le n+1$,

$$\ell_{ij} \;=\; \left\{\; z \in \mathbb{C}^2 \;\middle|\; f_{ij}(z) = 0 \;\right\}$$

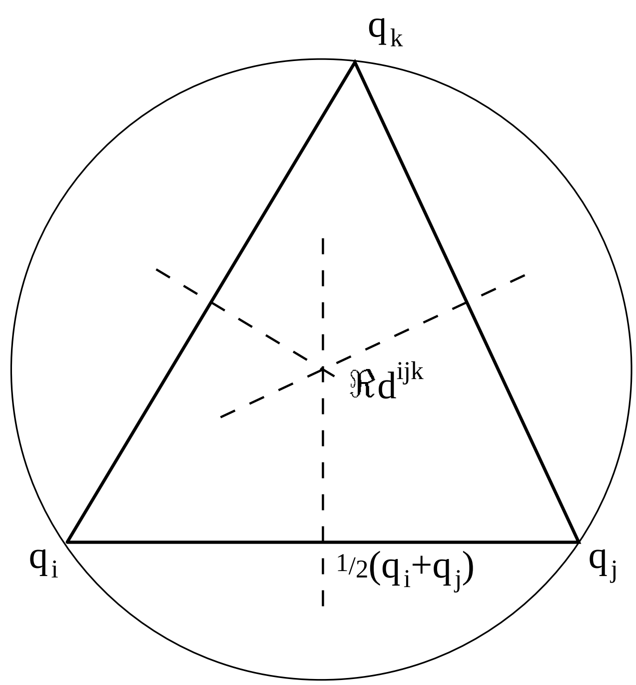
is a line in $\mathbb{C}^2$. We have

$$\ell_{ij} \;=\; \ell_{ji}$$

since $f_{ij} = -f_{ji}$. If $q_{i\,0} = q_{j\,0}$ the line $\ell_{ij} \cap \mathbb{R}^2$ is the perpendicular bisector of the points $\mathbf{q}_i$ and $\mathbf{q}_j$. Observe that for any three different indices $1 \le i,j,k \le n+1$ the lines ℓ_{ij} and ℓ_{jk} intersect in exactly one point $\mathbf{d}^{ijk}$ because $\mathbf{q}_i$, $\mathbf{q}_j$ and $\mathbf{q}_k$ do not lie on a line. Furthermore, $\mathbf{d}^{ijk}$ is the common intersection point of the three lines $\ell_{ij} = \ell_{ji}$, $\ell_{jk} = \ell_{kj}$, $\ell_{ki} = \ell_{ik}$, since

$$f_{ij} + f_{jk} + f_{ki} = 0$$

Observe that $\Re\,\mathbf{d}^{ijk}$ is the center of the circle circumscribing the triangle with vertices $\mathbf{q}_i$, $\mathbf{q}_j$, $\mathbf{q}_k$. If $q_{i\,0} = q_{i\,0} = q_{i\,0}$, then the point $\mathbf{d}^{ijk}$ is real.



For any three different indices $1 \le i, j, k \le n+1$, put

$$r_{ijk} \;=\; \frac{1}{\det\big(\mathbf{q}_j - \mathbf{q}_i, \mathbf{q}_k - \mathbf{q}_i\big)} \, \frac{1}{\prod\limits_{\substack{\nu=1 \\ \nu \neq i,j,k}}^{n+1} f_{i\nu}(\mathbf{d}^{ijk})}$$

where $\det\big(\mathbf{q}_j - \mathbf{q}_i, \mathbf{q}_k - \mathbf{q}_i\big)$ is the determinant of the matrix with rows $\mathbf{q}_j - \mathbf{q}_i$ and $\mathbf{q}_k - \mathbf{q}_i$. Observe that $\big|\det\big(\mathbf{q}_j - \mathbf{q}_i, \mathbf{q}_k - \mathbf{q}_i\big)\big|$ is twice the area of the triangle with vertices $\mathbf{q}_i,\ \mathbf{q}_j,\ \mathbf{q}_k$. By construction,

$$0 \;=\; f_{ij}(\mathbf{d}^{ijk}) \;=\; -f_{j\nu}(\mathbf{d}^{ijk}) - f_{\nu i}(\mathbf{d}^{ijk}) \;=\; -f_{j\nu}(\mathbf{d}^{ijk}) + f_{i\nu}(\mathbf{d}^{ijk})$$

so that

$$r_{ijk} \;=\; -\,r_{jik}$$

Let $\overline{\ell}_{ij}$, $i, j = 1, \cdots, n+1$, be the projective closure of ℓ_{ij} in the complex projective plane $\mathbb{P}^2$ and, as in the last section, let L_∞ be the line at infinity. For each pair of indices $1 \le i \neq j \le n+1$, let η_{ij} be the unique meromorphic differential form on $\overline{\ell}_{ij}$ that is holomorphic outside the points $\mathbf{d}^{ijk}$, $k = 1, \cdots, n+1,\ k \neq i, j$, and $\overline{\ell}_{ij} \cap L_\infty$, that has a simple pole with residue r_{ijk} at the point $\mathbf{d}^{ijk}$, and that has a zero of order $n-3$ (respectively, a simple pole for $n = 2$) at $\overline{\ell}_{ij} \cap L_\infty$. By construction,

$$\eta_{ij} = -\eta_{ji}$$

If $1 \le i \ne j \le n+1$, $q_{i0} \ne q_{j0}$ and $0 < s < k_F$, let $c_{ij}^{\pm}(s)$ be the point where the line ℓ_{ij} meets the quadric

$$Q_i^{\pm}(s) \;=\; \left\{\; z \in \mathbb{C}^2 \;\middle|\; (z_1 - \mathbf{q}_{i\,1})^2 + (z_2 - \mathbf{q}_{i\,2})^2 = s^2 \text{ and } \pm\,\Im\tfrac{z_2 - \mathbf{q}_{i\,2}}{z_1 - \mathbf{q}_{i\,1}} > 0 \;\right\}$$

In general $c_{ij}^{\pm}(s) \ne c_{ji}^{\pm}(s)$.

Theorem III.1 *Let $n \ge 2$. Suppose that no three of the points $\mathbf{q}_1, \cdots, \mathbf{q}_{n+1}$ in $\mathbb{R}^2$ lie on a line, that $q_{i0} \ne q_{j0}$ for $i \ne j$, and that $\mathbf{d}^{ijk} \ne \mathbf{d}^{ijk'}$ for $k \ne k'$. Further, suppose that $\mathbf{d}^{ijk} \ne c_{ij}^{\pm}(s)$, $0 < s < k_F$, for all $i \ne j$ and all $k \ne i, j$. Then,*

$$\begin{aligned} I(q_1, \cdots, q_{n+1}) \;&=\; +\tfrac{m^n}{2\pi\imath} \sum_{\substack{i,j=1\\ i\ne j}}^{n+1} \int_{\left\{\, c_{ij}^{+}(s) \;\middle|\; 0<s<k_F \,\right\}} \eta_{ij} \\ &=\; -\tfrac{m^n}{2\pi\imath} \sum_{\substack{i,j=1\\ i\ne j}}^{n+1} \int_{\left\{\, c_{ij}^{-}(s) \;\middle|\; 0<s<k_F \,\right\}} \eta_{ij} \end{aligned}$$

Proof: By Theorem II.1,

$$\begin{aligned} \int_{|\mathbf{k}-\mathbf{q}_i|<k_F} \frac{dk_1 dk_2}{\prod\limits_{\substack{j=1\\ j\ne i}}^{n+1} f_{ij}(\mathbf{k})} \;&=\; -\,2\pi\imath \sum_{j\ne i} \int_{\left\{\, c_{ij}^{+}(s) \;\middle|\; 0<s<k_F \,\right\}} \eta_{ij} \\ &=\; +\,2\pi\imath \sum_{j\ne i} \int_{\left\{\, c_{ij}^{-}(s) \;\middle|\; 0<s<k_F \,\right\}} \eta_{ij} \end{aligned}$$

for $i = 1, \cdots n+1$. ■

Remark. For each pair $1 \le i, j \le n+1$, $i \ne j$, choose $k \ne i, j$. We parameterize the line ℓ_{ij}, by

$$\pi_{ij}(z) \;=\; z\left((\mathbf{q}_j - \mathbf{q}_i)_2, -(\mathbf{q}_j - \mathbf{q}_i)_1\right) \;+\; \mathbf{d}^{ijk} \quad , \quad z \in \mathbb{C} \;,$$

The pull back $\pi_{ij}^* \eta_{ij}$ of η_{ij} is a rational form on $\mathbb{C}$ with simple poles. Consequently, there is a rational function $\varphi_{ij}(z)$ such that

$$\varphi_{ij}\; dz \;=\; \pi_{ij}^*\, \eta_{ij}$$

We can therefore reformulate the statement of Theorem III.1 as

$$I(q_1,\cdots,q_{n+1}) \;=\; \tfrac{m^n}{2\pi\imath}\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\int_{w_{ij}}\varphi_{ij}(z)\;dz$$

where

$$w_{ij} \;=\; \pi_{ij}^{-1}\Big(\big\{\, c_{ij}^{+}(s)\;\big|\; 0<s<k_F\,\big\}\Big)$$

Remark. For $n=2$, that is the 3-loop, we parameterize the line ℓ_{ij}, $i,j=1,2,3$, $i\neq j$, by

$$\pi_{ij}(z) \;=\; z\,\big((\mathbf{q}_j-\mathbf{q}_i)_2,\,-(\mathbf{q}_j-\mathbf{q}_i)_1\big)\;+\;\mathbf{d}\quad,\quad z\in\mathbb{C}\;,$$

where $\mathbf{d}=\mathbf{d}^{123}$ is the solution to the pair of equations

$$(\mathbf{q}_2-\mathbf{q}_1)\cdot\mathbf{k}+\tfrac{1}{2}\big(\mathbf{q}_1^2-\mathbf{q}_2^2\big)+\imath\, m\,(q_{1\,0}-q_{2\,0})=0$$
$$(\mathbf{q}_3-\mathbf{q}_1)\cdot\mathbf{k}+\tfrac{1}{2}\big(\mathbf{q}_1^2-\mathbf{q}_3^2\big)+\imath\, m\,(q_{1\,0}-q_{3\,0})=0$$

Observe that $\pi_{ij}^{*}\,\eta_{ij}$ is a rational form on $\mathbb{C}$ that has simple poles at 0 and ∞ and no others. By construction,

$$\pi_{ij}^{*}\,\eta_{ij} \;=\; \pm\,\frac{1}{\det\big(\mathbf{q}_2-\mathbf{q}_1,\mathbf{q}_3-\mathbf{q}_1\big)}\,\frac{dz}{z} \;=\; \pm\,\frac{1}{\det\big(\mathbf{q}_2-\mathbf{q}_1,\mathbf{q}_3-\mathbf{q}_1\big)}\,d\log z$$

with $+$ for $(i,j)=(1,2),(2,3),(3,1)$ and $-$ for $(i,j)=(2,1),(3,2),(1,3)$. Let $w_{ij}(s)$ be the unique root of

$$\big(z(\mathbf{q}_j-\mathbf{q}_i)_2+\mathbf{d}_1-\mathbf{q}_{i1}\big)^2\;+\;\big(-z(\mathbf{q}_j-\mathbf{q}_i)_1+\mathbf{d}_2-\mathbf{q}_{i2}\big)^2\;=\;s^2$$

with

$$\Im\,\frac{-w_{ij}(s)(\mathbf{q}_j-\mathbf{q}_i)_1+\mathbf{d}_2-\mathbf{q}_{i2}}{w_{ij}(s)(\mathbf{q}_j-\mathbf{q}_i)_2+\mathbf{d}_1-\mathbf{q}_{i1}}\;>\;0$$

We have

$$I(q_1,q_2,q_3) \;=\; \tfrac{1}{2\pi\imath}\,\frac{m^2}{\det\big(\mathbf{q}_2-\mathbf{q}_1,\mathbf{q}_3-\mathbf{q}_1\big)}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{\{\,w_{ij}(s)\,|\,0\leq s\leq k_F\,\}}\pm\;d\log z$$

Clearly, each of the six integrals on the right hand side can be performed explicitly.

Remark. For each pair $1 \le i,j \le n+1$, $i \ne j$, and $k \ne i,j$ let η_{ijk} be the unique meromorphic differential form on $\overline{\ell}_{ij}$ that is holomorphic outside the points $\mathbf{d}^{ijk}$, and $\overline{\ell}_{ij} \cap L_\infty$, that has a simple pole with residue r_{ijk} at the point $\mathbf{d}^{ijk}$, and that has a simple pole at $\overline{\ell}_{ij} \cap L_\infty$. Then

$$\eta_{ij} = \sum_{k \ne i,j} \eta_{ijk}$$

Consequently

$$I(q_1, \cdots, q_{n+1}) = \frac{m^n}{2\pi\imath} \sum_{\substack{i,j,k=1 \\ \text{pairwisedifferent}}}^{n+1} \int_{\left\{ c_{ij}^+(s) \;\middle|\; 0<s<k_F \right\}} \eta_{ijk}$$

So $I(q_1, \cdots, q_{n+1})$ is decomposed into a sum of terms each of which only depends on a triple $\{q_i, q_j, q_k\}$ of momenta. The fact that such a decomposition is possible follows directly from the identity

$$\frac{1}{\prod\limits_{i=1}^{n+1} \big(\imath(k-q_i)_0 - e(\mathbf{k}-\mathbf{q}_i)\big)}$$
$$= \sum_{1 \le i<j<k \le n+1} \frac{1}{\prod\limits_{\substack{\nu=1 \\ \nu \ne i,j,k}}^{n+1} f_{i\nu}(\mathbf{d}^{ijk})} \frac{1}{\big(\imath(k-q_i)_0 - e(\mathbf{k}-\mathbf{q}_i)\big)\big(\imath(k-q_j)_0 - e(\mathbf{k}-\mathbf{q}_j)\big)\big(\imath(k-q_k)_0 - e(\mathbf{k}-\mathbf{q}_k)\big)}$$

This identity was pointed out to us by A.Cavalli and P.Wagner.

Fix $\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}$ in $\mathbb{R}^2$. We assume that no three of these points lie on a line and no four of them lie on a circle. The main purpose of this section is to show that the limit

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) = \lim_{\substack{q_i \to (0,\mathbf{p}_i) \\ q_{i\,0} \ne q_{j\,0}}} I(q_1, \cdots, q_{n+1})$$

exists and to evaluate it. To do this, first set

$$F_{ij}(\mathbf{k}) = m\Big(e(\mathbf{k}-\mathbf{p}_i) - e(\mathbf{k}-\mathbf{p}_j)\Big) = (\mathbf{p}_j - \mathbf{p}_i)\cdot \mathbf{k} + \tfrac{1}{2}\left(\mathbf{p}_i^2 - \mathbf{p}_j^2\right)$$

for all $i,j = 1, \cdots, n+1$, and observe that

$$\lim_{q_i \to (0,\mathbf{p}_i)} f_{ij}(\mathbf{k}) = F_{ij}(\mathbf{k})$$

Furthermore, for each $1 \le i \ne j \le n+1$ the line ℓ_{ij} converges to the perpendicular bisector

$$L_{ij} \; = \; \left\{ \; z \in \mathbb{C}^2 \;\middle|\; F_{ij}(z) = 0 \; \right\}$$

of $\mathbf{p}_i$ and $\mathbf{p}_j$. Also, let $\mathbf{D}^{ijk}$ be the center of the circle circumscribing the triangle with vertices $\mathbf{p}_i$, $\mathbf{p}_j$ and $\mathbf{p}_k$. Then,

$$\lim_{q_i \to (0,\mathbf{p}_i)} \mathbf{d}^{ijk} \; = \; \mathbf{D}^{ijk}$$

We have $\mathbf{D}^{ijk} \ne \mathbf{D}^{ijk'}$ for all $k \ne k'$, since no four of the points $\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}$ lie on a circle.

For any three different indices $1 \le i, j, k \le n+1$, let

$$R_{ijk} \; = \; \frac{1}{\det\left(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i\right)} \, \frac{1}{\prod\limits_{\substack{\nu=1 \\ \nu \ne i,j,k}}^{n+1} F_{i\nu}(\mathbf{D}^{ijk})}$$

Then,

$$\lim_{q_i \to (0,\mathbf{p}_i)} r_{ijk} \; = \; R_{ijk}$$

Observe that

$$R_{ijk} \; = \; R_{jki} \; = \; R_{kij} \; = \; -R_{jik}$$

Let

$$\triangle_{(\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k)} \; = \; \left\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\right\}$$

be the triangle with vertices $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$. We can, using the antisymmetry of R_{ijk}, define

$$R_\triangle \; = \; \begin{cases} R_{ijk}, & \text{if } \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k \text{ are oriented counter clockwise} \\ -R_{ijk}, & \text{if } \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k \text{ are oriented clockwise} \end{cases}$$

for every triangle $\triangle = \triangle_{(\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k)}$. Let

$$\rho_\triangle \; = \; \left|\mathbf{D}^{ijk} - \mathbf{p}_i\right| \; = \; \left|\mathbf{D}^{ijk} - \mathbf{p}_j\right| \; = \; \left|\mathbf{D}^{ijk} - \mathbf{p}_k\right|$$

be the radius of the circle circumscribing $\triangle$. Since

$$\begin{aligned} F_{i\nu}(\mathbf{D}^{ijk}) \; &= \; m\left(e(\mathbf{D}^{ijk} - \mathbf{p}_i) - e(\mathbf{D}^{ijk} - \mathbf{p}_\nu)\right) \\ &= \; \tfrac{1}{2}\left(\rho_\triangle^2 - |\mathbf{D}^{ijk} - \mathbf{p}_\nu|^2\right) \end{aligned}$$

we have the geometric description of R_{ijk}

$$R_{ijk} \; = \; \frac{1}{\det\left(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i\right)} \, \frac{2^{n-2}}{\prod\limits_{\substack{\nu=1 \\ \nu \ne i,j,k}}^{n+1} \left(\rho_\triangle^2 - |\mathbf{D}^{ijk} - \mathbf{p}_\nu|^2\right)}$$

Theorem III.2 *Let* $n \geq 2$ *and let* $\mathbf{p}_1, \cdots .\mathbf{p}_{n+1}$ *be points in* $\mathbb{R}^2$ *such that no three lie on a line and no four lie on a circle. Furthermore, assume that for every triple* $1 \leq i, j, k \leq n+1$ *of pairwise distinct indices the triangle* $\triangle = \triangle_{(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k)}$ *is not a right triangle and* $\rho_\triangle \neq k_F$ *. Let* $\mathcal{T}$ *be the set of all triangles* $\triangle$ *with vertices in the set* $\{\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}\}$ *, and let* $\mathcal{T}_{\mathrm{ac}}$ *be the set of those triangles in* $\mathcal{T}$ *that have no angle bigger than* 90° *, i.e. acute triangles. Then,*

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) \;=\; m^n \sum_{\substack{\triangle \in \mathcal{T}_{\mathrm{ac}} \\ \rho_\triangle > k_F}} R_\triangle \;-\; 2m^n \sum_{\substack{\triangle \in \mathcal{T} \\ s \text{ edge of } \triangle \\ |s| > 2k_F}} \epsilon_{\triangle, s}\, R_\triangle \operatorname{arccot} \sqrt{\frac{4\rho_\triangle^2 - |s|^2}{|s|^2 - 4k_F^2}}$$

Here, for an edge s *of a triangle* $\triangle$ *we denote by* $|s|$ *the length of* s *and set*

$$\epsilon_{\triangle, s} \;=\; \begin{cases} +1 & \text{if the angle of } \triangle \text{ at the vertex opposite to } s \text{ is acute} \\ -1 & \text{if the angle of } \triangle \text{ at the vertex opposite to } s \text{ is obtuse} \end{cases}$$

Corollary III.3

i) *If* $|\mathbf{p}_i - \mathbf{p}_j| < 2k_F$ *for all* $i, j = 1, \cdots, n+1$ *, then*

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) \;=\; m^n \sum_{\substack{\triangle \in \mathcal{T}_{\mathrm{ac}} \\ \rho_\triangle > k_F}} R_\triangle$$

If, in particular, $k_F < k_F'$ *such that* $k_F > \frac{1}{2} \min_{k=1,\cdots,n+1} |\mathbf{p}_i - \mathbf{p}_j|$ *and such that there is no triangle* $\triangle \in \mathcal{T}_{\mathrm{ac}}$ *with* $k_F < \rho_\triangle < k_F'$ *, then*

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}; k_F) \;=\; J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}; k_F')$$

ii) *If all* $n+1$ *disks with radius* k_F *around the points* $\mathbf{p}_1, \cdots .\mathbf{p}_{n+1}$ *have at least one point in common, then*

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) \;=\; 0$$

(iii) *Suppose* $n = 2$ *and* $|\mathbf{p}_i - \mathbf{p}_j| < 2k_F$ *for all* $i, j = 1, 2, 3$ *. Let* $\triangle = \triangle_{(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}$ *. Then,*

$$J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \;=\; \begin{cases} 0 \,, & \text{if the disks of radius } k_F \text{ around } \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \text{ have at least one point in common} \\ \frac{m^2}{2 \times \text{area of } \triangle} \,, & \text{otherwise} \end{cases}$$

Proof of the Corollary: Part (i) follows immediately from Theorem III.2. If $\triangle = \triangle_{(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}$ is a triangle with only acute angles and the disks around the points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ with radius k_F have at least one point in common, then $\rho_\triangle \le k_F$. So, part (ii) follows from part (i). Finally, part (iii) is a special case of (i) and (ii). ■

The rest of this section is devoted to the proof of Theorem III.2. For each pair of indices $1 \le i \ne j \le n+1$, let ω_{ij} be the unique meromorphic differential form on $\overline{L}_{ij}$ (= projective closure of L_{ij}) that is holomorphic outside the points $\mathbf{D}^{ijk}$, $k = 1, \cdots, n+1$, $k \ne i,j$, and $\overline{L}_{ij} \cap L_\infty$, that has a simple pole with residue R_{ijk} at the point $\mathbf{D}^{ijk}$, and that has a zero of order $n-3$ (respectively, a simple pole for $n = 2$) at $\overline{L}_{ij} \cap L_\infty$. By construction,

$$\omega_{ij} = -\omega_{ji}$$

If $0 \le k_F \le \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$, let $C_{ij}^{\pm}$ be the intersection of the line L_{ij} with

$$\mathcal{Q}_i^{\pm}(k_F) \;=\; \left\{ \, z \in \mathbb{C}^2 \;\middle|\; (z_1 - \mathbf{p}_{i\,1})^2 + (z_2 - \mathbf{p}_{i\,2})^2 = k_F^2 \text{ and } \pm\Im \tfrac{z_2 - \mathbf{p}_{i\,2}}{z_1 - \mathbf{p}_{i\,1}} > 0 \, \right\}$$

Observe that $C_{ij}^{+} = C_{ji}^{-}$. It is useful to introduce the notation

$$p \;=\; \Big((0,\mathbf{p}_1), \cdots, (0,\mathbf{p}_{n+1}) \Big) \;\in\; \left(\mathbb{R} \times \mathbb{R}^2 \right)^{n+1}$$

and

$$\mathcal{M} \;=\; \left\{ q \;=\; \Big((q_{1\,0},\mathbf{q}_1), \cdots, (q_{n+1\,0},\mathbf{q}_{n+1}) \Big) \;\in\; \left(\mathbb{R} \times \mathbb{R}^2 \right)^{n+1} \;\middle|\; q_{i\,0} \ne q_{j\,0} \text{ for } i \ne j \right\}$$

Now, by Theorem III.1,

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) \;=\; \lim_{\substack{q \to p \\ q \in \mathcal{M}}} \frac{m^n}{2\pi\imath} \sum_{\substack{i,j=1 \\ i \ne j}}^{n+1} \int_{\left\{ \, c_{ij}^{+}(s) \;\middle|\; 0 < s < k_F \, \right\}} \eta_{ij}$$

It is easy to see that

$$\lim_{\substack{q \to p \\ q \in \mathcal{M}}} \eta_{ij} \;=\; \omega_{ij}$$

pointwise. However, it takes some work to determine the limiting behavior

$$\lim_{\substack{q \to p \\ q \in \mathcal{M}}} \left\{ \, c_{ij}^{+}(s) \;\middle|\; 0 < s < k_F \, \right\}$$

of the path $\left\{ c_{ij}^{+}(s) \;\middle|\; 0 < s < k_F \right\}$. There are two cases, $0 \le s < \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$ and $k_F \ge s \ge \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$.

Suppose $0 \le s < \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$ for some $1 \le i, j \le n+1$. Then, the line L_{ij} intersects

$$\mathcal{Q}_i^{\pm}(s) \;=\; \left\{ z \in \mathbb{C}^2 \;\middle|\; (z_1 - \mathbf{p}_{i\,1})^2 + (z_2 - \mathbf{p}_{i\,2})^2 = s^2 \text{ and } \pm \Im \tfrac{z_2 - \mathbf{p}_{i\,2}}{z_1 - \mathbf{p}_{i\,1}} > 0 \right\}$$

in exactly one point which is denoted by $C_{ij}^{\pm}(s)$. Again, $C_{ij}^{+}(s) = C_{ji}^{-}(s)$ and

$$C_{ij}^{\pm}(s) \;=\; \lim_{\substack{q \to p \\ q \in \mathcal{M}}} c_{ij}^{\pm}(s)$$

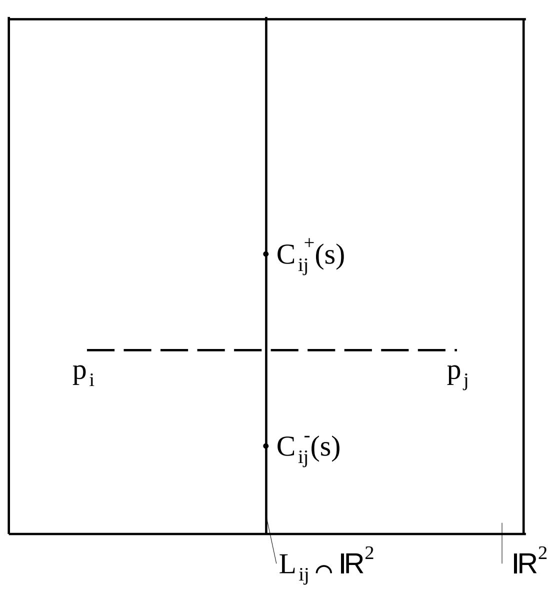
Also,

$$\Re\, C_{ij}^{\pm}(s) \;=\; \tfrac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$$

Suppose that $s > \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$. Then,

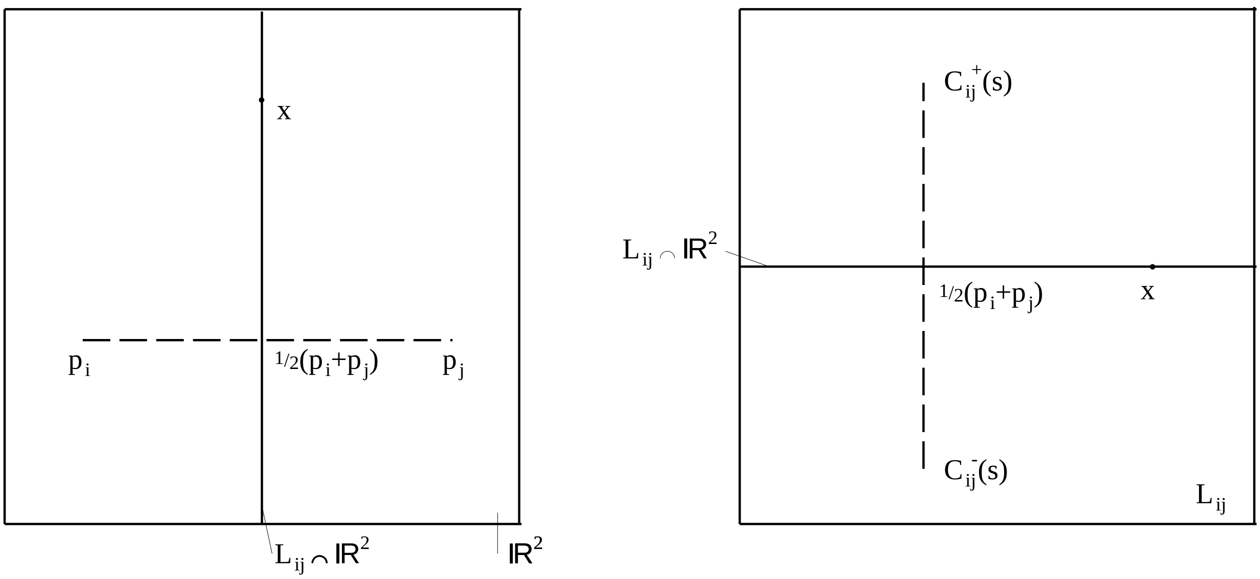
$$L_{ij} \cap \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; |\mathbf{x} - \mathbf{p}_i| = s \right\} \;=\; L_{ij} \cap \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; |\mathbf{x} - \mathbf{p}_j| = s \right\}$$

consists of two points which we again denote by $C_{ij}^{\pm}(s)$. By convention, $C_{ij}^{+}(s)$ is the point that makes the triangle with vertices $\mathbf{p}_i$, $\mathbf{p}_j$ and $C_{ij}^{+}(s)$ counter clockwise oriented, and $C_{ij}^{-}(s)$ is the point that makes the triangle with vertices $\mathbf{p}_i$, $\mathbf{p}_j$ and $C_{ij}^{-}(s)$ clockwise oriented.



Lemma III.4 *Let* $1 \le i, j \le n+1$.

(i) *Suppose that* $0 \le s < \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$. *Let* $\mathbf{x} \neq \frac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$ *in* $L_{jk} \cap \mathbb{R}^2$ *be a real point on the perpendicular bisector of* $\mathbf{p}_i$ *and* $\mathbf{p}_j$. *Then, the triangle in the complex line* L_{ij} *with the vertices* $C^+_{ij}(s)$, $C^-_{ij}(s)$ *and* $\mathbf{x}$ *has the same orientation as the triangle in* $\mathbb{R}^2$ *with the vertices* $\mathbf{p}_i$, $\mathbf{p}_j$ *and* $\mathbf{x}$.



(ii) *Suppose that* $s > \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$. *Then,* $C^{\pm}_{ij}(s) = C^{\mp}_{ji}(s)$ *and*

$$\begin{aligned} C^{\pm}_{ij}(s) &= \lim_{\substack{q \to p \\ q \in \mathcal{M} \\ q_{i\,0} - q_{j\,0} > 0}} c^{\pm}_{ij}(s) = \lim_{\substack{q \to p \\ q \in \mathcal{M} \\ q_{i\,0} - q_{j\,0} > 0}} c^{\pm}_{ji}(s) \\ &= \lim_{\substack{q \to p \\ q \in \mathcal{M} \\ q_{i\,0} - q_{j\,0} < 0}} c^{\mp}_{ji}(s) = \lim_{\substack{q \to p \\ q \in \mathcal{M} \\ q_{i\,0} - q_{j\,0} < 0}} c^{\mp}_{ij}(s) \end{aligned}$$

Proof: For convenience, we set $m = 1$. We may assume, by applying an orientation preserving isometry, that

$$\mathbf{p}_i = (-a, 0) \quad \text{and} \quad \mathbf{p}_j = (a, 0)$$

with $a > 0$. Now,

$$L_{ij} = \left\{ (0, z_2) \,\middle|\, z_2 \in \mathbb{C} \right\}$$

and

$$\mathcal{Q}^{\pm}_i(s) = \left\{ z \in \mathbb{C}^2 \;\middle|\; (z_1 + a)^2 + z_2^2 = s^2 \text{ and } \pm \Im \tfrac{z_2}{z_1 + a} > 0 \right\}$$

If $s < a$, then the first statement of Lemma III.4 follows from

$$C_{ij}^{\pm}(s) \;=\; (0,\; \pm i\sqrt{a^2 - s^2})$$

If, on the other hand, $s > a$,

$$C_{ij}^{\pm}(s) \;=\; (0,\; \pm\sqrt{s^2 - a^2}) \;=\; C_{ji}^{\mp}(s)$$

Suppose $q \in \mathcal{M}$ is close to p. Then, by a q-dependent orientation preserving isometry close to the identity, we may assume that

$$\mathbf{q}_i \;=\; (-a', 0) \quad \text{and} \quad \mathbf{q}_j \;=\; (a', 0)$$

with $a' > 0$ close to a. Now,

$$\ell_{ij} \;=\; \left\{ z \in \mathbb{C}^2 \;\middle|\; z_1 = -i\,\tfrac{q_{i\,0} - q_{j\,0}}{2a'} \right\}$$

and the second statement of Lemma III.4 follows directly from

$$c_{ij}^{\pm}(s) \;=\; \left(-i\,\tfrac{q_{i\,0}-q_{j\,0}}{2a'}\;,\; \pm\,\mathrm{sgn}\,(q_{i\,0}-q_{j\,0})\,\sqrt{s^2 - a'^2 + \tfrac{1}{4a'^2}(q_{i\,0}-q_{j\,0})^2 + i(q_{i\,0}-q_{j\,0})} \right)$$

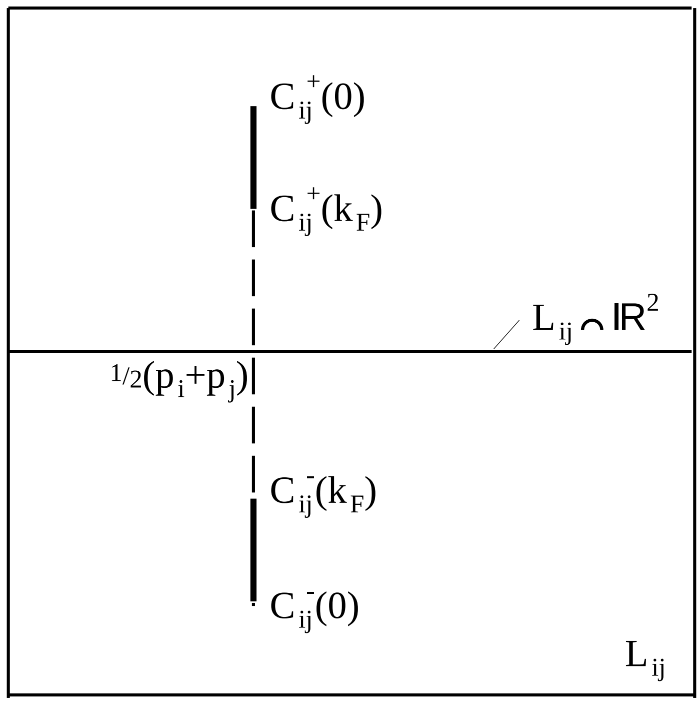
$$c_{ji}^{\pm}(s) \;=\; \left(-i\,\tfrac{q_{i\,0}-q_{j\,0}}{2a'}\;,\; \pm\,\mathrm{sgn}\,(q_{i\,0}-q_{j\,0})\,\sqrt{s^2 - a'^2 + \tfrac{1}{4a'^2}(q_{i\,0}-q_{j\,0})^2 - i(q_{i\,0}-q_{j\,0})} \right)$$

where we make a branch cut for the square root along the negative real axis. ■

If $0 \le k_F < \frac{1}{2}|\mathbf{p}^{(j)} - \mathbf{p}^{(k)}|$, then, by construction,

$$\lim_{\substack{q \in \mathcal{M} \\ q \to p}} \left\{\, c_{ij}^{\pm}(s) \;\middle|\; 0 \le s \le k_F \,\right\} \;=\; \left\{\, C_{ij}^{\pm}(s) \;\middle|\; 0 \le s \le k_F \,\right\}$$

is the straight line segment $[C_{ij}^{\pm}(0), C_{ij}^{\pm}]$ joining the point $C_{ij}^{\pm}(0)$ to the point $C_{ij}^{\pm} = C_{ij}^{\pm}(k_F)$ in L_{ij}.



If $k_F > \frac{1}{2}|\mathbf{p}^{(j)} - \mathbf{p}^{(k)}|$, then by Lemma III.4 (ii),

$$\lim_{\substack{q \to p \\ q \in \mathcal{M} \\ \pm(q_{i\,0} - q_{j\,0}) > 0}} \left\{ \, c^{+}_{ij}(s) \;\middle|\; 0 \le s \le k_F \, \right\} \cup \left\{ \, c^{+}_{ji}(s) \;\middle|\; k_F \ge s \ge 0 \, \right\}$$

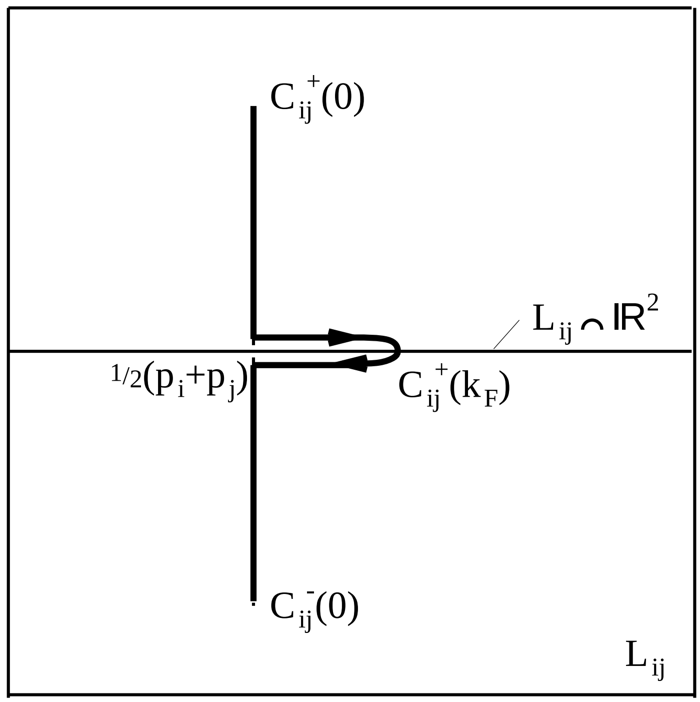
is the path obtained by composing the three paths:

(1) the straight segment in L_{ij} joining $C^{+}_{ij}(0)$ to $\frac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$

(2) the segment on $L_{ij} \cap \mathbb{R}^2$ joining $\frac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$ to $C^{\pm}_{ij}(k_F)$, followed by the inverse of this segment

and

(3) the straight segment in L_{ij} joining $\frac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$ to $C^{-}_{ij}(0)$.



We encounter two problems in calculating

$$\lim_{\substack{q \to p \\ q \in \mathcal{M} \\ \pm(q_{i\,0} - q_{j\,0}) > 0}} \left(\int_{\{\ c_{ij}^{+}(s)\ |\ 0 < s < k_F\ \}} \eta_{ij} + \int_{\{\ c_{ji}^{+}(s)\ |\ 0 < s < k_F\ \}} \eta_{ji} \right)$$

$$= \lim_{\substack{q \to p \\ q \in \mathcal{M} \\ \pm(q_{i\,0} - q_{j\,0}) > 0}} \int_{\{\ c_{ij}^{+}(s)\ |\ 0 \le s \le k_F\ \} \cup \{\ c_{ji}^{+}(s)\ |\ k_F \ge s \ge 0\ \}} \eta_{ij}$$

First, the path (2) depends on the sign of $q_{i\,0} - q_{j\,0}$. Second, the path (2) may contain one or more of the points $\mathbf{D}^{ijk}$, that are the poles of the limiting meromorphic form ω_{ij}.

Let $q \in \mathcal{M}$ be close to p. For all indices $1 \le j \ne k \le n+1$ with $k_F > \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$, let $[c_{ij}^{+}(k_F),\, c_{ji}^{+}(k_F)]$ be the straight line segment in ℓ_{ij} joining $c_{ij}^{+}(k_F)$ to $c_{ji}^{+}(k_F)$. The composition of the paths

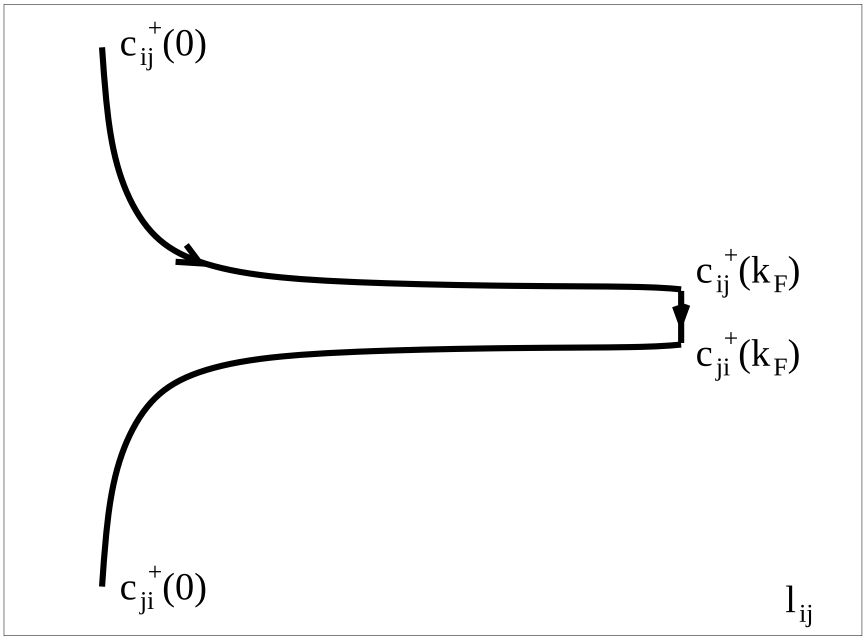
$$\{\ c_{ij}^{+}(s)\ |\ 0 \le s \le k_F\ \}$$

$$[c_{ij}^{+}(k_F),\, c_{ji}^{+}(k_F)]$$

and

$$\{\ c_{ji}^{+}(s)\ |\ k_F \ge s \ge 0\ \}$$

is a path joining $c_{ij}^{+}(0)$ to $c_{ji}^{+}(0)$.



For any three different indices $1 \leq i, j, k \leq n+1$, recall that $\Re\, \mathbf{d}^{ijk}$ is the center of the circle circumscribing the triangle with the vertices $\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k$. Let

$$\alpha_{ijk} \;=\; \tfrac{1}{2} \det(\mathbf{q}_i - \Re\, \mathbf{d}^{ijk}, \mathbf{q}_j - \Re\, \mathbf{d}^{ijk})$$

be the oriented area of the triangle with vertices $\mathbf{q}_i, \mathbf{q}_j, \Re\, \mathbf{d}^{ijk}$, and

$$A_{ijk} \;=\; \tfrac{1}{2} \det(\mathbf{q}_i - \mathbf{q}_k, \mathbf{q}_j - \mathbf{q}_k)$$

the oriented area of the full triangle $\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k$.

For all $1 \leq i \neq j \leq n+1$, set

$$\begin{aligned}
\mathcal{I}_{ij} &= \left\{ 1 \leq k \leq n+1 \;\middle|\; |\mathbf{D}^{ijk} - \mathbf{p}_i| < k_F \right\} \\
&= \left\{ 1 \leq k \leq n+1 \;\middle|\; \rho_{\triangle(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k)} < k_F \right\} \\
\mathcal{I}^+_{ij}(q) &= \left\{ k \in \mathcal{I}_{ij} \;\middle|\; \begin{array}{l} q_{i\,0} - q_{j\,0} > 0 \text{ and the triangle with vertices} \\ \mathbf{p}_i, \mathbf{p}_j, \mathbf{D}^{ijk} \text{ is counterclockwise oriented} \end{array} \right\} \\
\mathcal{I}^-_{ij}(q) &= \left\{ k \in \mathcal{I}_{ij} \;\middle|\; \begin{array}{l} q_{i\,0} - q_{j\,0} < 0 \text{ and the triangle with vertices} \\ \mathbf{p}_i, \mathbf{p}_j, \mathbf{D}^{ijk} \text{ is clockwise oriented} \end{array} \right\}
\end{aligned}$$

Finally, recall that for any three different indices $1 \leq i, j, k \leq n+1$, r_{ijk} is the residue of η_{ij} at $\mathbf{d}^{ijk}$, and if $\triangle = \triangle_{(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)}$ is the triangle with vertices $\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k$, set

$$r_\triangle \;=\; \begin{cases} r_{ijk}, & \text{if } \mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k \text{ are oriented counter clockwise} \\ -r_{ijk}, & \text{if } \mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k \text{ are oriented clockwise} \end{cases}$$

Lemma III.5 *Let $q \in \mathcal{M}$ be close to p and suppose that for all pairwise distinct indices $1 \leq i, j, k \leq n+1$,*

$$\frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k\,0} - q_{i\,0}}{q_{i\,0} - q_{j\,0}} \neq \frac{\alpha_{kij}}{A_{kij}}$$

(i) Fix $1 \leq i \neq j \leq n+1$ with $k_F > \frac{1}{2}|\mathbf{p}_i - \mathbf{p}_j|$. Then,

$$\int_{\{c_{ij}^{+}(s)\,|\,0 \leq s \leq k_F\}} \eta_{ij} + \int_{[c_{ij}^{+}(k_F),\, c_{ji}^{+}(k_F)]} \eta_{ij} + \int_{\{c_{ji}^{+}(s)\,|\,0 \leq s \leq k_F\}} \eta_{ji}$$
$$= \int_{c_{ij}^{+}(0)}^{c_{ji}^{+}(0)} \eta_{ij} \; - \; 2\pi i \sum_{\substack{k=1 \\ k \neq i,j}}^{n+1} \chi_{ijk}\, r_{ijk}$$

Here, $\int_{c_{ij}^{+}(0)}^{c_{ji}^{+}(0)} \eta_{ij}$ is the integral along the straight line segment $[c_{ij}^{+}(0),\, c_{ji}^{+}(0)]$ in ℓ_{ij}, and

$$\chi_{ijk} = \begin{cases} 0 & \text{unless } k \in \mathcal{I}_{ij}^{+}(q) \cup \mathcal{I}_{ij}^{-}(q) \text{ and} \\ & \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k\,0} - q_{i\,0}}{q_{i\,0} - q_{j\,0}} < \frac{\alpha_{kij}}{A_{kij}} \text{ and } \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{j\,0} - q_{k\,0}}{q_{i\,0} - q_{j\,0}} < \frac{\alpha_{jki}}{A_{jki}} \\ \pm 1 & \text{if } k \in \mathcal{I}_{ij}^{\pm}(q) \text{ and} \\ & \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k\,0} - q_{i\,0}}{q_{i\,0} - q_{j\,0}} < \frac{\alpha_{kij}}{A_{kij}} \text{ and } \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{j\,0} - q_{k\,0}}{q_{i\,0} - q_{j\,0}} < \frac{\alpha_{jki}}{A_{jki}} \end{cases}$$

(ii) Let $1 \leq i, j, k \leq n+1$ be any three different indices and let $\triangle = \triangle_{(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)}$. Then,

$$\sum_{\{i',j',k'\} = \{i,j,k\}} \chi_{i'j'k'}\, r_{i'j'k'} = \begin{cases} 2r_\triangle & \text{if } \rho_\triangle < k_F \text{ and all angles in the triangle } \triangle \text{ are smaller than } 90^\circ \\ 0 & \text{otherwise} \end{cases}$$

Lemma III.5 is now used to obtain an intermediate expression for the function $J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1})$ appearing in Theorem III.2. Recall that $\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}$ are points in $\mathbb{R}^2$ such that no three of them lie on a line and no four of them lie on a circle, and furthermore, that for every triple $1 \leq i, j, k \leq n+1$ of pairwise distinct indices the triangle $\triangle = \triangle_{(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k)}$ is not a right triangle and $\rho_\triangle \neq k_F$. Here, as above, $\rho_\triangle$ is the radius of the circle circumscribing the triangle $\triangle$.

Proposition III.6

$$J(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) = \frac{m^n}{4\pi\imath} \sum_{1 \leq i \neq j \leq n+1} \int_{C_{ij}^{+}(0)}^{C_{ij}^{-}(0)} \omega_{ij} + \frac{m^n}{4\pi\imath} \sum_{\substack{1 \leq i \neq j \leq n+1 \\ |\mathbf{p}_i - \mathbf{p}_j| > 2k_F}} \int_{C_{ij}^{-}}^{C_{ij}^{+}} \omega_{ij} - m^n \sum_{\substack{\triangle \in \mathcal{T}_{\mathrm{ac}} \\ \rho_\triangle < k_F}} R_\triangle$$

Proof: By Theorem III.1 and Lemma III.5 (i),

$$\begin{aligned}
\tfrac{2}{m^n}\, I(q_1,\cdots,q_{n+1}) \;&=\; \tfrac{1}{2\pi\imath} \sum_{1\le i\ne j\le n+1} \left(\int_{\{c_{ij}^+(s)\,|\,0<s<k_F\}} \eta_{ij} \;+\; \int_{\{c_{ji}^+(s)\,|\,0<s<k_F\}} \eta_{ji} \right) \\
&= \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \left(\int_{c_{ij}^+(0)}^{c_{ij}^+(k_F)} \eta_{ij} \;+\; \int_{c_{ji}^+(0)}^{c_{ji}^+(k_F)} \eta_{ji} \right) \\
&\quad + \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|<2k_F}} \left(\int_{c_{ij}^+(0)}^{c_{ji}^+(0)} \eta_{ij} \;-\; \int_{[c_{ij}^+(k_F),\,c_{ji}^+(k_F)]} \eta_{ij} \right) \\
&\quad - \sum_{1\le i\ne j\le n+1} \sum_{\substack{k=1\\ k\ne i,j}}^{n+1} \chi_{ijk}\, r_{ijk}
\end{aligned}$$

on an open, dense subset of $\mathcal{M}$. By Lemma III.5 (ii),

$$\begin{aligned}
\tfrac{2}{m^n}\, I(q_1,\cdots,q_{n+1}) \;&=\; \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \left(\int_{c_{ij}^+(0)}^{c_{ij}^+(k_F)} \eta_{ij} \;+\; \int_{c_{ji}^+(0)}^{c_{ji}^+(k_F)} \eta_{ji} \right) \\
&\quad + \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|<2k_F}} \left(\int_{c_{ij}^+(0)}^{c_{ji}^+(0)} \eta_{ij} \;-\; \int_{[c_{ij}^+(k_F),\,c_{ji}^+(k_F)]} \eta_{ij} \right) \\
&\quad - \;2 \sum_{\substack{\triangle\in\mathcal{T}_{\mathbf{ac}}\\ \rho_\triangle<k_F}} r_\triangle
\end{aligned}$$

on an open, dense subset of $\mathcal{M}$. In fact the last identity holds for all q in $\mathcal{M}$ near p, since both sides are continuous functions on $\mathcal{M}$ near p. Finally, by Lemma III.4,

$$\begin{aligned}
\tfrac{2}{m^n}\, J(\mathbf{p}_1,\cdots,\mathbf{p}_{n+1}) \;&=\; 2 \lim_{\substack{q\in\mathcal{M}\\ q\to p}} I(q_1,\cdots,q_{n+1}) \\
&= \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \left(\int_{C_{ij}^+(0)}^{C_{ij}^+(k_F)} \omega_{ij} \;-\; \int_{C_{ji}^+(0)}^{C_{ji}^+(k_F)} \omega_{ij} \right) \\
&\quad + \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|<2k_F}} \int_{C_{ij}^+(0)}^{C_{ji}^+(0)} \omega_{ij} \;-\; 2 \sum_{\substack{\triangle\in\mathcal{T}_{\mathbf{ac}}\\ \rho_\triangle<k_F}} R_\triangle \\
&= \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \left(\int_{C_{ij}^+(0)}^{C_{ji}^+(0)} \omega_{ij} - \int_{C_{ij}^+(k_F)}^{C_{ji}^+(k_F)} \omega_{ij} \right) \\
&\quad + \tfrac{1}{2\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|<2k_F}} \int_{C_{ij}^+(0)}^{C_{ji}^+(0)} \omega_{ij} \;-\; 2 \sum_{\substack{\triangle\in\mathcal{T}_{\mathbf{ac}}\\ \rho_\triangle<k_F}} R_\triangle
\end{aligned}$$

■

The Proof of Lemma III.5 : For convenience, we again set $m = 1$. To prove part (i), let $\gamma_{ij}(q)$ be the closed path obtained by composing $\{\ c^+_{ij}(s) \mid 0 \le s \le k_F\ \}$, $[c^+_{ij}(k_F),\, c^+_{ji}(k_F)]$, $\{\ c^+_{ji}(s) \mid k_F \ge s \ge 0\ \}$ and $[c^+_{ji}(0),\, c^+_{ij}(0)]$. We have

$$\begin{aligned}
&\int_{\{c^+_{ij}(s)\,|\,0\le s\le k_F\}} \eta_{ij} \;+\; \int_{[c^+_{ij}(k_F),\,c^+_{ji}(k_F)]} \eta_{ij} \;+\; \int_{\{c^+_{ji}(s)\,|\,0\le s\le k_F\}} \eta_{ji} \\
&\quad = \int_{c^+_{ij}(0)}^{c^+_{ji}(0)} \eta_{ij} \;+\; \int_{\gamma_{ij}(q)} \eta_{ij}
\end{aligned}$$

since, $\eta_{ij} = -\eta_{ji}$. By the residue theorem

$$\begin{aligned}
\int_{\gamma_{ij}(q)} \eta_{ij} &= 2\pi\imath \sum_{\substack{k=1\\ k\ne i,j}}^{n+1} \big(\text{winding number of } \gamma_{ij} \text{ around } \mathbf{d}^{ijk}\big) \times \operatorname{res}_{\eta_{ij}}(\mathbf{d}^{ijk}) \\
&= 2\pi\imath \sum_{\substack{k=1\\ k\ne i,j}}^{n+1} \big(\text{winding number of } \gamma_{ij} \text{ around } \mathbf{d}^{ijk}\big)\, r_{ijk}
\end{aligned}$$

We shall show that the winding number of $\gamma_{ij}(q)$ around $\mathbf{d}^{ijk}$ is $-\chi_{ijk}$.

Fix $1 \le k \le n+1,\ k \ne i,j$. A necessary condition that the winding number of $\gamma_{ij}(q)$ around $\mathbf{d}^{ijk}$ is not zero is that $\mathbf{D}^{ijk}$ be a point on the limiting path (see, Figure 6)

$$\lim_{\substack{q'\to p\\ q'\in\mathcal{M}\\ \operatorname{sgn}(q'_{i\,0}-q'_{j\,0})=\operatorname{sgn}(q_{i\,0}-q_{j\,0})}} \gamma_{ij}(q')$$

By Lemma III.4 (ii), our criterion becomes

$$\mathbf{D}^{ijk} \in \begin{cases} \big[\tfrac{1}{2}(\mathbf{p}_i+\mathbf{p}_j)\,,\, C^+_{ij}(k_F)\big]\ , & q_{i\,0}-q_{j\,0} > 0 \\[4pt] \big[\tfrac{1}{2}(\mathbf{p}_i+\mathbf{p}_j)\,,\, C^-_{ij}(k_F)\big]\ , & q_{i\,0}-q_{j\,0} < 0 \end{cases}$$

since, $\mathbf{D}^{ijk}$ is real. Observe that, by construction, the distance between $\mathbf{D}^{ijk}$ and the midpoint $\frac{1}{2}(\mathbf{p}_i+\mathbf{p}_j)$ is smaller than the distance between $C^\pm_{ij}(k_F)$ and $\frac{1}{2}(\mathbf{p}_i+\mathbf{p}_j)$ if and only if $|\mathbf{D}^{ijk}-\mathbf{p}_i| < k_F$. Also, observe that the point $\mathbf{D}^{ijk}$ lies on the same side of $\frac{1}{2}(\mathbf{p}_i+\mathbf{p}_j)$ as $C^+_{ij}(k_F)$ if and only if the triangle with vertices $\mathbf{p}_i$, $\mathbf{p}_j$, $\mathbf{D}^{ijk}$ is counter clockwise oriented, and on the same side of $\frac{1}{2}(\mathbf{p}_i+\mathbf{p}_j)$ as $C^-_{ij}(k_F)$ if and only if the triangle with vertices $\mathbf{p}_i$, $\mathbf{p}_j$, $\mathbf{D}^{ijk}$ is clockwise oriented. Thus, the winding number of $\gamma_{ij}(q)$ around $\mathbf{d}^{ijk}$ is zero when $k \notin \mathcal{I}^+_{ij}(q) \cup \mathcal{I}^-_{ij}(q)$.

Now, suppose that $k \in \mathcal{I}^{+}_{ij}(q) \cup \mathcal{I}^{-}_{ij}(q)$. As in the proof of Lemma III.4 we may assume that

$$\mathbf{p}_i \;=\; (-a, 0) \quad \text{and} \quad \mathbf{p}_j \;=\; (a, 0)$$

with $a > 0$ and

$$\mathbf{q}_i \;=\; (-a', 0) \quad \text{and} \quad \mathbf{q}_j \;=\; (a', 0)$$

with a' close to a. We have

$$\mathbf{d}^{ijk} \;=\; \left(-\,\imath\,\frac{q_{i\,0}-q_{j\,0}}{2a'} \;,\; \frac{1}{2\mathbf{q}_{k2}}\left(|\mathbf{q}_k|^2-a'^2\right) + \imath\,\frac{1}{\mathbf{q}_{k2}}\left(q_{k\,0}+\frac{\mathbf{q}_{k\,1}-a'}{2a'}q_{i\,0}-\frac{\mathbf{q}_{k\,1}+a'}{2a'}q_{j\,0}\right)\right)$$

The function

$$g(t) \;=\; \left(-\,\imath\,\frac{q_{i\,0}-q_{j\,0}}{2a'} \;,\; \frac{1}{2\mathbf{q}_{k2}}\left(|\mathbf{q}_k|^2-a'^2\right) + \imath\, t\right)$$

defined for $t \in \mathbb{R}$ parameterizes the real line in ℓ_{ij} passing through $\mathbf{d}^{ijk}$ parallel to the imaginary z_2-axis. By construction $g(t_0) = \mathbf{d}^{ijk}$, where

$$t_0 \;=\; \frac{1}{\mathbf{q}_{k2}}\left(q_{k\,0}+\frac{\mathbf{q}_{k\,1}-a'}{2a'}q_{i\,0}-\frac{\mathbf{q}_{k\,1}+a'}{2a'}q_{j\,0}\right)$$

It is easy to see that

$$t_i \;=\; \frac{\mathbf{q}_{k\,2}}{|\mathbf{q}_k|^2-a'^2}\,(q_{i\,0} - q_{j\,0})$$

is the only real number for which there exists an $s > 0$ such that $g(t) \in Q^{+}_{i}(s)$. Similarly,

$$t_j \;=\; -\frac{\mathbf{q}_{k\,2}}{|\mathbf{q}_k|^2-a'^2}\,(q_{i\,0} - q_{j\,0})$$

is the only real number for which there exists an $s > 0$ such that $g(t) \in Q^{+}_{j}(s)$. Our hypothesis (see, the statement of Theorem III.2) implies that $\mathbf{q}_i$, $\mathbf{q}_j$ and $\mathbf{q}_k$ are not the vertices of a right triangle. By Thales' theorem (see, Euclid, Book III, §31), the denominator $|\mathbf{q}_k|^2 - a'^2$ does not vanish and

$$\frac{\mathbf{q}_{k\,2}}{|\mathbf{q}_k|^2-a'^2} \;>\; 0$$

when $k \in \mathcal{I}^{+}_{ij}(q)$ and

$$\frac{\mathbf{q}_{k\,2}}{|\mathbf{q}_k|^2-a'^2} \;<\; 0$$

when $k \in \mathcal{I}_{ij}^{-}(q)$. Consequently,

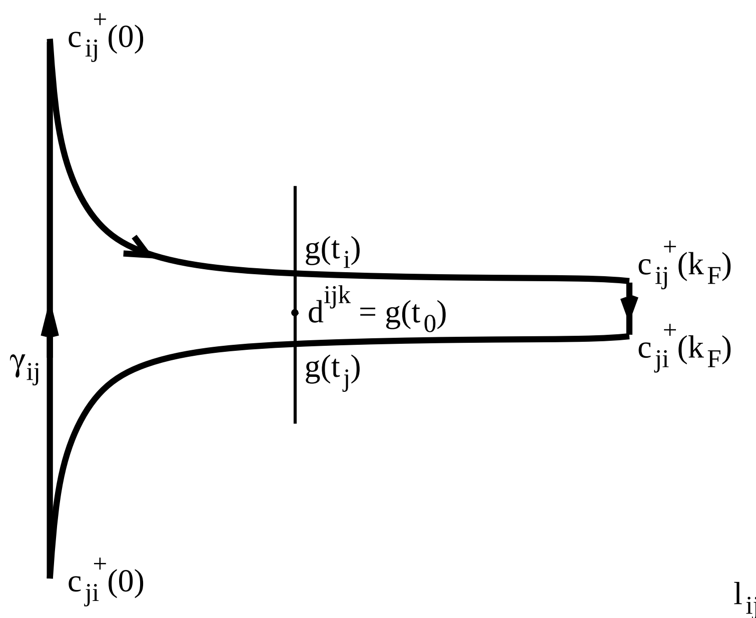
$$t_j \; < 0 < t_i$$

since $k \in \mathcal{I}_{ij}^{+}(q) \cup \mathcal{I}_{ij}^{-}(q)$.

The line $g(t)$, $-\infty < t < \infty$, meets the loop γ_{ij} in exactly the two points $g(t_i)$ and $g(t_j)$. Furthermore, the winding number of γ_{ij} around $\mathbf{d}^{ijk} = g(t_0)$ is

$$\begin{cases} 0 & \text{when } t_i < t_0 \text{ or } t_0 < t_j \\ \mp 1 & \text{if } t_j < t_0 < t_i \text{ and } k \in \mathcal{I}_{ij}^{\pm}(q) \end{cases}$$

The first case is immediate from the figure



If, on the other hand, $t_j < t_0 < t_i$, then the winding number is -1 when the triangle with vertices $c_{ij}^{+}(0)$, $c_{ji}^{+}(0)$, $\mathbf{d}^{ijk}$ is counter clockwise oriented. This occurs, if and only if the limiting triangle with vertices $C_{ij}^{+}(0)$, $C_{ij}^{-}(0) = C_{ji}^{+}(0)$, $\mathbf{D}^{ijk}$ is counter clockwise oriented. By Lemma III.4 (i), this happens if and only if the triangle with vertices $\mathbf{p}_i$, $\mathbf{p}_j$, $\mathbf{D}^{ijk}$ is counter clockwise oriented. That is, $k \in \mathcal{I}_{ij}^{+}(q)$. The case of winding number $+1$ is similar.

The last step in the proof of part (i) is to express the conditions on t_i, t_0 and t_j that determine the winding number of γ_{ij} around $\mathbf{d}^{ijk}$ in terms of the quantities α_{ijk} and A_{ijk}. To start with,

$$t_0 \;=\; \tfrac{1}{\mathbf{q}_{k\,2}}\Big(q_{k\,0} + \tfrac{\mathbf{q}_{k\,1}-a'}{2a'}\,q_{i\,0} - \tfrac{\mathbf{q}_{k\,1}+a'}{2a'}\,q_{j\,0}\Big) \;<\; \tfrac{\mathbf{q}_{k\,2}}{|\mathbf{q}_k|^2-a'^2}\,(q_{i\,0}-q_{j\,0}) \;=\; t_i$$

is equivalent to

$$\frac{\alpha_{ijk}}{A_{ijk}} \frac{1}{q_{i\,0}-q_{j\,0}} \left(q_{k\,0}+\frac{\mathbf{q}_{k\,1}-a'}{2a'} q_{i\,0}-\frac{\mathbf{q}_{k\,1}+a'}{2a'} q_{j\,0} \right) \;<\; \tfrac{1}{2}$$

since,

$$\alpha_{ijk} \;=\; \frac{a'}{2\mathbf{q}_{k\,2}} \left(|\mathbf{q}_k|^2 - a'^2 \right)$$

$$A_{ijk} \;=\; a' \, \mathbf{q}_{k\,2}$$

and $t_i > 0$. This inequality is in turn equivalent to

$$\begin{aligned} \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k\,0}-q_{i\,0}}{q_{i\,0}-q_{j\,0}} \;&<\; \tfrac{1}{2} \;-\; \frac{\alpha_{ijk}}{A_{ijk}} \frac{1}{q_{i\,0}-q_{j\,0}} \left(q_{i\,0}+\frac{\mathbf{q}_{k\,1}-a'}{2a'} q_{i\,0}-\frac{\mathbf{q}_{k\,1}+a'}{2a'} q_{j\,0} \right) \\ &=\; \tfrac{1}{2} \left(1 \;-\; \frac{\alpha_{ijk}}{A_{ijk}} \frac{\mathbf{q}_{k\,1}+a'}{a'} \right) \end{aligned}$$

We will show that

$$\tfrac{1}{2} \left(1 \;-\; \frac{\alpha_{ijk}}{A_{ijk}} \frac{\mathbf{q}_{k\,1}+a'}{a'} \right) \;=\; \frac{\alpha_{kij}}{A_{kij}}$$

In other words,

$$t_0 \;<\; t_i \;\Leftrightarrow\; \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{k\,0}-q_{i\,0}}{q_{i\,0}-q_{j\,0}} \;<\; \frac{\alpha_{kij}}{A_{kij}}$$

Let $2e_\ell > 0$, $\ell = i, j, k$, be the length of the edge opposite to the vertex $\mathbf{q}_\ell$ in the triangle with vertices $\mathbf{q}_i$, $\mathbf{q}_j$, $\mathbf{q}_k$, and let h_ℓ be the (signed) height of the triangle over the edge opposite to the vertex $\mathbf{q}_\ell$. Clearly,

$$A_{ijk} \;=\; e_\ell \cdot h_\ell$$

for $\ell = i, j, k$. We also let s_ℓ be the (signed) height of $\Re\, \mathbf{d}^{ijk}$ over the edge opposite to the vertex $\mathbf{q}_\ell$. Clearly,

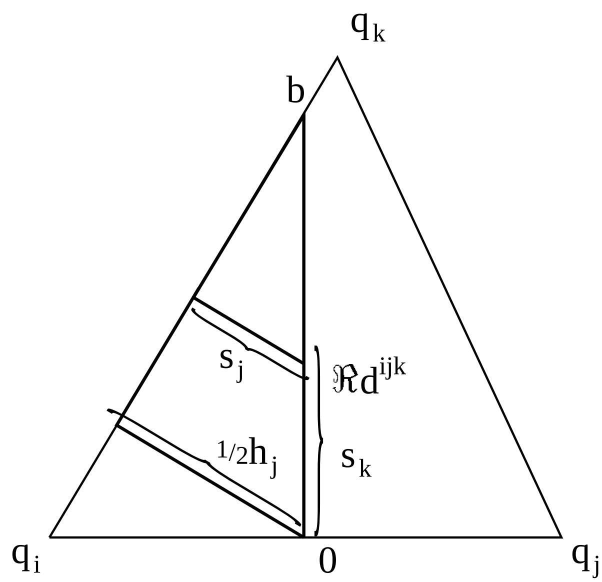
$$\alpha_{ijk} \;=\; e_k \cdot s_k \;=\; a' \cdot s_k$$

$$\alpha_{jki} \;=\; e_i \cdot s_i$$

$$\alpha_{kij} \;=\; e_j \cdot s_j$$

Finally, let $\mathbf{b}$ be the point where the perpendicular bisector of $\mathbf{q}_i, \mathbf{q}_j$ meets the line through $\mathbf{q}_i$ and $\mathbf{q}_k$.

The two triangles in the figure below with vertex $\mathbf{b}$ and solid edges are similar.



Therefore,

$$\frac{s_j}{\frac{1}{2}h_j} = \frac{\mathbf{b}_2 - s_k}{\mathbf{b}_2} = 1 - \frac{s_k}{\mathbf{b}_2}$$

Also, the triangle with vertices $\mathbf{q}_i = (-a', 0), 0, \mathbf{b}$ is similar to the triangle with vertices $\mathbf{q}_i = (-a', 0), (\mathbf{q}_{k1}, 0), \mathbf{q}_k$. Therefore,

$$\frac{\mathbf{q}_{k\,2}}{\mathbf{b}_2} = \frac{a' + \mathbf{q}_{k\,1}}{a'}$$

It follows that

$$\frac{\alpha_{kij}}{A_{kij}} = \frac{s_j}{h_j} = \frac{1}{2}\left(1 - \frac{s_k}{\mathbf{b}_2}\right) = \frac{1}{2}\left(1 - \frac{s_k}{\mathbf{q}_{k\,2}}\frac{a'+\mathbf{q}_{k\,1}}{a'}\right) = \frac{1}{2}\left(1 - \frac{\alpha_{ijk}}{A_{ijk}}\frac{a'+\mathbf{q}_{k\,1}}{a'}\right)$$

as claimed above.

By the same sort of argument,

$$t_j < t_0 \quad \Leftrightarrow \quad \frac{\alpha_{ijk}}{A_{ijk}}\frac{q_{j\,0} - q_{k\,0}}{q_{i\,0} - q_{j\,0}} < \frac{\alpha_{jki}}{A_{jki}}$$

The proof of the first part of Lemma III.5 is now complete.

(ii) If $k_F < \rho_\triangle$, then, by definition, $\chi_{i'j'k'} = 0$ for all permutations i', j', k' of i, j, k. We may therefore assume that $\rho_\triangle < k_F$. We may further assume that the triangle with vertices

$\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k$ is counter clockwise oriented since, by direct inspection, $\chi_{ijk} = -\chi_{jik}$ and, as observed before, $r_{ijk} = -r_{jik}$. It thus suffices to show that

$$\chi_{ijk} + \chi_{jki} + \chi_{kij} = \begin{cases} 1\ , & \text{if } \triangle \text{ is an acute triangle} \\ 0\ , & \text{if } \triangle \text{ is an obtuse triangle} \end{cases}$$

Clearly, one or two of the differences

$$q_{i\,0} - q_{j\,0}\ ,\ q_{j\,0} - q_{k\,0}\ ,\ q_{k\,0} - q_{i\,0}$$

is positive, since $\big(q_{i\,0} - q_{j\,0}\big) + \big(q_{j\,0} - q_{k\,0}\big) + \big(q_{k\,0} - q_{i\,0}\big) = 0$. We first verify the statement of the last paragraph in the case that $\triangle_{(\mathbf{q}_i,\mathbf{q}_j,\mathbf{q}_k)}$ is an acute triangle whose vertices $\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k$ are counter clockwise oriented and exactly one of the differences is positive. Observe that $0 < A_{ijk} = A_{jki} = A_{kij}$ since $\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k$ are counter clockwise oriented and that $\alpha_{ijk}, \alpha_{jki}, \alpha_{kij} > 0$ since $\triangle_{(\mathbf{q}_i,\mathbf{q}_j,\mathbf{q}_k)}$ is acute.

We may assume, without loss of generality, that

$$q_{i\,0} - q_{j\,0} > 0\ ,\ q_{j\,0} - q_{k\,0} < 0\ ,\ q_{k\,0} - q_{i\,0} < 0$$

By construction, $i \notin \mathcal{I}^+_{jk}(q) \cup \mathcal{I}^-_{jk}(q)$ and $j \notin \mathcal{I}^+_{ki}(q) \cup \mathcal{I}^-_{ki}(q)$ and therefore, by definition,

$$\chi_{jki} = \chi_{kij} = 0$$

On the other hand, $k \in \mathcal{I}^+_{ij}(q)$ and

$$\begin{aligned} \tfrac{\alpha_{ijk}}{A_{ijk}}\,\tfrac{q_{k\,0}-q_{i\,0}}{q_{i\,0}-q_{j\,0}} &< 0 < \tfrac{\alpha_{kij}}{A_{kij}} \\ \tfrac{\alpha_{ijk}}{A_{ijk}}\,\tfrac{q_{j\,0}-q_{k\,0}}{q_{i\,0}-q_{j\,0}} &< 0 < \tfrac{\alpha_{jki}}{A_{jki}} \end{aligned}$$

so that, by definition,

$$\chi_{ijk} = 1$$

We next treat the case in which $\triangle_{(\mathbf{q}_i,\mathbf{q}_j,\mathbf{q}_k)}$ is an acute triangle whose vertices $\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k$ are counter clockwise oriented and exactly two of the differences are positive. We assume, without loss of generality, that

$$q_{i\,0} - q_{j\,0} > 0\ ,\ q_{j\,0} - q_{k\,0} > 0\ ,\ q_{k\,0} - q_{i\,0} < 0$$

As above, the third inequality implies $\chi_{kij} = 0$. On the other hand, the first two inequalities imply $k \in \mathcal{I}_{ij}^{+}(q)$, $i \in \mathcal{I}_{jk}^{+}(q)$ and consequently,

$$\chi_{ijk} = \begin{cases} 1 & \text{if } \frac{\alpha_{ijk}}{A_{ijk}} \frac{q_{j\,0} - q_{k\,0}}{q_{i\,0} - q_{j\,0}} < \frac{\alpha_{jki}}{A_{jki}} \\ 0 & \text{otherwise} \end{cases}$$
$$\chi_{jki} = \begin{cases} 1 & \text{if } \frac{\alpha_{jki}}{A_{jki}} \frac{q_{i\,0} - q_{j\,0}}{q_{j\,0} - q_{k\,0}} < \frac{\alpha_{ijk}}{A_{ijk}} \\ 0 & \text{otherwise} \end{cases}$$

This shows that

$$\chi_{ijk} + \chi_{jki} = 1$$

The proof that

$$\chi_{ijk} + \chi_{jki} + \chi_{kij} = 1$$

when $\triangle$ is an accute, counter clockwise oriented triangle is now complete.

The case of an obtuse triangle is similar. ■

The proof of Theorem III.2 requires two ingredients in addition to Proposition III.6.

Proposition III.7 *For each pair $1 \le i, j \le n+1$, $i \ne j$, and $k \ne i, j$ let ω_{ijk} be the unique meromorphic differential form on $\overline{L}_{ij}$ that is holomorphic outside the points $\mathbf{D}^{ijk}$, and $\overline{L}_{ij} \cap L_\infty$, that has a simple pole with residue 1 at the point $\mathbf{D}^{ijk}$, and that has a simple pole at $\overline{L}_{ij} \cap L_\infty$. If $|\mathbf{p}_i - \mathbf{p}_j| > 2k_F$ then*

$$\int_{C_{ij}^-}^{C_{ij}^+} \omega_{ijk} = -4\pi\imath\, \epsilon_{\triangle,(\mathbf{p}_i,\mathbf{p}_j)}\, R_\triangle \operatorname{arccot}\sqrt{\frac{4\rho_\triangle^2 - |\mathbf{p}_i - \mathbf{p}_j|^2}{|\mathbf{p}_i - \mathbf{p}_j|^2 - 4k_F^2}}$$

where $\triangle = \triangle(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k)$ and

$$\epsilon_{\triangle,(\mathbf{p}_i,\mathbf{p}_j)} = \begin{cases} +1 & \text{if the angle of } \triangle \text{ at } \mathbf{p}_k \text{ is acute} \\ -1 & \text{if the angle of } \triangle \text{ at } \mathbf{p}_k \text{ is obtuse} \end{cases}$$

Proposition III.8

$$\sum_{1 \le i \ne j \le n+1} \int_{C_{ij}^+(0)}^{C_{ij}^-(0)} \omega_{ij} = 4\pi i \sum_{\triangle \in \mathcal{T}_{\mathrm{ac}}} R_\triangle$$

Before proving Proposition III.7 and Proposition III.8, we give the

Proof of Theorem III.2 : By Proposition III.6 and Proposition III.8,

$$\begin{aligned}
\tfrac{1}{m^n} J(\mathbf{p}_1,\cdots,\mathbf{p}_{n+1}) &= \tfrac{1}{4\pi\imath} \sum_{1\le i\ne j\le n+1} \int_{C_{ij}^+(0)}^{C_{ij}^-(0)} \omega_{ij} + \tfrac{1}{4\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \int_{C_{ij}^-}^{C_{ij}^+} \omega_{ij} - \sum_{\substack{\triangle\in\mathcal{T}_{\mathrm{ac}}\\ \rho_\triangle<k_F}} R_\triangle \\
&= \sum_{\triangle\in\mathcal{T}_{\mathrm{ac}}} R_\triangle + \tfrac{1}{4\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \int_{C_{ij}^-}^{C_{ij}^+} \omega_{ij} - \sum_{\substack{\triangle\in\mathcal{T}_{\mathrm{ac}}\\ \rho_\triangle<k_F}} R_\triangle \\
&= \tfrac{1}{4\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \int_{C_{ij}^-}^{C_{ij}^+} \omega_{ij} + \sum_{\substack{\triangle\in\mathcal{T}_{\mathrm{ac}}\\ \rho_\triangle>k_F}} R_\triangle
\end{aligned}$$

By construction for $1\le i,j\le n+1$, $i\ne j$

$$\omega_{ij} = \sum_{k\ne i,j} R_{ijk}\,\omega_{ijk}$$

Therefore by Proposition III.7

$$\begin{aligned}
&\tfrac{1}{4\pi\imath} \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \int_{C_{ij}^-}^{C_{ij}^+} \omega_{ij} \\
&\quad = - \sum_{\substack{1\le i\ne j\le n+1\\ |\mathbf{p}_i-\mathbf{p}_j|>2k_F}} \sum_{\substack{1\le k\le n+1\\ k\ne i,j}} \epsilon_{\triangle(\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k),(\mathbf{p}_i,\mathbf{p}_j)}\, R_{\triangle(\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k)} \operatorname{arccot}\sqrt{\tfrac{4\rho^2_{\triangle(\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k)}-|\mathbf{p}_i-\mathbf{p}_j|^2}{|\mathbf{p}_i-\mathbf{p}_j|^2-4k_F^2}} \\
&\quad = -2 \sum_{\substack{\triangle\in\mathcal{T}\\ s\ \text{edge of}\ \triangle\\ |s|>2k_F}} \epsilon_{\triangle,s}\, R_\triangle \operatorname{arccot}\sqrt{\tfrac{4\rho_\triangle^2-|s|^2}{|s|^2-4k_F^2}}
\end{aligned}$$

■

Proof of Proposition III.7: By possibly interchanging i and j we may assume that the triangle with vertices $\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k$ is counterclockwise oriented. As in the proof of Lemma III.4 we may assume that $\mathbf{p}_i=(-a,0)$ and $\mathbf{p}_j=(a,0)$ with $a>0$. Then there is $b\ne 0$ such that $\mathbf{D}^{ijk}=(0,b)$. Then

$$\begin{aligned}
a^2+b^2 &= \rho_\triangle^2 & b &= \epsilon_{\triangle,(\mathbf{p}_i,\mathbf{p}_j)}\,|b| \\
L_{ij} &= \{\ (z_1,z_2)\in\mathbb{C}^2 \ \big|\ z_1=0\ \} & C_{ij}^\pm &= (0,\pm\imath\sqrt{a^2-k_F^2}) \\
&& \omega_{ijk} &= \tfrac{dz_2}{z_2-b}
\end{aligned}$$

Consequently

$$\int_{C_{ij}^-}^{C_{ij}^+} \omega_{ijk} = \int_{-\imath\sqrt{a^2-k_F^2}}^{+\imath\sqrt{a^2-k_F^2}} \frac{dz}{z-b} = \int_{-b-\imath\sqrt{a^2-k_F^2}}^{-b+\imath\sqrt{a^2-k_F^2}} \frac{dz}{z}$$

$$= -4\pi\imath \operatorname{arccot} \frac{b}{\sqrt{a^2-k_F^2}} = -4\pi\imath\, \epsilon_{\triangle,(\mathbf{p}_i,\mathbf{p}_j)} \operatorname{arccot} \sqrt{\frac{\rho_\triangle^2-a^2}{a^2-k_F^2}}$$

The proof of Proposition III.8 requires a lemma. To prepare for Lemma III.9 pick a generic vector $(v_1, v_2) \in \mathbb{R}^2$ and, for each $i = 1, \cdots, n+1$, let

$$g_i = \left\{ z \in \mathbb{C}^2 \;\middle|\; v_1 z_1 + v_2 z_2 = v_1 \mathbf{p}_{i1} + v_2 \mathbf{p}_{i2} \right\}$$

be the line through $\mathbf{p}_i$ whose slope is determined by (v_1, v_2). For all pairs $i \neq j$, let E_{ij} be the point where the lines g_i and L_{ij} meet. Furthermore, let $\omega_{i\infty}$ be the rational differential form on the line L_∞ at infinity that is holomorphic outside the points of $L_\infty \cap \overline{L}_{ij}$ and has a pole of order one at $L_\infty \cap \overline{L}_{ij}$ with residue $-\sum_{k\neq i,j} R_{ijk}$. Finally, let γ_∞ be the union of the paths

$$\left\{ [0\,, -(1-t)i - tv_2\,, 1-t+tv_1] \;\middle|\; 0 \le t \le 1 \right\}$$

$$\left\{ [0\,, +(1-t)i - tv_2\,, 1-t+tv_1] \;\middle|\; 1 \ge t \ge 0 \right\}$$

lying in L_∞.

Lemma III.9 *For all* $1 \le i \le n+1$,

$$\sum_{j\neq i} \int_{C_{ij}^+(0)}^{C_{ij}^-(0)} \omega_{ij} = \int_{\gamma_\infty} \omega_{i\infty} + 2\pi\imath \sum_{\substack{j,k\neq i \\ j\neq k}} \beta_{ijk} R_{ijk}$$

where

$$\beta_{ijk} = \begin{cases} +1 & \text{if } \mathbf{D}^{ijk} \text{ lies between } \tfrac{1}{2}(\mathbf{p}_i+\mathbf{p}_j) \text{ and } E_{ij}, \\ & \text{and } \mathbf{p}_i, \mathbf{p}_j, E_{ij} \text{ are counterclockwise oriented} \\ -1 & \text{if } \mathbf{D}^{ijk} \text{ lies between } \tfrac{1}{2}(\mathbf{p}_i+\mathbf{p}_j) \text{ and } E_{ij}, \\ & \text{and } \mathbf{p}_i, \mathbf{p}_j, E_{ij} \text{ are clockwise oriented} \\ 0 & \text{otherwise} \end{cases}$$

Proof: Put

$$F_t^\pm(z_1, z_2) = (1-t)\Big((z_1 - \mathbf{p}_{i1}) \pm i(z_2 - \mathbf{p}_{i2})\Big) + t\Big(v_1 z_1 + v_2 z_2 - v_1 \mathbf{p}_{i1} - v_2 \mathbf{p}_{i2}\Big)$$

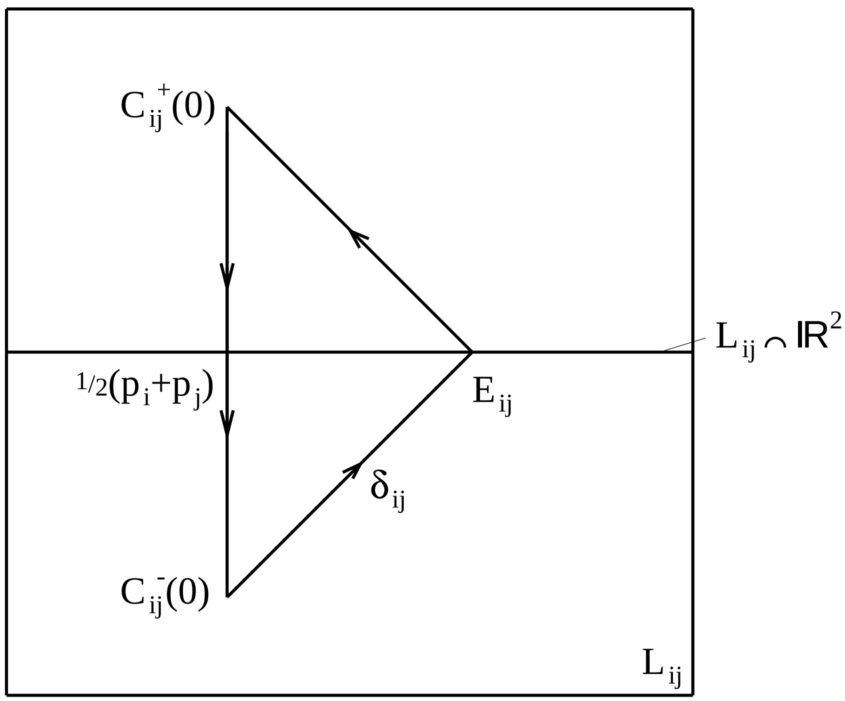
and let $E_{ij}^{\pm}(t)$ be the root of $F_t^{\pm}$ on L_{ij}. Clearly, $E_{ij}^{\pm}(1) = E_{ij}$ and

$$\begin{aligned} g_i \;&=\; \left\{ \, (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; F_1^{\pm}(z_1, z_2) = 0 \, \right\} \\ \overline{Q_i^{\pm}(0)} \;&=\; \left\{ \, (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; F_0^{\pm}(z_1, z_2) = 0 \, \right\} \end{aligned}$$

and consequently,

$$\delta_{ij} \;=\; \left[C_{ij}^{+}(0)\,,C_{ij}^{-}(0)\right] \cup \left\{ \; E_{ij}^{-}(t) \;\middle|\; 0 \le t \le 1 \; \right\} \cup \left\{ \; E_{ij}^{+}(t) \;\middle|\; 1 \ge t \ge 0 \; \right\}$$

is a loop that meets $L_{ij} \cap \mathbb{R}^2$ only in the points $\frac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$ and E_{ij}.



Observe that the projective closure of $\left\{ \, (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; F_t^{\pm}(z_1, z_2) = 0 \, \right\}$ meets the line L_∞ at the point

$$E_{i\infty}(t) \;=\; \left[0\,, \mp(1-t)\imath - tv_2\,, 1-t+tv_1\right]$$

We have

$$\gamma_\infty \;=\left\{ \; E_{i\infty}^{+}(t) \;\middle|\; 0 \le t \le 1 \; \right\} \cup \left\{ \; E_{i\infty}^{-}(t) \;\middle|\; 1 \ge t \ge 1 \; \right\}$$

for all $i = 1, \cdots, n+1$.

The union of lines

$$X_i \;=\; L_\infty \cup \bigcup_{j \neq i} \overline{L}_{ij}$$

is a singular curve in $\mathbb{P}^2$. The forms $\omega_{ij}, j = 1, \cdots, n+1, \infty, j \neq i$, define a Rosenlicht differential on X_i. For each $t \in [0,1]$ the divisor of the meromorphic function

$$\frac{F_t^- \cdot F_0^+}{F_0^- \cdot F_t^+}$$

on the curve X_i is

$$\sum_{\substack{j \neq i \\ j=1,\cdots,n+1,\,\infty}} \left(E_{ij}^-(t) - E_{ij}^-(0)\right) + \left(E_{ij}^+(0) - E_{ij}^+(t)\right)$$

By Abel's Theorem ([S], Ch. V.10, Proposition 5),

$$\sum_{\substack{j \neq i \\ j=1,\cdots,n+1,\,\infty}} \left(\int_{\left\{ E_{ij}^-(t) \,\middle|\, 0 \leq t \leq s \right\}} \omega_{ij} + \int_{\left\{ E_{ij}^+(t) \,\middle|\, s \geq t \geq 0 \right\}} \omega_{ij} \right)$$

is independent of s. For $s = 0$ this quantity is zero. Therefore,

$$\sum_{\substack{j \neq i \\ j=1,\cdots,n+1,\,\infty}} \int_{\left\{ E_{ij}^-(t) \,\middle|\, 0 \leq t \leq 1 \right\} \cup \left\{ E_{ij}^+(t) \,\middle|\, 1 \geq t \geq 0 \right\}} \omega_{ij} = 0$$

By the definition of the loop δ_{ij} and the conclusion of the last paragraph,

$$\begin{aligned}
\sum_{\substack{j=1 \\ j\neq i}}^{n+1} \int_{C_{ij}^+(0)}^{C_{ij}^-(0)} \omega_{ij} &= \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \int_{\delta_{ij}} \omega_{ij} + \int_{\gamma_\infty} \omega_{i\infty} \\
&\quad - \sum_{\substack{j \neq i \\ j=1,\cdots,n+1,\,\infty}} \int_{\left\{ E_{ij}^-(t) \,\middle|\, 0 \leq t \leq 1 \right\} \cup \left\{ E_{ij}^+(t) \,\middle|\, 1 \geq t \geq 0 \right\}} \omega_{ij} \\
&= 2\pi\imath \sum_{\substack{j,k \neq i \\ j \neq k}} \beta_{ijk} R_{ijk} + \int_{\gamma_\infty} \omega_{i\infty}
\end{aligned}$$

since, by Lemma III.4 (i), the winding number of δ_{ij} around $\mathbf{D}^{ijk}$ is equal to β_{ijk}. ■

Finally, we give the

Proof of Proposition III.8 : Observe that

$$\sum_{i=1}^{n+1} \omega_{i\infty} = 0$$

since, by construction, the residues at each pole add up to zero. Now, by Lemma III.9,

$$\sum_{1\le j\ne i\le n+1} \int_{C_{ij}^+(0)}^{C_{ij}^-(0)} \omega_{ij} \;=\; \int_{\gamma_\infty} \sum_{i=1}^{n+1} \omega_{i\infty} \;+\; 2\pi\imath \sum_{\substack{\text{pairwise different}\\ 1\le i,j,k\le n+1}} \beta_{ijk}\, R_{ijk}$$
$$= \; 2\pi\imath \sum_{\substack{\text{pairwise different}\\ 1\le i,j,k\le n+1}} \beta_{ijk}\, R_{ijk}$$

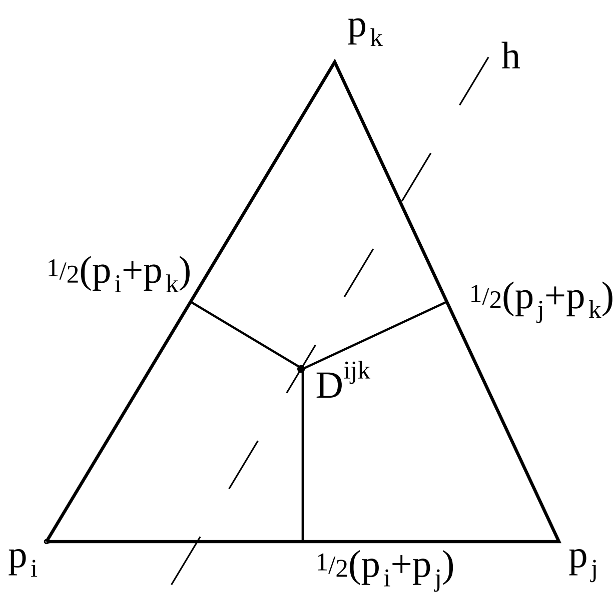
Fix three pairwise different indices i,j,k and let $\triangle = \triangle_{(\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k)}$. Let

$$h \;=\; \left\{\, (x_1,x_2) \in \mathbb{R}^2 \;\middle|\; v_1 x_1 + v_2 x_2 = v_1 \mathbf{D}_1^{ijk} + v_2 \mathbf{D}_2^{ijk} \,\right\}$$

be the real line through the center $\mathbf{D}^{ijk}$ of the circle circumscribing $\triangle$ whose slope is determined by (v_1,v_2). Then,

$$\beta_{ijk} \;=\; \begin{cases} 0 & \text{unless } h \text{ meets the segment between } \mathbf{p}_i \text{ and } \tfrac{1}{2}(\mathbf{p}_i+\mathbf{p}_j) \\ \pm 1 & \text{if } h \text{ meets the segment between } \mathbf{p}_i \text{ and } \tfrac{1}{2}(\mathbf{p}_i+\mathbf{p}_j) \\ & \text{and the triangle } \mathbf{p}_i,\mathbf{p}_j,\mathbf{D}^{ijk} \text{ is counterclockwise (clockwise) oriented} \end{cases}$$

In particular, if $\triangle$ is acute, there are exactly two permutations i',k',j' of i,j,k for which $\beta_{i'k'j'} \ne 0$.



Furthermore, if $\beta_{i'k'j'} \ne 0$, then

$$\beta_{i'k'j'} = \pm 1 \quad\Leftrightarrow\quad \mathbf{p}_{i'},\mathbf{p}_{j'},\mathbf{p}_{k'} \text{ counterclockwise (clockwise) oriented}$$
$$\Leftrightarrow\quad R_{i'k'j'} = \pm R_\triangle$$

Consequently,

$$\sum_{\{i',j',k'\}=\{i,j,k\}} \beta_{i'j'k'}\, R_{i'j'k'} \;=\; 2R_{\Delta}$$

Similarly, if $\triangle$ is obtuse,

$$\sum_{\{i',j',k'\}=\{i,j,k\}} \beta_{i'j'k'}\, R_{i'j'k'} \;=\; 0$$

■

Appendix

In this appendix we assume that $J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ exists and give a direct proof of

$$J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \;=\; 0$$

when the circle circumscribing the triangle with vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ has radius less than k_F. If the triangle is acute, the previous condition is equivalent to the statement that the three open disks K_1, K_2, K_3 with radius k_F around the points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ have a point in common. By translation invariance, the center of the circle circumscribing the triangle can be placed at the origin.

By definition,

$$J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \;=\; \tfrac{1}{(2\pi)^3} \int_{\mathbb{R}^2} dk_1\, dk_2 \int_{-\infty}^{+\infty} \frac{dk_0}{\prod\limits_{i=1}^{3} \big(\imath k_0 - e(\mathbf{k}-\mathbf{p}_i)\big)}$$

The integral

$$\int_{-\infty}^{+\infty} \frac{dk_0}{\prod\limits_{i=1}^{3} \big(\imath k_0 - e(\mathbf{k}-\mathbf{p}_i)\big)}$$

is evaluated by closing the contour in the upper half plane when the point $\mathbf{k} \in \mathbb{R}^2$ belongs to at most one of the disks K_1, K_2, K_3 and closing in the lower half plane when it belongs to at least two of the disks. In particular, the integral vanishes when $\mathbf{k} \notin K_1 \cup K_2 \cup K_3$ or $\mathbf{k} \in K_1 \cap K_2 \cap K_3$. We obtain

$$\begin{aligned}
\tfrac{(2\pi)^2}{m^2}\, J(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \;=\;& \int_{K_1\setminus(K_2\cup K_3)} \frac{dk_1\, dk_2}{F_{12}(\mathbf{k})\, F_{13}(\mathbf{k})} - \int_{(K_2\cup K_3)\setminus K_1} \frac{dk_1\, dk_2}{F_{12}(\mathbf{k})\, F_{13}(\mathbf{k})} \\
&+ \int_{K_3\setminus(K_1\cup K_2)} \frac{dk_1\, dk_2}{F_{31}(\mathbf{k})\, F_{32}(\mathbf{k})} - \int_{(K_1\cup K_2)\setminus K_3} \frac{dk_1\, dk_2}{F_{31}(\mathbf{k})\, F_{32}(\mathbf{k})} \\
&+ \int_{K_2\setminus(K_3\cup K_1)} \frac{dk_1\, dk_2}{F_{23}(\mathbf{k})\, F_{21}(\mathbf{k})} - \int_{(K_3\cup K_1)\setminus K_2} \frac{dk_1\, dk_2}{F_{23}(\mathbf{k})\, F_{21}(\mathbf{k})}
\end{aligned}$$

where, for all $i, j = 1, 2, 3$,

$$F_{ij}(\mathbf{k}) \;=\; m\Big(e(\mathbf{k}-\mathbf{p}_i) - e(\mathbf{k}-\mathbf{p}_j)\Big) \;=\; (\mathbf{p}_j - \mathbf{p}_i)\cdot \mathbf{k}$$

since, $\mathbf{p}_1^2 = \mathbf{p}_2^2 = \mathbf{p}_3^2$.

Let $2r$ be the length of the secant to the circle ∂K_1 through 0 that is perpendicular to the line through 0 and $\mathbf{p}_1$. Then the map

$$\begin{aligned} \phi : \ \mathbb{R}^2 \setminus \{0\} &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ \mathbf{k} &\longmapsto -\tfrac{r^2}{|\mathbf{k}|^2}\, \mathbf{k} \end{aligned}$$

maps each of the circles ∂K_1, ∂K_2, ∂K_3 to itself ([Be], 10.8). By hypothesis, $0 \in K_i$, $i = 1, 2, 3$, and therefore, ϕ maps K_i to the exterior of ∂K_i and conversely. In particular,

$$\phi\left(K_1 \setminus \left(\overline{K}_2 \cup \overline{K}_3\right)\right) \ = \ (K_2 \cap K_3) \setminus \overline{K}_1$$

Substituting,

$$F_{12}\left(\phi(\mathbf{k})\right) F_{13}\left(\phi(\mathbf{k})\right) \ = \ \tfrac{r^4}{|\mathbf{k}|^4}\, F_{12}(\mathbf{k})\, F_{13}(\mathbf{k})$$

Also,

$$\phi^*\left(dk_1 \wedge dk_2\right) \ = \ \tfrac{r^4}{|\mathbf{k}|^4}\, dk_1 \wedge dk_2$$

The last three equations imply that

$$\int_{K_1 \setminus (K_2 \cup K_3)} \frac{dk_1\, dk_2}{F_{12}(\mathbf{k})\, F_{13}(\mathbf{k})} \ - \ \int_{(K_2 \cap K_3) \setminus K_1} \frac{dk_1\, dk_2}{F_{12}(\mathbf{k})\, F_{13}(\mathbf{k})} \ = \ 0$$

The other pairs also cancel.

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