An Intrinsic 1/N Expansion for Many Fermion Systems

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Abstract The $1/N$ expansion is a popular tool for investigating non-perturbative long range phenomena. In many body models, discretization of the Fermi surface naturally introduces a many component picture. If, for example, number symmetry is broken, $N \approx (e^{\frac{1}{\lambda}})^{(d1)}$ where $\lambda > 0$ is the bare coupling constant. We expect that this intrinsic $1/N$ expansion appears whenever the "free propagator" of the system is singular along a hypersurface (for instance, in the Anderson model for localized and extended states).

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§I Introduction

Most interesting long range physical phenomena, such as quark confinement, or pairing in the BCS theory of superconductivity, are non-perturbative. To analyze non-perturbative effects one usually consider models with a variable number, N, of components. The strategy is to resum the set of graphs that dominate when $N \to \infty$, then to develop an expansion in $1/N$.

This approach works best for "vector models". In such models with a $(\vec{\phi} \cdot \vec{\phi})^2$ or four fermi interaction, the $N = \infty$ set of graphs is a geometric series of "bubbles" that can be resummed. Applying this remark, mass generation in two-dimensional non-linear σ models can be studied order by order in $1/N$ and presumably be established rigorously for N large enough. In the more complicated case of a matrix model (e.g. confinement), the $N = \infty$ set of graphs consists of all planar graphs. It cannot easily be resummed. In these two examples, the $1/N$ expansion is only a useful artifice because the number of components is not a physical variable and is not particularly large.

We argue, in §II, that the Fermi surface induces an intrinsic $1/N$ expansion for the BCS model, in which the number $N = (1/\Delta)^{d1}$ of components is the number of elements in a discretization [2a] of the Fermi surface. Here $\Delta \approx e^{-1/\lambda}$ is the BCS gap. The number of components can be increased by decreasing the coupling constant $\lambda > 0$. We argue that this is why the usual BCS gap equation, the result of a one-loop computation, is a good approximation. Similar remarks apply to many other approximations of solid state physics in which the "free propagator" of the system is singular along a hypersurface, for instance the Anderson model for a single electron in a random potential or with random spin-orbit coupling.

In this letter, we restrict ourselves for simplicity to $\ell = 0$ superconductivity in $d \geq 2$. Consider the action $\mathcal{A}(\psi, \bar{\psi}) = -\int \! đk \, \big(ik_0 e(\mathbf{k})\big)\bar{\psi}(k)\psi(k) - \mathcal{V}(\psi, \bar{\psi})$ in which $k = (k_0, \mathbf{k}) \in \mathbb{R}^{d+1}$, $đk = \frac{d^{d+1}k}{(2\pi)^{d+1}}$ and $e(\mathbf{k}) = \frac{1}{2\mathbf{m}}\mathbf{k}^2 - \mu$. In the interaction

$$\mathcal{V}(\psi, \bar{\psi}) = \frac{\lambda}{2} \int \prod_{i=1}^{4} đk_i \; (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4) \; \bar{\psi}(k_1)\psi(k_3) \, \langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle \, \bar{\psi}(k_2)\psi(k_4) \tag{I.1}$$

we assume that $\lambda > 0$ and $\langle s', -s' | \mathrm{V} | t', -t' \rangle$ is attractive and dominant in the $\ell = 0$ sector.

Here, $k' = (0, \frac{\mathbf{k}}{|\mathbf{k}|}\sqrt{2\mathbf{m}\mu})$ is the projection of k onto the Fermi surface. The electron fields are vectors $\psi(k) = \begin{pmatrix} \psi_\uparrow(k) \\ \psi_\downarrow(k) \end{pmatrix}$ and $\bar\psi(k) = \left(\bar\psi_\uparrow(k), \bar\psi_\downarrow(k)\right)$ whose components $\psi_\sigma(k)$, $\bar\psi_\sigma(k)$ generate an infinite dimensional Grassmann algebra over $\mathbb{C}$.

§II The N Component Vertex in the Symmetry Breaking Regime

To investigate the long range behavior of correlation functions at low temperature, it is natural to use a renormalization group analysis ([1a,b]) near the Fermi surface. This entails slicing the free propagator around its singularity on the Fermi sphere. The renormalization group generates an effective slice-dependent interaction.

In Euclidean field theory, one defines the momentum k to be of scale j if $|k| \approx 2^j$. As $j \to \infty$ $(-\infty)$, the momentum k approachs the ultraviolet (infrared) end of the model.

In non-relativistic solid state physics the natural scales consist of finer and finer shells around the Fermi surface. For each negative integer $j = 0, -1, -2, ...$ the j-th slice contains all momenta in a shell of thickness 2^j a distance 2^j from the singular locus $\left\{ k \in \mathbb{R}^{d+1} \;\middle|\; k_0 = 0,\ |\mathbf{k}| = \sqrt{2\mathbf{m}\mu} \right\}$. The propagator for the j-th slice is

$$C^j(\xi_1, \xi_2) = \delta_{\sigma_1,\sigma_2} \int \text{đ}k \frac{e^{i\langle k, \xi_1 - \xi_2\rangle_-}}{ik_0 - e(\mathbf{k})} 1_j\left(k_0^2 + e(\mathbf{k})^2\right) \tag{II.1}$$

where $1_j\left(k_0^2 + e(\mathbf{k})^2\right)$ is the characteristic function for the set $2^j \le |ik_0 - e(\mathbf{k})| < 2^{j+1}$. (A smooth partition of unity is required for a technically correct analysis). Summing over $j \le 0$, we have the infrared propagator $C(\xi_1, \xi_2) = \sum_{j \le 0} C_j(\xi_1, \xi_2)$.

Each single scale propagator (II.1) is supported in momentum space on a $d+1$ dimensional manifold with boundary. The natural coordinates on this manifold are k_0, $\eta = e(\mathbf{k})$ and $\mathbf{k}' = \sqrt{2\mathbf{m}\mu}\frac{\mathbf{k}}{|\mathbf{k}|}$. In these coordinates the shell is $\left\{ k \;\middle|\; 2^j \le \sqrt{k_0^2 + \eta^2} \le \text{const}\, 2^j \right\}$ and is topologically $S^{d-1} \times S^1 \times [0,1]$. The first factor, the Fermi sphere S^{d-1}, has a macroscopic diameter of order 1 while S^1 and $[0,1]$ have small diameters of order 2^j.

The fact that this manifold has two length scales, 1 and 2^j, of radically different size reflects the basic anisotropy between frequency k_0 and momentum $\mathbf{k}$. Consequently the behavior of $C^j(0,\xi)$ at large $\xi = (\mathbf{x}, t)$ cannot be simply characterized as 'decay at length scale 2^{-j}'. Rather, smoothly cutoff, C^j obeys $\left|C^j(0,\xi)\right| \le \text{const}\, 2^j \left[1 + |\mathbf{x}|\right]^{(1-d)/2} \left[1 + 2^j|\xi|\right]^{-n}$.

Shells induce an infrared renormalization group flow. The important part (see [1a]) of (I.1) comes from the reduced interaction $-\lambda\,\langle s',-s'|\mathrm{V}|t',-t'\rangle$. Expanding in spherical harmonics $-\lambda\,\langle s',-s'|\mathrm{V}|t',-t'\rangle \;=\; \sum_{\ell\geq 0}\lambda_\ell(0)\,\pi_\ell(s',t')$ our assumption becomes $\lambda_0(0)>0$ and $\lambda_0(0)\gg|\lambda_\ell(0)|\,,\,\ell\geq 1\,.$ The renormalization group flows the set of coupling constants $\{\ \lambda_\ell\ \big|\ \ell\geq 0\ \}$. Let $\{\ \lambda_\ell(j)\ \big|\ \ell\geq 0\ \}$ be their values at scale j. In the second order ladder approximation, the flow equation is (see [1a,I.85])

$$\lambda_\ell(j-1) \;=\; \lambda_\ell(j)+\beta(j)\,\lambda_\ell(j)^2$$

where $\beta(j)>0$ and $\lim_{j\to-\infty}\beta(j)=\beta>0$. In this approximation, $\lambda_0(j)$ grows slowly as j goes down to the symmetry breaking scale $\delta=-[1/\lambda_0(0)]$ and then quickly takes off to infinity. The other coupling constants remain much smaller than λ_0. This approximation breaks down at about scale δ. The divergence of a flow generated by a "Fermi surface" away from a Gaussian fixed point towards a nontrivial fixed point is typical of many symmetry breaking or mass generation phenomena in condensed matter physics.

This renormalization group analysis reveals three distinct energy regimes. Fix $a\gg 1$ and let Δ be the BCS gap. In the first regime at scales j for which $2^j>a\Delta$ the effective coupling constant $\lambda_0(j)$ can be used as a small parameter. Symmetry breaking takes place in the second regime where $\frac{1}{a}\Delta<2^j<a\Delta\,.$ In the third regime $2^j<\frac{1}{a}\Delta$ the physics of the Goldstone boson dominates. As explained above the effective coupling constant is not small in the latter two regimes. We will show that there is another small parameter there.

Consider now the middle, symmetry breaking, regime. The scale at the top is $\delta=\log_2 a\Delta\,.$ We make the Ansatz, based on the above picture, that the effective vertex at scale δ is given by the BCS interaction for Cooper pairs

$$\mathcal{V}_{\text{eff}} \;=\; -\lambda_0(\delta)\int_{|q|<\text{const}\,\Delta} đq\,đt\,đs\ \bar\psi_\uparrow(t+\tfrac{q}{2})\bar\psi_\downarrow(-t+\tfrac{q}{2})\psi_\downarrow(-s+\tfrac{q}{2})\psi_\uparrow(s+\tfrac{q}{2}) \tag{II.2}$$

Here, $0<\lambda_0(\delta)\simeq O(1)\,.$ All the fields are at scale δ so that the integrals are implicitly constrained by $|e(\pm t+\frac{q}{2})|\,,\ |e(\pm s+\frac{q}{2})|\,,\ |t_0|\,,\ |s_0|<\text{const}\,\Delta\,.$ This Ansatz can be nonperturbatively justified [1a, 2a] in two space dimensions and perturbatively justified [1a] in three.

Now momenta lie in the shell $S=\{\ k\ \big|\ \frac{1}{a}\Delta\leq\sqrt{k_0^2+\eta^2}\leq a\Delta\ \}$. To obtain regions in momentum space all of whose dimensions are of order Δ it is natural to further divide this shell, through a partition of unity, into $N = \Delta^{-(d-1)}$ pieces, each having longest and shortest diameters of order Δ. We call each piece a sector. They are related to the patches in Haldane's innovative, phenomenological theory of the Fermi surface. Sectors function as colors in a many component model. The free propagator in the middle regime in sector Σ is given by

$$C_\Sigma(\xi_1, \xi_2) \;=\; \delta_{\sigma_1,\sigma_2} \int \, đk \, \frac{e^{i\langle k,\xi_1-\xi_2\rangle_-}}{ik_0 - e(\mathbf{k})} S(k)\, \Sigma(\mathbf{k})$$

where $\Sigma(\mathbf{k})$ is supported on the sector Σ and by abuse of notation $S(k)$ is the characteristic function of S. Of course, $\sum_\Sigma C_\Sigma$ is the full propagator in the symmetry breaking regime. There is a corresponding decomposition of the fields.

By the Pauli exclusion principle, at most two spin one half fields can be localized in a position space box of side $(a\Delta)^{-1}$ when their momenta are in $S \cap \Sigma$. Thus, there can be $O(\Delta^{-(d1)})$ fields with momenta in S. That is, sectors enforce the Pauli principle, while the whole shell allows an accumulation of fields, as in Bosonic models. We emphasize that sectors are not required for a rigorous analysis of any finite order of perturbation theory. They are essential [2a] for the non-perturbative control of the first regime in $d = 2$. When $d = 2$ the full interaction has a vector structure, even in the first regime [2b]. Here, we will demonstrate the $1/N$ character of the effective vertex in the symmetry breaking regime, irrespective of dimension.

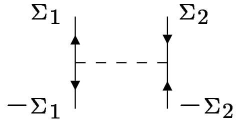
Expanding in sectors, the effective vertex (II.2) becomes

$$\sum_{\Sigma_1,\cdots,\Sigma_4} -\lambda_0(\delta) \int_{|q|<\mathrm{const}\,\Delta} đq\, đs_1\, đs_2\; \bar\psi_{\Sigma_1,\uparrow(s_1+\frac{q}{2})} \bar\psi_{\Sigma_3,\downarrow(-s_1+\frac{q}{2})} \psi_{\Sigma_4,\downarrow(-s_2+\frac{q}{2})} \psi_{\Sigma_2,\uparrow(s_2+\frac{q}{2})}$$

Here, a priori, all four sums run over $(\mathrm{const}\,\Delta)^{-(d1)}$ sectors. However, $|q| < \mathrm{const}\,\Delta$ and the momenta $s_1 + \frac{q}{2}$, $-s_1 + \frac{q}{2}$ must be in the sectors Σ_1 and Σ_3 respectively. Therefore, Σ_1 and Σ_3 must be antipodal, up to nearest neighbors, and similarly for Σ_2 and Σ_4. Consequently, it is no more difficult to treat (II.2) than the vertex

$$\sum_{\Sigma_1,\Sigma_2} -\lambda_0(\delta) \int đq\, đs_1\, đs_2\; \bar\psi_{\Sigma_1,\uparrow(s_1+\frac{q}{2})} \bar\psi_{-\Sigma_1,\downarrow(-s_1+\frac{q}{2})} \psi_{-\Sigma_2,\downarrow(-s_2+\frac{q}{2})} \psi_{\Sigma_2,\uparrow(s_2+\frac{q}{2})} \tag{II.3}$$

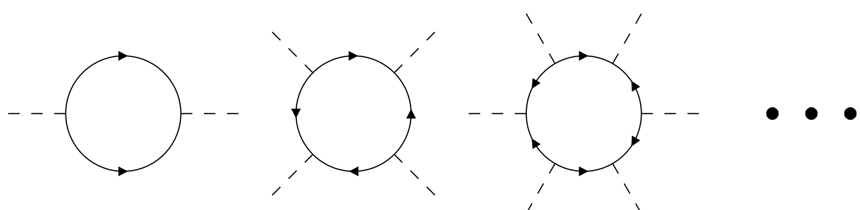
where $-\Sigma$ is the sector antipodal to Σ and nearest neighbors are ignored. Observe that the constraint $|q| < \mathrm{const}\,\Delta$ is now automatically fulfilled.



The last vertex clearly has the structure typical of an N-component vector model with $N = \text{const}\,\Delta^{-(d1)}$. Pictorially, sectors ("colors") Σ_1 and $-\Sigma_1$ enter at the left of an interaction line and sectors Σ_2 and $-\Sigma_2$ emerge at the right. Thus, sector is conserved up to flips as it flows along particle lines in a graph.

We now check, by power counting, that the effective coupling constant of the vertex is $1/N$. Let $G_{n,p}$ be a connected graph with n vertices (II.3) and $2p$ external lines. First assume that each internal line has a single sector propagator $\frac{S(k)\,\Sigma(\mathbf{k})}{ik_0-e(\mathbf{k})}$. Each such propagator is bounded in magnitude by $\frac{1}{\Delta}$ and supported on a set of volume Δ^{d+1}. Since the number of lines in this graph is $l = 2n-p$ and the number of independent loops is $L = l-n+1 = n-p+1$, the perturbative power counting for $G_{n,p}$ is $\left(\frac{1}{\Delta}\right)^l \Delta^{L(d+1)} = \Delta^{n(d-1)}\Delta^{-pd+d+1} = O(N^{-n})$. In other words each vertex has weight $1/N$.

As usual, in a vector model, we must perform the sum over sector assignments to the lines of $G_{n,p}$. The vector structure of the vertex implies that there is one such sum, containing N terms, for each particle loop. Each m-loop, containing m half vertices and one



sector sum, is $O(N^{1-m/2})$. Thus, the bubble is $O(1)$, while other loops are higher order in $1/N$. Our conclusion is that in the symmetry breaking regime the weights in powers of the small parameter $1/N$ are those of a vector model.

§III The Goldstone Boson Regime

Consider now the third, Goldstone boson, regime. We reexpress the model (II.2), using a Hubbard-Stratonovich transformation, which introduces two bosonic fields γ_1, γ_2 that are real valued in position space. The transformed action is

$$\begin{aligned}
\mathcal{A}(\psi,\bar\psi,\gamma_1,\gamma_2) &= -\int_{|ik_0-e(\mathbf{k})|<a\Delta} đk\ \bar\psi(k)(ik_0-e(\mathbf{k}))\psi(k) - \sum_{j=1}^{2}\tfrac{1}{2}\int_{|q|<a\Delta} đq\ \ \bar\gamma_j(q)\gamma_j(q) \\
&\quad + g\int đt\, đq\ \big(\Gamma(q)\bar\psi_\uparrow(t+\tfrac{q}{2})\bar\psi_\downarrow(-t+\tfrac{q}{2}) + \bar\Gamma(q)\psi_\downarrow(-t+\tfrac{q}{2})\psi_\uparrow(t+\tfrac{q}{2})\big)
\end{aligned}$$

where $\lambda_0(\delta)=2g^2$ and $\Gamma(q)=\gamma_1(q)+i\gamma_2(q)$. The effective potential is gotten by integrating out the fermions while holding Γ fixed at some constant value. The result is a Mexican hat with minimum at $g|\Gamma(\xi)|=\Delta$, where Δ is the BCS gap. The phase of Γ must be determined by boundary conditions. Suppose that it is zero. Change variables to the components of Γ that are tangential and normal to $g|\Gamma(\xi)|=\Delta$ at $g\Gamma=\Delta$. Namely,

$$\Phi(\xi)=\gamma_\nu(\xi)+i\gamma_\tau(\xi)=\gamma_1(\xi)-\Delta/g+i\gamma_2(\xi)$$

The action becomes

$$\begin{aligned}
\mathcal{A}(\psi,\bar\psi,\Phi) &= -\int_{|ik_0-e(\mathbf{k})|<a\Delta} đk\ \ [\bar\psi(k)(ik_0-e(\mathbf{k}))\psi(k) - \Delta\bar\psi_\uparrow(k)\bar\psi_\downarrow(-k) - \Delta\psi_\downarrow(-k)\psi_\uparrow(k)] \\
&\quad -\tfrac{1}{2}\int_{|q|<a\Delta} đq\ \ [\bar\gamma_\nu(q)\gamma_\nu(q)+\bar\gamma_\tau(q)\gamma_\tau(q)] - \Delta\gamma_\nu(0)/g + \text{const} \qquad \text{(III.1)} \\
&\quad + g\int đt\, đq\ \big(\Phi(q)\bar\psi_\uparrow(t+\tfrac{q}{2})\bar\psi_\downarrow(-t+\tfrac{q}{2}) + \bar\Phi(q)\psi_\downarrow(-t+\tfrac{q}{2})\psi_\uparrow(t+\tfrac{q}{2})\big)
\end{aligned}$$

When the fermions are integrated out, the resulting model has generalized vertices of the same form as in the previous figure (but with loops of odd orders also since number symmetry has been broken). The solid lines in these generalized vertices no longer have arrows and are evaluated using the propagators

$$\begin{aligned}
\langle\psi_\uparrow(k)\bar\psi_\uparrow(p)\rangle = \langle\psi_\downarrow(k)\bar\psi_\downarrow(p)\rangle &= -\frac{ik_0+e(\mathbf{k})}{k_0^2+e(\mathbf{k})^2+\Delta^2}(2\pi)^{d+1}\delta(k-p)\Theta(k_0^2+e(\mathbf{k})^2\le a^2\Delta^2) \\
\langle\psi_\uparrow(k)\psi_\downarrow(-p)\rangle = \langle\bar\psi_\downarrow(-k)\bar\psi_\uparrow(p)\rangle &= -\frac{\Delta}{k_0^2+e(\mathbf{k})^2+\Delta^2}(2\pi)^{d+1}\delta(k-p)\Theta(k_0^2+e(\mathbf{k})^2\le a^2\Delta^2)
\end{aligned}$$

All other combinations are zero. In momentum space these lines are bounded by $\frac{\text{const}}{\Delta}$ and have support of volume const Δ^2.

Now form propagators for γ_ν, γ_τ by combining the constant and $O(q^2)$ terms from the Taylor expansion of the 2-loop with the second line of (III.1). The propagator for γ_ν is a constant. For γ_τ it is given by $\frac{\text{const}\,\Delta^2}{q_0^2+\text{const}\,\mathbf{q}^2}$. Thus, a many Fermion system in the Goldstone boson regime power counts like a local field theory with one massive and one massless boson and an overall ultraviolet cutoff. There are vertices $\gamma_\nu^m \gamma_\tau^n$ for all m and n. All of these vertices are superrenormalizable with the exception of $m = 0\,,\, n \le 6$ and $m = 1\,,\, n \le 3$ in two dimensions and $m = 0\,,\, n \le 4$ and $m = 1\,,\, n \le 2$ in three dimensions. For the other vertices approximate Ward identities [2c] must be used.

A similar analysis can be made for the Anderson model. Its four point function is

$$\langle G_+(x,y,E+\omega+i\varepsilon) G_-(x,y,E-\omega-i\varepsilon)\rangle = \int d\mu(V)\ \frac{1}{-\Delta-(E+\omega+i\varepsilon)+\lambda V}\ (x,y)\ \frac{1}{-\Delta-(E-\omega-i\varepsilon)+\lambda V}\ (x,y)$$

where $V(x)$, $x \in \mathbb{Z}^d$, is a family of Gaussian random variables. Again one has to split the hypersurface singularity of the free propagator $G_0(x,y,\varepsilon) = \int_{T^d} \frac{e^{-i\langle k,xy\rangle}}{k^2-E-i\varepsilon}$ into sectors.

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