Superconductivity in a Repulsive Model

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Introduction

Let

$$e(\mathbf{k}) \;=\; \tfrac{\mathbf{k}^2}{2m} \;-\; \mu$$

with the chemical potential $\mu > 0$ and let u be a short range rotation invariant pair potential. The corresponding Hamiltonian for the d-dimensional system of fermions with dispersion relation $e(\mathbf{k})$ and pair potential u is

$$\int đ\mathbf{k}\; e(\mathbf{k})\; a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} \;+\; \tfrac{1}{2}\int \prod_{i=1}^{4} đ\mathbf{k}_i \;(2\pi)^d\, \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\, \lambda\, u(\mathbf{k}_1-\mathbf{k}_3)\, a^\dagger_{\mathbf{k}_1,\sigma} a^\dagger_{\mathbf{k}_2,\tau} a_{\mathbf{k}_4,\tau} a_{\mathbf{k}_3,\sigma}$$

where, $đ\mathbf{k} = \frac{d^d\mathbf{k}}{(2\pi)^d}$ and repeated spin indices are summed over $\{\uparrow,\downarrow\}$. In three dimensions, Kohn and Luttinger [KL,L] made the surprising observation that for any *purely repulsive* short range rotation invariant pair potential u the second order Bethe-Salpeter equation for the Cooper channel always has a solution in some odd angular momentum sector. This result suggests that the Fermi sea is unstable and further, that in the true ground state number symmetry is broken and higher angular momentum Cooper pairs form. In [FKLT] we showed that reflection invariance of the dispersion relation $e(\mathbf{k})$ is essential for this instability.

We consider the two dimensional system with the special pair potential

$$\lambda\, u(\mathbf{k}_1-\mathbf{k}_3) \;=\; \lambda$$

where, $\lambda > 0$. That is, a purely repulsive delta function in position space. Let

$$K_N(s,t) \;=\; \sum_{n=1}^{N} \lambda^n\, K^{(n)}(s,t)$$

be the Bethe-Salpeter kernel (see, I.1) for the Cooper channel up to order N, and let $K_N(s',t')$ be its restriction to the Fermi surface. Here, for each $k = (k_0,\mathbf{k}) \in \mathbb{R}\times\left(\mathbb{R}^2\setminus\{0\}\right)$,

$$k' \;=\; \left(0\,,\; \tfrac{\mathbf{k}}{|\mathbf{k}|}\, k_F\right)$$

and $k_F \;=\; (2m\mu)^{\frac{1}{2}}$. In the Appendix we show that there is a solution of the associated Bethe-Salpeter equation when $K_N(s',t')$, regarded as the kernel of an integral operator on $L^2\left(\text{Fermi surface}\right)$, has a strictly negative eigenvalue. By rotation invariance,

$$K_N(s',t') \;=\; \sum_{\ell\ge 0} \Lambda_{N,\ell}(\lambda)\; \cos 2\ell\theta$$

and therefore, the Fourier coefficients $\Lambda_{N,\ell}(\lambda)$, $\ell \geq 0$, are the eigenvalues of $K_N(s',t')$. By convention, 2θ is the angle between s' and t' on the Fermi surface.

The kernel $K^{(n)}(s,t)$, $n \geq 1$, is a sum of two particle irreducible diagrams. The particle lines of a diagram represent the free propagator

$$C(k) \;=\; \frac{f\big(|\mathbf{k}|/\mathfrak{C}\big)}{ik_0 - e(\mathbf{k})}$$

The numerator $f\big(|\mathbf{k}|/\mathfrak{C}\big)$ cuts the ultraviolet end of the system off at $\mathfrak{C}$. In this expression, f is a nonnegative smooth function that is identically one between 0 and 1, decreases monotonically between 1 and 2 and is identically zero to the right of 2.

It is easy to compute the second order contribution $K^{(2)}(s,t)$ in the limit $\mathfrak{C} \to \infty$. One obtains (Lemma II.5), using the fact (Corollary II.2) that the the two dimensional polarization bubble is constant for $k_0 = 0$ and $|\mathbf{k}| \leq 2k_F$,

$$K^{(2)}(s',t') \;=\; \frac{m}{2\pi}$$

In particular, $K^{(2)}(s',t')$ is independent of the angle between any momenta s' and t' on the Fermi surface. It follows immediately that $\Lambda_{2,\ell}(\lambda) = 0$, $\ell \geq 1$. Thus, in contrast to three dimensions, one must compute at least the third order contribution $K^{(3)}(s,t)$ to determine whether there is an attractive angular momentum sector.

The third author [Si] evaluated $K^{(3)}(s,t)$ numerically and found that

$$\Lambda_{3,1}(\lambda) \;<\; \Lambda_{3,\ell}(\lambda) \;<\; 0$$

for all $2 \leq \ell \leq 100$. The numerical results led us to the rather surprising conclusion that it is possible to explicitly calculate all of the Fourier coefficients $\Lambda_{3,\ell}(\lambda)$, $\ell \geq 1$, of $K_3(s',t')$ in the limit $\mathfrak{C} \to \infty$. We obtain (see, Corollary II.8),

$$\Lambda_{3,\ell}(\lambda) \;=\; -\,\lambda^3\,\tfrac{m^2}{\pi^2}\left[\tfrac{1}{2\ell} - \left(\tfrac{1}{\ell+1} - \tfrac{1}{\ell+2} + \tfrac{1}{\ell+3} - \tfrac{1}{\ell+4} \pm \cdots\right)\right]$$

for all $\ell \geq 1$. In particular,

$$\Lambda_{3,1}(\lambda) \;=\; -\,\lambda^3\,\tfrac{m^2}{\pi^2}\left(\log 2 - \tfrac{1}{2}\right)$$

and for all $\ell \geq 2$,

$$\Lambda_{3,1}(\lambda) \;<\; \Lambda_{3,\ell}(\lambda) \;<\; 0$$

In other words, a two dimensional system of fermions with dispersion relation $e(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$ and a purely repulsive delta function pair potential generates the dominant attractive coupling $\Lambda_{3,1}(\lambda)$ in the third order Bethe-Salpeter approximation for the Cooper channel. This result suggests that the true ground state of our system is an $\ell = 1$ superconductor.

To verify that the ground state is indeed an $\ell = 1$ superconductor is not straight forward. In another paper we intend to rigorously implement, with the aid of a computer, a renormalization group analysis of our system around the Fermi sphere that shows that the $\ell = 1$ sector of the whole model is attractive and even dominates the $\ell = 0$ sector.

It is our pleasure to thank Franz Merkl for a number of useful suggestions.

§1 The Bethe-Salpeter Equation

Consider the many-fermion model with propagator

$$C(k) = \frac{f(|\mathbf{k}|/\mathfrak{C})}{ik_0 - e(\mathbf{k})}$$

and interaction

$$\frac{\lambda}{2} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \int \left(\prod_{i=1}^{4} \frac{d^{d+1}k_i}{(2\pi)^{d+1}} \right) (2\pi)^{d+1}\, \delta(k_1 + k_2 - k_3 - k_4) u(|\mathbf{k}_1 - \mathbf{k}_3|)\, \bar\psi_{\mathbf{k}_1,\sigma} \bar\psi_{\mathbf{k}_2,\tau} \psi_{\mathbf{k}_4,\tau} \psi_{\mathbf{k}_3,\sigma}$$

Here $e(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m} - \mu$ with μ being the chemical potential. We use the smooth function $0 \le f(x) \le 1$, which is identically one for $0 \le x \le 1$ and identically zero for $x \ge 2$, to impose an ultraviolet cutoff at $\mathfrak{C}$. We will ultimately set the two–body interaction $u(|\mathbf{k}_1 - \mathbf{k}_3|) = 1$, that is, a delta function interaction in position space. We could equally well treat a model having propagator $C(k) = \frac{1}{ik_0 - e(\mathbf{k})}$ and two–body interaction $u(|\mathbf{k}|) = f(|\mathbf{k}|/\mathfrak{C})$.

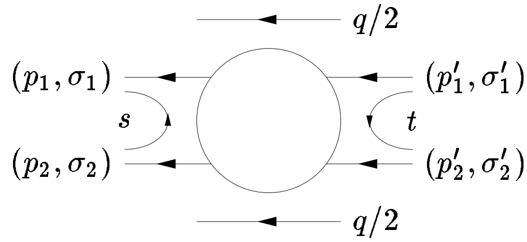
Let

$$\left\langle \psi_{\sigma p} \bar\psi_{\sigma' p'} \right\rangle_\beta = \beta \delta_{p_0,p_0'} (2\pi)^d \delta(\mathbf{p} - \mathbf{p}') \delta_{\sigma,\sigma'} G(p)$$

resp.

$$\begin{aligned} &\left\langle \psi_{\sigma_1 p_1} \psi_{\sigma_2 p_2} \bar\psi_{\sigma_2' p_2'} \bar\psi_{\sigma_1' p_1'} \right\rangle_\beta \\ &\qquad = \beta \delta_{p_{10}+p_{20}-p_{10}'-p_{20}'} (2\pi)^d \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2') S_{\sigma_1\sigma_2\sigma_1'\sigma_2'} \left(\tfrac{p_1 - p_2}{2}, p_1 + p_2, \tfrac{p_1' - p_2'}{2} \right) \end{aligned}$$

be the one– resp. two–particle Schwinger functions at temperature $T = \frac{1}{k\beta}$. Diagrammatically, $G(p)$ is the sum of all Feynman diagrams (with appropriate signs and combinatorial factors) having one incoming and one outgoing particle line, each with momentum p. The energy-momentum conserving delta function is not included in the value of the diagram. Similarly, S is the sum of all Feynman diagrams having two incoming particle lines with momenta and spins p_1', σ_1' and p_2', σ_2' and two outgoing particle lines with momenta and spins p_1, σ_1 and p_2, σ_2. The interpretation of the arguments s, q, t of $S_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s, q, t)$ is as follows: $q = p_1 + p_2 = p_1' + p_2'$ is the transfer momentum, $s = \frac{p_1 - p_2}{2}$ is the relative momentum of the outgoing particles and $t = \frac{p_1' - p_2'}{2}$ is the relative momentum of the incoming particles. The diagrams need not be connected, though every connected component must have at least one external line.



The two-particle Schwinger function is related to the vertex $\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t)$ by

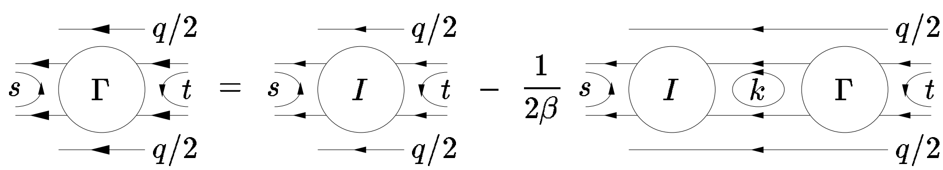
$$\begin{aligned} S_{\sigma_1\sigma_2\sigma_1'\sigma_2'}&(s,q,t) \\ &= \beta G\big(s+\tfrac{q}{2}\big)G\big(-s+\tfrac{q}{2}\big)\left[\delta_{s_0,t_0}(2\pi)^d\delta(\mathbf{s}-\mathbf{t})\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} - \delta_{s_0,-t_0}(2\pi)^d\delta(\mathbf{s}+\mathbf{t})\delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}\right] \\ &\quad - G\big(s+\tfrac{q}{2}\big)G\big(-s+\tfrac{q}{2}\big)\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t) \end{aligned}$$

Diagrammatically, Γ is the negative of the sum of all connected Feynman diagrams having two incoming and two outgoing particle lines. All four external lines are amputated by the interacting propagator. Γ is normalized so that in first order

$$\Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t) = \lambda u(\mathbf{s}-\mathbf{t})\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} - \lambda u(\mathbf{s}+\mathbf{t})\delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}$$

The Bethe-Salpeter kernel $I_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t)$ is the sum of all diagrams from Γ that are two particle irreducible in the channel from (p_1',p_2') to (p_1,p_2). We have

$$\begin{aligned} \Gamma_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t) &= I_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t) \\ &\quad - \tfrac{1}{2\beta}\sum_{k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\sum_{\sigma_1''\sigma_2''}\int \tfrac{d^d\mathbf{k}}{(2\pi)^d} I_{\sigma_1\sigma_2\sigma_1''\sigma_2''}(s,q,k)G\big(k+\tfrac{q}{2}\big)G\big(-k+\tfrac{q}{2}\big)\Gamma_{\sigma_1''\sigma_2''\sigma_1'\sigma_2'}(k,q,t) \end{aligned}$$



If, for some q, there exists a nontrivial solution ψ of the **Bethe-Salpeter equation**

$$\psi_{\sigma_1\sigma_2}(s) = -\tfrac{1}{2\beta}\sum_{k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\sum_{\sigma_1''\sigma_2''}\int \tfrac{d^d\mathbf{k}}{(2\pi)^d} I_{\sigma_1\sigma_2\sigma_1''\sigma_2''}(s,q,k)G\big(k+\tfrac{q}{2}\big)G\big(-k+\tfrac{q}{2}\big)\psi_{\sigma_1''\sigma_2''}(k)$$

there will be a corresponding pole in Γ. Kohn and Luttinger use the onset of such a singularity to signal the formation of Cooper pairs and consequently the breaking of number symmetry.

At the critical temperature the binding energy for Cooper pairs with momentum $\mathbf{q} = 0$ is $q_0 = 0$. This corresponds to a nontrivial solution of the Bethe-Salpeter equation for $q = 0$ and $\beta = \beta_c$.

$$\psi_{\sigma_1\sigma_2}(s) = -\tfrac{1}{2\beta_c} \sum_{k_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \sum_{\sigma_1''\sigma_2''} \int \tfrac{d^d\mathbf{k}}{(2\pi)^d} I_{\sigma_1\sigma_2\sigma_1''\sigma_2''}(s,0,k) G(k) G(-k) \psi_{\sigma_1''\sigma_2''}(k)$$

Because the interaction is spin independent, I is of the form

$$I_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t) = I_1(s,q,t)\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} + I_2(s,q,t)\delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}$$

By construction

$$I_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,0,t) = -I_{\sigma_2\sigma_1\sigma_1'\sigma_2'}(-s,0,t) = -I_{\sigma_1\sigma_2\sigma_2'\sigma_1'}(s,0,-t)$$

Put $K(s,t) = I_1(s,0,t)$. From the second equation it follows that $I_2(s,0,t) = -K(s,-t)$ so that

$$I_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,0,t) = K(s,t)\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} - K(s,-t)\delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}$$

One checks that the four spaces of functions

$$\{\ \psi_{\sigma_1\sigma_2}(s) = \delta_{\sigma_1\tau}\delta_{\sigma_2\tau}\chi(s)\ \big|\ \chi(s) = -\chi(-s)\ \}, \qquad \tau \in \{\uparrow,\downarrow\}$$
$$\{\ \psi_{\sigma_1\sigma_2}(s) = \tfrac{1}{2}\left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right)\chi(s)\ \big|\ \chi(s) = -\chi(-s)\ \}$$
$$\{\ \psi_{\sigma_1\sigma_2}(s) = \tfrac{1}{2}\left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right)\chi(s)\ \big|\ \chi(s) = \chi(-s)\ \}$$

are invariant under the integral operator with kernel I, and that the restriction of the Bethe-Salpeter equation to each of these subspaces is

$$\chi(s) = -\tfrac{1}{\beta_c} \sum_{t_0 \in \frac{\pi}{\beta_c}(2\mathbb{Z}+1)} \int \tfrac{d^d\mathbf{t}}{(2\pi)^d} K(s,t)|G(t)|^2\chi(t) \tag{I.1}$$

In (I.1), $K(s,t)$ is the sum of all diagrams in $I_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,0,t)$ that have a fermion string joining σ_1 to σ_1'. If $\chi(s)$ is a nontrivial solution of this equation then at least one of the two functions $(\delta_{\sigma_1\uparrow} + \delta_{\sigma_2\uparrow})\big(\chi(s) - \chi(-s)\big)$ and $(\delta_{\sigma_1\uparrow} - \delta_{\sigma_2\uparrow})\big(\chi(s) + \chi(-s)\big)$ is a nontrivial solution of the Bethe-Salpeter equation.

Observe that $I_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(s,q,t) = I_{\sigma_2\sigma_1\sigma_1'\sigma_2'}(Rs,Rq,Rt)$ for all $R \in SO(d)$. Therefore

$$K(s,t) = K(Rs,Rt) \qquad \text{for all } R \in SO(d)$$

Hence the integral operator with kernel K commutes with the action of $SO(d)$ on the space of χ's and it suffices to look for solutions of (I.1) in each angular momentum subspace. Write

$$K(s,t) = \sum_{\ell=0}^{\infty} K_\ell(s_0, |\mathbf{s}|, t_0, |\mathbf{t}|; \beta) \cdot \pi_\ell\left(\tfrac{\mathbf{s}}{|\mathbf{s}|}, \tfrac{\mathbf{t}}{|\mathbf{t}|}\right)$$

where for each $\ell \geq 0$, π_ℓ is the orthogonal projection from $L^2(S^{d-1})$ onto the subspace of all sphereical harmonics of degree ℓ. For emphasis, we have made explicit the dependence of K_ℓ on the inverse temperature β. In the angular momentum sector ℓ, equation (I.1) is

$$\chi(s_0, |\mathbf{s}|) = -\tfrac{1}{\beta_c} \sum_{t_0 \in \frac{\pi}{\beta_c}(2\mathbb{Z}+1)} \int \tfrac{d^d\mathbf{t}}{(2\pi)^d} K_\ell(s_0, |\mathbf{s}|, t_0, |\mathbf{t}|; \beta) |G(t)|^2 \chi(t_0, |\mathbf{t}|) \tag{I.2}$$

The important feature of this equation is the fact that in the integral $|G(t)|^2 \approx \frac{f(\mathbf{t}/\mathfrak{C})^2}{t_0^2 + e(\mathbf{t})^2}$ has a non-integrable singularity at $k_0 = 0$, $|\mathbf{k}| = k_F = \sqrt{2m\mu}$. Hence it is reasonable to expect that (I.2) should be, up to higher order corrections, equivalent to

$$1 = -\frac{1}{\beta_c} \sum_{t_0 \in \frac{\pi}{\beta_c}(2\mathbb{Z}+1)} \int \frac{d^d\mathbf{t}}{(2\pi)^d} \Lambda_\ell \frac{f(|\mathbf{t}|/\mathfrak{C})^2}{t_0^2 + e(\mathbf{t})^2} \tag{I.3}$$

Here $\Lambda_\ell = K_\ell(0, k_F, 0, k_F; \infty)$. A more precise statement of this nature is given in Proposition A.1 of the Appendix. Observe that

$$\frac{1}{\beta_c} \sum_{t_0 \in \frac{\pi}{\beta_c}(2\mathbb{Z}+1)} \int \frac{d^d\mathbf{t}}{(2\pi)^d} \frac{f(|\mathbf{t}|/\mathfrak{C})^2}{t_0^2 + e(\mathbf{t})^2} = \frac{m}{2\pi} \ln \beta_c + O(1) \tag{I.4}$$

Therefore, whenever Λ_ℓ is small and negative, equation (I.3) has a solution with

$$\beta_c \approx e^{2\pi/(m|\Lambda_\ell|)}$$

If the original two-body interaction is attractive then Λ_0 is negative even in first order perturbation theory. Kohn and Luttinger [KL,L] observe that, for $d = 3$, even for a repulsive two-body interaction, Λ_ℓ is negative in second order perturbation theory for some sufficiently large ℓ. We show

Theorem. *Let* $d = 2$ *and let the two-body interaction* $u(|\mathbf{k}|) = 1$ *be a delta function in position space. Then, the perturbation expansion of* Λ_1 *is*

$$\Lambda_1 = -\alpha\lambda^3 + O(\lambda)^4$$

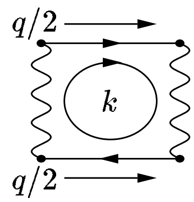
with $\alpha > 0$ *for all sufficiently large* $\mathfrak{C}$.

The Theorem is an immediate consequence of Lemma II.5 and Corollary II.8. It strengthens the results of [BCK] who show that $\Lambda_\ell < 0$ for some sufficiently large ℓ.

§II Evaluation of Diagrams

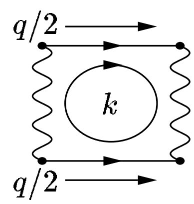
For the rest of this paper we restrict to $d = 2$. In this section we use explicit formulae for the values of the polarization bubble

$$W(q_0, |\mathbf{q}|) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{1}{i(k_0 + q_0/2) - e(\mathbf{k} + \mathbf{q}/2)} \frac{1}{i(k_0 - q_0/2) - e(\mathbf{k} - \mathbf{q}/2)}$$

$=$ 

and the particle-particle bubble

$$R(q_0, |\mathbf{q}|; \mathfrak{C}) = \int_{|\mathbf{k}| \le \mathfrak{C}} \frac{d^3k}{(2\pi)^3} \frac{1}{i(k_0 + q_0/2) - e(\mathbf{k} + \mathbf{q}/2)} \frac{1}{i(-k_0 + q_0/2) - e(-\mathbf{k} + \mathbf{q}/2)}$$

$=$ 

with a δ–function interaction to evaluate all second and third order graphs. Note that in the definition of W, the fermion propagators have no ultraviolet cutoff, while in the definition of R, they have a sharp ultraviolet cutoff at $|\mathbf{k}| = \mathfrak{C}$.

For $q_0, r \in \mathbb{C},\ r \neq 0$ let

$$a(q_0, r) \;=\; \frac{r}{2k_F} - \frac{im}{k_F r} q_0$$

If $a(q_0, r) \notin [-1, 1]$ the quadratic equation

$$z^2 - 2a(q_0, r)z + 1 = 0$$

has two different roots whose product is one, but which are not complex conjugates of each other. Denote $\alpha(q_0, r)$ the root with absolute value bigger than one. Then, by definition

$$\alpha(q_0, r) - 2a(q_0, r) + \frac{1}{\alpha(q_0, r)} \;=\; 0$$

and

$$z - 2a(q_0, r) + \frac{1}{z} \;=\; \Big(z - \alpha(q_0, r)\Big)\Big(1 - \frac{1}{z\alpha(q_0, r)}\Big)$$

Proposition II.1 *If* $q_0, r \neq 0$ *or* $r \geq 2k_F$ *then*

a)

$$W(q_0,r) = -\frac{m}{2\pi} + \frac{m}{4\pi r}\left(\alpha(q_0,r) - \frac{1}{\alpha(q_0,r)} + \alpha(-q_0,r) - \frac{1}{\alpha(-q_0,r)}\right)$$
$$= -\frac{m}{2\pi} + \frac{m}{2\pi r}\mathrm{Re}\left(\alpha(q_0,r) - \frac{1}{\alpha(q_0,r)}\right)$$

b)

$$R(q_0,r;\mathfrak{C}) = \frac{m}{2\pi}\ln\left(1 - \frac{2k_F}{r\alpha(q_0,r)}\right) + \frac{m}{4\pi}\ln\left(1 + \frac{4\mathfrak{C}^2}{r^2 - 4imq_0 - 4k_F^2}\right)$$

where ln *is the standard branch of the logarithm with cut along the negative real axis.*

Since $W(q_0,r)$ is continuous in q_0 we can evaluate $W(0,r)$ as a limit $W(q_0,r)$. For $q_0 = 0$ and $0 < r \leq 2k_F$, $a(r,0)$ is a real number between 0 and 1. Consequently the roots of $z^2 - 2a(0,r)z + 1 = 0$ are complex conjugates and $\lim_{q_0\to 0}\mathrm{Re}\left(\alpha(q_0,r) - \frac{1}{\alpha(q_0,r)}\right) = 0$. This implies

Corollary II.2 *For* $0 < r \leq 2k_F$

$$W(0,r) = -\frac{m}{2\pi}$$

Remark II.3 Proposition II.1b implies that $R(q_0,r;\mathfrak{C})$ diverges as the ultraviolet cutoff $\mathfrak{C} \to \infty$. On the other hand $W(q_0,r)$ is well defined even in the absence of an ultraviolet cutoff. Moreover, if $f : \mathbb{R}^2 \to [0,1]$ is a smooth function which is one on the unit disk, then for all $\mathfrak{C}$ sufficiently large and $|\mathbf{q}| < \mathfrak{C} - k_F$

$$\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{f\big((\mathbf{k}+\mathbf{q}/2)/\mathfrak{C}\big)}{i(k_0+q_0/2) - e(\mathbf{k}+\mathbf{q}/2)} \frac{f\big((\mathbf{k}-\mathbf{q}/2)/\mathfrak{C}\big)}{i(k_0-q_0/2) - e(\mathbf{k}-\mathbf{q}/2)} = W(q_0,|\mathbf{q}|)$$

To see this, evaluate the k_0 integral by residues to get

$$\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{f\big((\mathbf{k}+\mathbf{q}/2)/\mathfrak{C}\big)}{i(k_0+q_0/2) - e(\mathbf{k}+\mathbf{q}/2)} \frac{f\big((\mathbf{k}-\mathbf{q}/2)/\mathfrak{C}\big)}{i(k_0-q_0/2) - e(\mathbf{k}-\mathbf{q}/2)}$$
$$= \int_{e(\mathbf{k}+\mathbf{q}/2)e(\mathbf{k}-\mathbf{q}/2)<0} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\mathrm{sign}\, e(\mathbf{k}+\mathbf{q}/2)}{iq_0 + e(\mathbf{k}-\mathbf{q}/2) - e(\mathbf{k}+\mathbf{q}/2)} f\big((\mathbf{k}+\mathbf{q}/2)/\mathfrak{C}\big) f\big((\mathbf{k}-\mathbf{q}/2)/\mathfrak{C}\big)$$
$$= \int_{e(\mathbf{k}+\mathbf{q}/2)e(\mathbf{k}-\mathbf{q}/2)<0} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\mathrm{sign}\, e(\mathbf{k}+\mathbf{q}/2)}{iq_0 + e(\mathbf{k}-\mathbf{q}/2) - e(\mathbf{k}+\mathbf{q}/2)}$$
$$= \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{1}{i(k_0+q_0/2) - e(\mathbf{k}+\mathbf{q}/2)} \frac{1}{i(k_0-q_0/2) - e(\mathbf{k}-\mathbf{q}/2)} = W(q_0,|\mathbf{q}|)$$

In the transition from line two to line three, we used the fact that for $e(\mathbf{k}\pm\mathbf{q}/2)<0$, we have $|\mathbf{k}\pm\mathbf{q}/2|\le k_F\le\mathfrak{C}$ and $|\mathbf{k}\mp\mathbf{q}/2|\le|\mathbf{k}\pm\mathbf{q}/2|+|\mathbf{q}|\le\mathfrak{C}$ provided $|\mathbf{q}|\le\mathfrak{C}-k_F$. Consequently, on the domain of integration $f\big((\mathbf{k}+\mathbf{q}/2)/\mathfrak{C}\big)=f\big((\mathbf{k}-\mathbf{q}/2)/\mathfrak{C}\big)=1$.

The formulae for R,W stated in Proposition II.1 are well-known [FHN,St]. We include here a possibly nonstandard evaluation, by residues, of W. The evaluation uses

Lemma II.4 *Fix* $a\in\mathbb{C}$ *with* $\operatorname{Im}a\ne 0$ *or* $|\operatorname{Re}a|\ge 1$ *. Let* α_+ *be the root of*

$$z^2-2az+1\;=\;0$$

determined by $|\alpha_+|\ge 1$ *. Then,*

$$\int_{x^2+y^2\le 1}\frac{1}{x-a}\,dx\wedge dy\;=\;\pi\big(\alpha_+-\alpha_+^{-1}-2a\big)$$

Proof: By Stokes' theorem,

$$\int_{x^2+y^2\le 1}\frac{1}{x-a}\,dx\wedge dy\;=\;-\int_{x^2+y^2\le 1}d\,\frac{y\,dx}{x-a}\;=\;-\int_{x^2+y^2=1}\frac{y\,dx}{x-a}$$

Substituting,

$$\begin{aligned}x&=\tfrac12(z+z^{-1})\\ y&=\tfrac{1}{2i}(z-z^{-1})\\ dx&=\tfrac12(1-z^{-2})\,dz\end{aligned}$$

our integral becomes

$$\int_{x^2+y^2\le 1}\frac{1}{x-a}\,dx\wedge dy\;=\;-\tfrac{1}{2i}\int_{|z|=1}\frac{(z-z^{-1})(1-z^{-2})}{z+z^{-1}-2a}\,dz\;=\;-\tfrac{1}{2i}\int_{|z|=1}\frac{(z^2-1)^2}{z^2(z^2-2az+1)}\,dz$$

or

$$\int_{x^2+y^2\le 1}\frac{1}{x-a}\,dx\wedge dy\;=\;-\tfrac{1}{2i}\int_{|z|=1}\frac{(z^2-1)^2}{z^2(z-\alpha_+)(z-\alpha_+^{-1})}\,dz$$

We have

$$\begin{aligned}\operatorname{Res}\frac{(z^2-1)^2}{z^2(z^2-2az+1)}\Big|_{z=0}&=\frac{d}{dz}\frac{(z^2-1)^2}{(z^2-2az+1)}\Big|_{z=0}\;=\;2a\\ \operatorname{Res}\frac{(z^2-1)^2}{z^2(z-\alpha_+)(z-\alpha_+^{-1})}\Big|_{z=\alpha_+^{-1}}&=\frac{(\alpha_+^{-2}-1)^2}{\alpha_+^{-2}(\alpha_+^{-1}-\alpha_+)}\;=\;\frac{(\alpha_+^{-1}-\alpha_+)^2}{(\alpha_+^{-1}-\alpha_+)}\;=\;\alpha_+^{-1}-\alpha_+\end{aligned}$$

Applying the residue theorem,

$$\int_{x^2+y^2\le 1}\frac{1}{x-a}\,dx\wedge dy\;=\;\pi\big(\alpha_+-\alpha_+^{-1}-2a\big)$$

■

Proof of Proposition II.1a): By rotation invariance, we may assume, without loss of generality that $\mathbf{q} = (r, 0)$. We first do the integral over k_0, closing the contour in the upper half plane.

$$\begin{aligned}
W(q_0, |\mathbf{q}|) &= \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{1}{i(k_0+q_0/2)-e(\mathbf{k}+\mathbf{q}/2)} \frac{1}{i(k_0-q_0/2)-e(\mathbf{k}-\mathbf{q}/2)} \\
&= \frac{1}{(2\pi)^2} \int_{e(\mathbf{k}+\mathbf{q}/2)<0} \frac{dk_1 \wedge dk_2}{-iq_0+e(\mathbf{k}+\mathbf{q}/2)-e(\mathbf{k}-\mathbf{q}/2)} + \frac{1}{(2\pi)^2} \int_{e(\mathbf{k}-\mathbf{q}/2)<0} \frac{dk_1 \wedge dk_2}{iq_0+e(\mathbf{k}-\mathbf{q}/2)-e(\mathbf{k}+\mathbf{q}/2)} \\
&= \frac{1}{(2\pi)^2} \int_{e(\mathbf{k})<0} \frac{dk_1 \wedge dk_2}{e(\mathbf{k})-e(\mathbf{k}-\mathbf{q})-iq_0} + \frac{1}{(2\pi)^2} \int_{e(\mathbf{k})<0} \frac{dk_1 \wedge dk_2}{e(\mathbf{k})-e(\mathbf{k}+\mathbf{q})+iq_0} \\
&= \frac{m}{2\pi^2} \int_{\mathbf{k}^2<k_F^2} \frac{dk_1 \wedge dk_2}{2\mathbf{k}\cdot\mathbf{q}-\mathbf{q}^2-i2mq_0} - \frac{m}{2\pi^2} \int_{\mathbf{k}^2<k_F^2} \frac{dk_1 \wedge dk_2}{2\mathbf{k}\cdot\mathbf{q}+\mathbf{q}^2-i2mq_0} \\
&= \frac{m}{2\pi^2} \int_{\mathbf{k}_1^2+\mathbf{k}_2^2<k_F^2} \frac{dk_1 \wedge dk_2}{2r\mathbf{k}_1-r^2-i2mq_0} - \frac{m}{2\pi^2} \int_{\mathbf{k}_1^2+\mathbf{k}_2^2<k_F^2} \frac{dk_1 \wedge dk_2}{2r\mathbf{k}_1+r^2-i2mq_0} \\
&= \frac{mk_F^2}{2\pi^2} \int_{x^2+y^2<1} \frac{dx \wedge dy}{2rk_F x-r^2-i2mq_0} - \frac{mk_F^2}{2\pi^2} \int_{x^2+y^2<1} \frac{dx \wedge dy}{2rk_F x+r^2-i2mq_0} \\
&= \frac{mk_F}{4\pi^2} \frac{1}{r} \int_{x^2+y^2<1} \frac{dx \wedge dy}{x-a(-q_0,r)} - \frac{mk_F}{4\pi^2} \frac{1}{r} \int_{x^2+y^2<1} \frac{dx \wedge dy}{x+a(q_0,r)}
\end{aligned}$$

By Lemma II.4, since the roots of $z^2-2az+1 = 0$ are the negatives of the roots of $z^2+2az+1 = 0$, we have

$$\begin{aligned}
W(q_0, |\mathbf{q}|) &= \frac{mk_F}{4\pi} \frac{1}{r} \left(\alpha(-q_0, r) - \frac{1}{\alpha(-q_0,r)} - 2a(-q_0, r) + \alpha(q_0, r) - \frac{1}{\alpha(q_0,r)} - 2a(q_0, r) \right) \\
&= \frac{mk_F}{4\pi} \frac{1}{r} \left(-\frac{2r}{k_F} + \alpha(q_0, r) - \frac{1}{\alpha(q_0,r)} + \alpha(-q_0, r) - \frac{1}{\alpha(-q_0,r)} \right)
\end{aligned}$$

This proves the first line of Proposition II.1a). The second follows from the observation that $a(q_0, r)$ and $a(-q_0, r)$ are complex conjugates, which implies that $\alpha(q_0, r)$ and $\alpha(-q_0, r)$ are complex conjugates. ■

Proof of Proposition II.1b): [FHN] show that

$$R(q_0, r; \mathfrak{C}) = \frac{m}{4\pi} \ln \frac{4(x_0 - iq_0)(x_c - iq_0)}{[-iq_0 + \sqrt{x_+ - iq_0}\,\sqrt{x_- - iq_0}]^2}$$

where

$$\begin{aligned}
x_0 &= \frac{(r - 2k_F)(r + 2k_F)}{4m} \\
x_c &= \frac{\mathfrak{C}^2 - k_F^2 + r^2/4}{m} \\
x_\pm &= \frac{r(r \pm 2k_F)}{2m}
\end{aligned}$$

and the square root is chosen to have positive real part. Writing

$$a(q_0, r) = a \qquad \alpha(q_0, r) = \alpha$$

we have

$$\begin{aligned}
x_\pm - iq_0 &= \frac{rk_F}{m}\left(\frac{r}{2k_F} \pm 1 - \frac{im}{k_F r} q_0\right) &&= \frac{rk_F}{m}(a \pm 1) \\
x_0 - iq_0 &= \frac{r^2}{4m} - iq_0 - \frac{k_F^2}{m} &&= \frac{rk_F}{m}\left(a - \frac{r}{4k_F}\right) - \frac{k_F^2}{m} \\
x_c - iq_0 &= x_0 - iq_0 + \frac{\mathfrak{C}^2}{m} &&= \frac{rk_F}{m}\left(a - \frac{r}{4k_F}\right) + \frac{\mathfrak{C}^2 - k_F^2}{m}
\end{aligned}$$

so that the denominator

$$\begin{aligned}
-iq_0 + \sqrt{x_+ - iq_0}\,\sqrt{x_- - iq_0} &= -iq_0 + \frac{rk_F}{m}\sqrt{a^2 - 1} \\
&= \frac{rk_F}{m}\left(a - \frac{r}{2k_F} + \sqrt{a^2 - 1}\right) \\
&= \frac{rk_F}{m}\left(\alpha - \frac{r}{2k_F}\right)
\end{aligned}$$

In the computation above we used

$$\sqrt{a-1}\sqrt{a+1} = \sqrt{a^2 - 1}$$

and

$$a + \sqrt{a^2 - 1} = \alpha$$

These are justified by the fact that a lies in the right half plane and consequently

$$\operatorname{sign}\operatorname{Im} a = \operatorname{sign}\operatorname{Im} a^2 = \operatorname{sign}\operatorname{Im}\sqrt{a^2 - 1} = \operatorname{sign}\operatorname{Im}\alpha = -\operatorname{sign} q_0$$

Observe that

$$\begin{aligned}
\left(\alpha - \frac{r}{2k_F}\right)\left(1 - \frac{2k_F}{r\alpha}\right) &= \alpha + \frac{1}{\alpha} - \frac{2k_F}{r} - \frac{r}{2k_F} \\
&= 2a - \frac{2k_F}{r} - \frac{r}{2k_F} \\
&= \frac{r}{2k_F} - \frac{2k_F}{r} - \frac{2im}{k_F r} q_0 \\
&= \frac{2m}{k_F r}\left(\frac{r^2 - 4k_F^2}{4m} - iq_0\right) \\
&= \frac{2m}{k_F r}(x_0 - iq_0)
\end{aligned}$$

Therefore

$$
\begin{aligned}
\frac{4(x_0 - iq_0)(x_c - iq_0)}{[-iq_0 + \sqrt{x_+ - iq_0}\,\sqrt{x_- - iq_0}]^2} &= \left(1 - \frac{2k_F}{r\alpha}\right)^2 \frac{x_c - iq_0}{x_0 - iq_0} \\
&= \left(1 - \frac{2k_F}{r\alpha}\right)^2 \left(1 + \frac{\mathfrak{C}^2}{m(x_0 - iq_0)}\right) \\
&= \left(1 - \frac{2k_F}{r\alpha}\right)^2 \left(1 + \frac{\mathfrak{C}^2}{rk_F\left(a - \frac{r}{4k_F}\right) - k_F^2}\right) \\
&= \left(1 - \frac{2k_F}{r\alpha}\right)^2 \left(1 + \frac{4\mathfrak{C}^2}{4rk_F a - r^2 - 4k_F^2}\right) \\
&= \left(1 - \frac{2k_F}{r\alpha}\right)^2 \left(1 + \frac{4\mathfrak{C}^2}{r^2 - 4k_F^2 - 4imq_0}\right)
\end{aligned}
$$

and

$$
R(q_0, r; \mathfrak{C}) \;=\; \frac{m}{2\pi} \ln\left(1 - \frac{2k_F}{r\alpha}\right) \;+\; \frac{m}{4\pi} \ln\left(1 + \frac{4\mathfrak{C}^2}{r^2 - 4k_F^2 - 4imq_0}\right)
$$

since

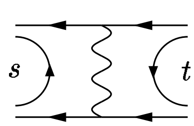
$$
\mathrm{Re}\,\left(\tfrac{r}{2k_F} - \tfrac{1}{\alpha}\right) = \mathrm{Re}\,\left(\mathrm{Re}\,a - \tfrac{1}{\alpha}\right) > 0
$$

and

$$
\mathrm{sign}\,\mathrm{Im}\,\left(\tfrac{r}{2k_F} - \tfrac{1}{\alpha}\right) = -\mathrm{sign}\,q_0 = -\mathrm{sign}\,\mathrm{Im}\,\left(1 + \tfrac{4\mathfrak{C}^2}{r^2 - 4k_F^2 - 4imq_0}\right)
$$

■

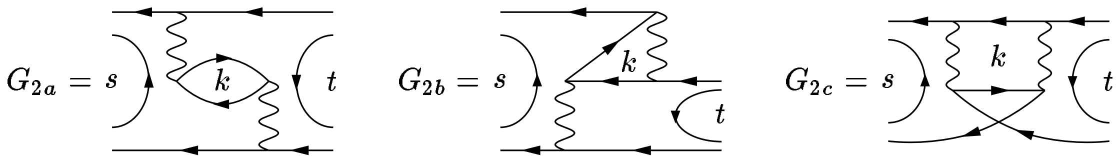
We now start the evaluation of Λ_ℓ in low order perturbation theory. The first order computation is trivial, because the only diagram which contributes is



$$= \lambda\, u(\mathbf{s} - \mathbf{t})$$

This is indepedent of $\mathbf{s}$ and $\mathbf{t}$ because $\hat{u} \equiv 1$. Hence this diagram contributes to Λ_0, but no Λ_ℓ with $\ell > 0$.

For a general interaction, the connected, amputated, second order diagrams that are two particle irreducible in the particle-particle to hole-hole channel are



and a fourth diagram, G_{2d}, which is the reflection of G_{2b} about a horizontal axis. Clearly $G_{2d}(s,t) = G_{2b}(-s,-t)$.

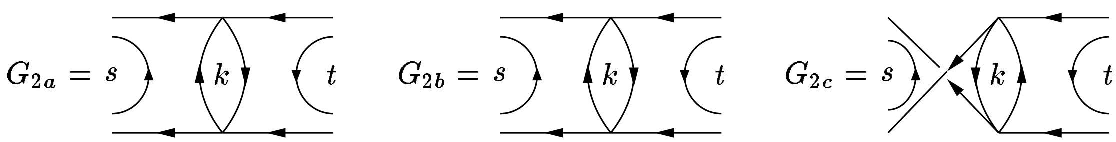
The second order contribution to $K(s,t)$ is

$$K^{(2)}(s,t) = 2G_{2a}(s,t) - G_{2b}(s,t) - G_{2b}(-s,-t) - G_{2c}(s,t)$$

It is computed as follows. In general, the sign of the combinatorial factor of a diagram contributing to $K^{(n)}$ is $(-1)(-1)^n(-1)^{\#\text{ Fermion loops}}$, with the first factor coming from the fact that K is the negative of the sum of all diagrams .... In this case $n = 2$ and the number of Fermion loops is 1 for G_{2a} and zero for G_{2b} and G_{2c}. For this model, as for quantum electrodynamics (see, for example, [IZ chapter 6-1-2]) there are no symmetry factors because diagrams have no symmetries that leave the external legs fixed. There is, however one spin sum for each Fermion loop. Hence any n^{th} order diagram comes with the combinatorial factor

$$(-1)^{n+1}(-2)^{\#\text{ Fermion loops}} \tag{II.1}$$

When there is a delta function interaction, the three diagrams collapse to



so that

$$G_{2a}(s,t) = G_{2b}(s,t) = G_{2c}(-s,t)$$

Lemma II.5 *Let $K^{(2)}(s,t)$ be the second order contribution to $K(s,t)$. Then for $s_0 = t_0 = 0$ and $|\mathbf{s}|, |\mathbf{t}| \le k_F$*

$$K^{(2)}(s,t) = \frac{m}{2\pi}$$

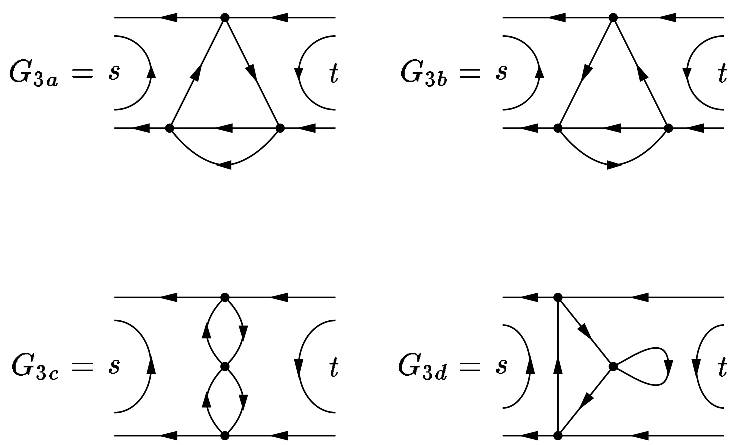
is independent of $\mathbf{s}$ *and* $\mathbf{t}$*.*

Proof: The value of $G_{2a}(s,t)$ is

$$G_{2a}(s,t) = \int_{\mathrm{IR}^3} \frac{d^3k}{(2\pi)^3} \frac{f\left((\mathbf{k}+\frac{\mathbf{t}-\mathbf{s}}{2})/\mathfrak{C}\right)}{i(k_0+\frac{t_0-s_0}{2}) - e(\mathbf{k}+\frac{\mathbf{t}-\mathbf{s}}{2})} \frac{f\left((\mathbf{k}-\frac{\mathbf{t}-\mathbf{s}}{2})/\mathfrak{C}\right)}{i(k_0-\frac{t_0-s_0}{2}) - e(\mathbf{k}-\frac{\mathbf{t}-\mathbf{s}}{2})}$$

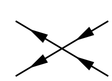
By Remark (II.3), this is $W(t_0 - s_0, |\mathbf{t}-\mathbf{s}|)$ for all $|\mathbf{t}-\mathbf{s}| < \mathfrak{C} - k_F$. Corollary II.2 yields the claim. ■

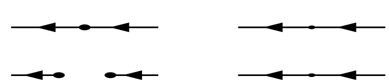
For a delta function interaction, the possible third order diagrams are

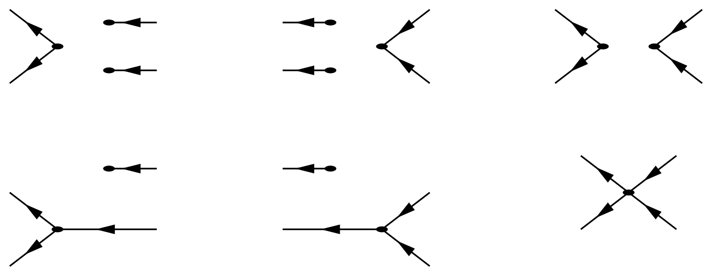


Lemma II.6 *The third order contribution to* $K(s,t)$ *is*

$$K^{(3)}(s,t) = G_{3a}(s,-t) + G_{3a}(-s,t) - G_{3b}(s,t) - G_{3b}(-s,-t) + G_{3c}(s,t) + G_{3c}(s,-t)$$

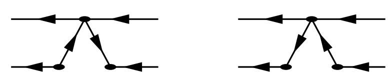
Proof: We need a list of all four–legged graphs having three  vertices that are connected, amputated and two particle irreducible in the two–particle to two–hole channel. We first draw only the external legs and the vertices to which they belong. The following basic configurations are possible:



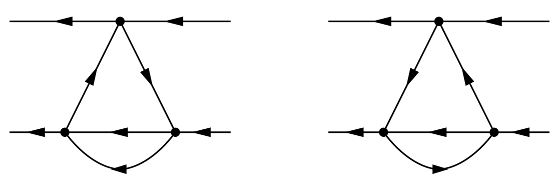


The configurations in the second line cannot be completed to two–particle irreducible diagrams because they each contain a vertex with two particle external lines and two hole internal lines or conversely. The first two configurations in the second line cannot be completed to amputated diagrams because they each contain a vertex having only one internal line. The last configuration always yields a disconnected diagram.

Let us now consider the first configuration. If both internal legs of the upper vertex are connected to the same vertex the resulting diagram is either disconnected or not amputated. If they are connected to different vertices one gets the following two configurations

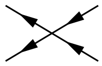
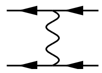
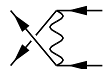


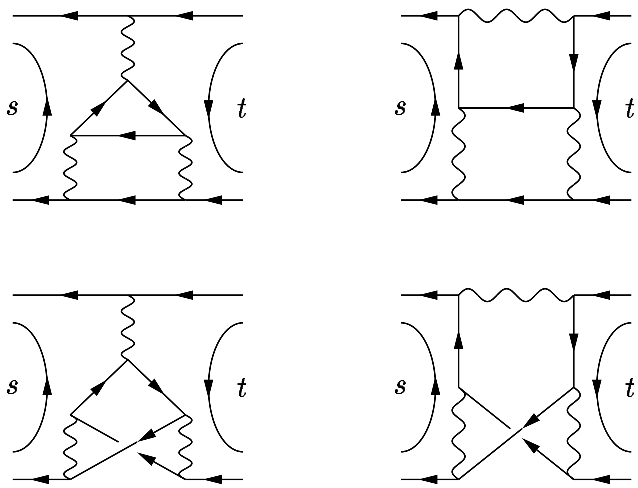
The remaining internal lines connect to form either two tadpoles, which do not yield amputated graphs, or a bubble.



For each of the two graphs, the momenta $\pm s$, $\pm t$ may be assigned to the outgoing, resp. ingoing, legs in four possible ways yielding $G_{3a}(\pm s, \pm t)$ and $G_{3b}(\pm s, \pm t)$.

Similarly the second basic configuration above yields $G_{3c}(\pm s, \pm t)$ and $G_{3d}(\pm s, \pm t)$.

Observe that $G_{3d}(s,t) \equiv 0$ because the tadpole renormalizes to zero exactly. Therefore $K^{(3)}(s,t)$ is a linear combination of $G_{3a}(\pm s, \pm t)$, $G_{3b}(\pm s, \pm t)$ and $G_{3c}(\pm s, \pm t)$. By symmetry the coefficient of $G_{3a}(s,t)$ in the linear combination will agree with that of $G_{3a}(-s,-t)$ and so on. We now determine the coefficients of $G_{3a}(s,t)$ and $G_{3a}(s,-t)$. We first find all graphs including interaction squiggles that collapse down to $G_{3a}(s,t)$ and then we determine the combinatorial coefficient for each. To find the graphs with squiggles, just replace each vertex  by  or  . This gives



By (II.1) the combinatorial factors for these diagrams are -2,1,1,1 respectively. The first three combine to give $G_{3a}(s,t)$ a net factor of zero, while the last gives $G_{3a}(-s,t)$ a coefficient of 1. The remaining terms are computed similarly. ■

Theorem II.7 *Let* $s_0 = t_0 = 0$ *and* $|\mathbf{s}|, |\mathbf{t}| = k_F$. *Denote by* $2\theta \in [0,\pi]$ *the angle between* $\mathbf{s}$ *and* $\mathbf{t}$. *Then, for sufficiently large ultraviolet cutoff* $\mathfrak{C}$,

a) there are constants $B_{\mathfrak{C}}, B'_{\mathfrak{C}}$ *depending on* $\mathfrak{C}$ *such that*

$$\begin{aligned} G_{3a}(s,t) &= B'_{\mathfrak{C}} + \frac{m^2}{4\pi^2}\theta\frac{\cos\theta}{\sin\theta} + O(1/\mathfrak{C}) \\ &= B_{\mathfrak{C}} - \tfrac{1}{2}\sum_{\ell=1}^{\infty}(-1)^\ell a_\ell \cos(2\ell\theta) + O(1/\mathfrak{C}) \end{aligned}$$

where

$$a_\ell = \frac{m^2}{\pi^2}\left[\frac{1}{2\ell} - \left(\frac{1}{\ell+1} - \frac{1}{\ell+2} + \frac{1}{\ell+3} - \frac{1}{\ell+4} \pm \cdots\right)\right]$$

b)

$$G_{3b}(s,t) = \frac{m^2}{(2\pi)^2}(1 - \ln 2) + O(1/\mathfrak{C})$$

c)

$$G_{3c}(s,t) = \frac{m^2}{(2\pi)^2}$$

Corollary II.8 *Let* $s_0 = t_0 = 0$ *and* $|\mathbf{s}|, |\mathbf{t}| = k_F$*. Denote by* $2\theta \in [0,\pi]$ *the angle between* $\mathbf{s}$ *and* $\mathbf{t}$ *and let*

$$K^{(3)}(s,t) = \frac{\alpha_0}{2} + \sum_{\ell=1}^{\infty} \alpha_\ell \cos(2\ell\theta)$$

be the Fourier series expansion of $K^{(3)}$*. Then,*

$$\alpha_\ell = -\frac{m^2}{\pi^2}\left[\frac{1}{2\ell} - \left(\frac{1}{\ell+1} - \frac{1}{\ell+2} + \frac{1}{\ell+3} - \frac{1}{\ell+4} \pm \cdots\right)\right] + O(1/\mathfrak{C})$$

with the error $O(1/\mathfrak{C})$ *uniform in* ℓ*. In particular,*

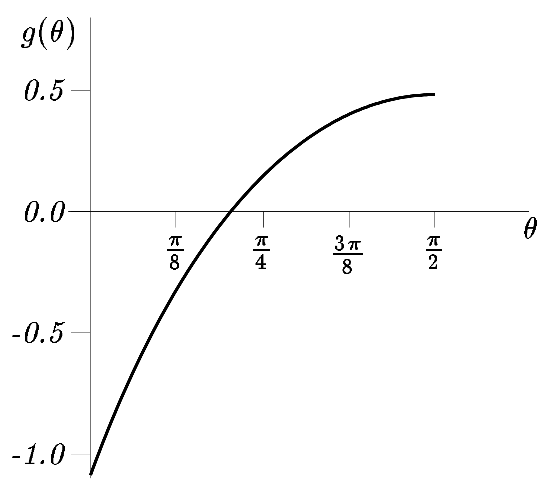
$$\alpha_1 = -\tfrac{m^2}{\pi^2}\left(\ln 2 - \tfrac{1}{2}\right) + O(1/\mathfrak{C})$$

and for all $\ell \geq 2$ *and sufficiently large* $\mathfrak{C}$

$$\alpha_1 < \alpha_\ell < 0$$

There is a constant $B''_{\mathfrak{C}}$ *such that*

$$K^{(3)}(s,t) = \frac{m^2}{4\pi^2}(\pi - 2\theta)\frac{\sin\theta}{\cos\theta} + B''_{\mathfrak{C}} + O(1/\mathfrak{C})$$



Proof: By Lemma II.6 and Theorem II.7 there is a constant $B'_{\mathfrak{C}}$ such that

$$\left\| K^{(3)}(s,t) + \sum_{\ell=1}^{\infty} a_\ell \cos(2\ell\theta) - B'_{\mathfrak{C}} \right\|_\infty = O(1/\mathfrak{C})$$

This proves the first two claims. The last one follows from

$$a_1 > a_2 > \cdots > 0$$

■

The rest of this section deals with the proof of Theorem II.7. By scaling we may, without loss of generality, choose $k_F = 1$. Part c) of Theorem II.7 is an immediate consequence of $G_{3c}(s,t) = W(s_0 - t_0, \mathbf{s} - \mathbf{t})^2$. See the proof of Lemma II.5. The proofs of the other parts depend on the following lemmata.

For $\mathbf{s}, \mathbf{t} \in \mathrm{I\!R}^2$ with $|\mathbf{s}| = |\mathbf{t}| = k_F = 1$ define

$$A(q_0, r, \theta) = \frac{1}{2\pi} \int_{|\mathbf{q}|=r} \frac{1}{iq_0 - e(\mathbf{q}-\mathbf{s})} \frac{1}{iq_0 - e(\mathbf{q}-\mathbf{t})}$$

where 2θ is the angle between $\mathbf{s}$ and $\mathbf{t}$. Here the integral is over the circle of radius r in $\mathrm{I\!R}^2$ with respect to the standard length element.

Lemma II.9 *Let $s_0 = t_0 = 0$ and $|\mathbf{s}|, |\mathbf{t}| = k_F$. Denote by 2θ the angle between $\mathbf{s}$ and $\mathbf{t}$. Then there is a constant* const_f, *depending only on the cutoff function f, such that*

$$\left\| G_{3a}(s,t) - \mathrm{const}_f - \frac{1}{(2\pi)^2} \int_0^\infty dr \int_{-\infty}^{\infty} dq_0 \; A(q_0, r, \theta) R(q_0, r; \mathfrak{C}) \right\|_\infty = O(1/\mathfrak{C})$$
$$\left\| G_{3b}(s,t) - \frac{1}{(2\pi)^2} \int_0^\infty dr \int_{-\infty}^{\infty} dq_0 \; A(q_0, r, \theta) W(q_0, r) \right\|_\infty = O(1/\mathfrak{C})$$

Proof: We must compute the difference between

$$G_{3a}(\mathbf{s},\mathbf{t}) = \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{f(\mathbf{k}/\mathfrak{C})}{-ik_0 - e(\mathbf{k})} \frac{f\big((\mathbf{k}+\mathbf{q})/\mathfrak{C}\big)}{i(k_0+q_0) - e(\mathbf{k}+\mathbf{q})} \frac{f\big((\mathbf{q}-\mathbf{s})/\mathfrak{C}\big)}{iq_0 - e(\mathbf{q}-\mathbf{s})} \frac{f\big((\mathbf{q}-\mathbf{t})/\mathfrak{C}\big)}{iq_0 - e(\mathbf{q}-\mathbf{t})}$$

and

$$\frac{1}{(2\pi)^2}\int_0^\infty dr\int_{-\infty}^{\infty} dq_0\ A(q_0,r,\theta)R(q_0,r;\mathfrak{C})$$
$$=\int\frac{dr}{(2\pi)^3}\int dq_0\int\limits_{|\mathbf{q}|=r}\ \int\limits_{|\mathbf{k}|\le\mathfrak{C}}\frac{d^3k}{(2\pi)^3}\frac{1}{i(-k_0+q_0/2)-e(-\mathbf{k}+\mathbf{q}/2)}\frac{1}{i(k_0+q_0/2)-e(\mathbf{k}+\mathbf{q}/2)}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{s})}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{t})}$$
$$=\int_{\mathrm{IR}^2}\frac{d^2\mathbf{q}}{(2\pi)^2}\int_{|\mathbf{k}+\mathbf{q}/2|\le\mathfrak{C}}\frac{d^2\mathbf{k}}{(2\pi)^2}\int\frac{dq_0}{2\pi}\int\frac{dk_0}{2\pi}\frac{1}{-ik_0-e(\mathbf{k})}\frac{1}{i(k_0+q_0)-e(\mathbf{k}+\mathbf{q})}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{s})}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{t})}$$

For fixed $\mathfrak{C}<\infty$ the integrand of G_{3a} is absolutely integrable and we may choose any order of integration. Exchanging the angular $\mathbf{q}$ and q_0 integrals as well as the q_0 and $\mathbf{k}$ integrals is justified by absolute integrability with respect to the exchanged variables for almost all values of the non-exchanged variables. Evaluating the k_0 and q_0 integrals by residues

$$\int\frac{dq_0}{2\pi}\int\frac{dk_0}{2\pi}\frac{1}{-ik_0-e(\mathbf{k})}\frac{1}{i(k_0+q_0)-e(\mathbf{k}+\mathbf{q})}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{s})}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{t})}$$
$$=[\chi_{++--}+\chi_{--++}]\frac{1}{e(\mathbf{k})+e(\mathbf{k}+\mathbf{q})-e(\mathbf{q}-\mathbf{s})}\frac{1}{e(\mathbf{k})+e(\mathbf{k}+\mathbf{q})-e(\mathbf{q}-\mathbf{t})}$$
$$+[\chi_{++-+}+\chi_{--+-}]\frac{1}{e(\mathbf{k})+e(\mathbf{k}+\mathbf{q})-e(\mathbf{q}-\mathbf{s})}\frac{1}{e(\mathbf{q}-\mathbf{s})-e(\mathbf{q}-\mathbf{t})}$$
$$+[\chi_{+++-}+\chi_{---+}]\frac{1}{e(\mathbf{k})+e(\mathbf{k}+\mathbf{q})-e(\mathbf{q}-\mathbf{t})}\frac{1}{e(\mathbf{q}-\mathbf{t})-e(\mathbf{q}-\mathbf{s})}$$

where $\chi_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ is the characteristic function of the set of $\mathbf{k}$ and $\mathbf{q}$ for which the signs of $e(\mathbf{k}), e(\mathbf{k}+\mathbf{q}), e(\mathbf{q}-\mathbf{s})$ and $e(\mathbf{q}-\mathbf{t})$ are $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4$ respectively.

First consider the χ_{--++} term. For this term we must have $|\mathbf{k}|,|\mathbf{k}+\mathbf{q}|\le k_F$. Consequently $|\mathbf{k}|,|\mathbf{q}|\le 2k_F$ so that

$$\chi_{--++}\left[f(\mathbf{k}/\mathfrak{C})f\big((\mathbf{k}+\mathbf{q})/\mathfrak{C}\big)f\big((\mathbf{q}-\mathbf{s})/\mathfrak{C}\big)f\big((\mathbf{q}-\mathbf{t})/\mathfrak{C}\big)-\chi_{|\mathbf{k}+\mathbf{q}/2|\le\mathfrak{C}}\right]=0$$

For all other terms $|\mathbf{q}|\le 2k_F$ so that $f(\mathbf{q}-\mathbf{s})f\big((\mathbf{q}-\mathbf{t})/\mathfrak{C}\big)=1$. In order for $f(\mathbf{k}/\mathfrak{C})f\big((\mathbf{k}+\mathbf{q})/\mathfrak{C}\big)-\chi_{|\mathbf{k}+\mathbf{q}/2|}$ to be nonzero, $|\mathbf{k}|$ must be bounded above and below by const $\mathfrak{C}$. Consequently all $\chi_{--\epsilon_3\epsilon_4}$ terms are zero. On this domain, the integral over $\mathbf{k}$ and $\mathbf{q}$ of the χ_{++--} term is bounded above by const $/\mathfrak{C}^2$.

So far, we have shown that

$$G_{3a}(s,t)-\frac{1}{(2\pi)^2}\int_0^\infty dr\int_{-\infty}^{\infty}dq_0\ A(q_0,r,\theta)R(q_0,r;\mathfrak{C})$$
$$=\int_{\mathrm{IR}^2}\frac{d^2\mathbf{q}}{(2\pi)^2}\int_{\mathrm{IR}^2}\frac{d^2\mathbf{k}}{(2\pi)^2}\left[\chi_{++-+}\frac{f(\mathbf{k}/\mathfrak{C})f((\mathbf{k}+\mathbf{q})/\mathfrak{C})-\chi_{|\mathbf{k}+\mathbf{q}/2|<\mathfrak{C}}}{[e(\mathbf{k})+e(\mathbf{k}+\mathbf{q})-e(\mathbf{q}-\mathbf{s})][e(\mathbf{q}-\mathbf{s})-e(\mathbf{q}-\mathbf{t})]}+\mathbf{s}\leftrightarrow\mathbf{t}\right]+O(1/\mathfrak{C})$$

The difference between then integral above and

$$\int_{\mathbb{R}^2} \tfrac{d^2\mathbf{q}}{(2\pi)^2} \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{k}}{(2\pi)^2} \left[\chi_{++-+} \tfrac{f(\mathbf{k}/\mathfrak{C})^2 - \chi_{|\mathbf{k}|<\mathfrak{C}}}{2e(\mathbf{k})[e(\mathbf{q}-\mathbf{s})-e(\mathbf{q}-\mathbf{t})]} + \mathbf{s} \leftrightarrow \mathbf{t}\right]$$

is bounded by $O(1/\mathfrak{C})$. Finally, note that, for $|\mathbf{q}| \le 2k_F$ and $\mathbf{k}$ in the support of the difference of cutoff functions $f(\mathbf{k}/\mathfrak{C})^2 - \chi_{|\mathbf{k}|<\mathfrak{C}}$, we have $|\mathbf{k}|, |\mathbf{k}+\mathbf{q}| > k_F$ so that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \tfrac{d^2\mathbf{q}}{(2\pi)^2} \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{k}}{(2\pi)^2} \left[\chi_{++-+} \tfrac{f(\mathbf{k}/\mathfrak{C})^2 - \chi_{|\mathbf{k}|<\mathfrak{C}}}{2e(\mathbf{k})[e(\mathbf{q}-\mathbf{s})-e(\mathbf{q}-\mathbf{t})]} + \mathbf{s} \leftrightarrow \mathbf{t}\right] \\
&= \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{k}}{(2\pi)^2} \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{q}}{(2\pi)^2} \left[\chi_{|\mathbf{q}-\mathbf{s}|<k_F}\chi_{|\mathbf{q}-\mathbf{t}|>k_F} - \chi_{|\mathbf{q}-\mathbf{t}|<k_F}\chi_{|\mathbf{q}-\mathbf{s}|>k_F}\right] \tfrac{f(\mathbf{k}/\mathfrak{C})^2 - \chi_{|\mathbf{k}|<\mathfrak{C}}}{2e(\mathbf{k})[e(\mathbf{q}-\mathbf{s})-e(\mathbf{q}-\mathbf{t})]} \\
&= \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{k}}{(2\pi)^2} \left(f(\mathbf{k}/\mathfrak{C})^2 - \chi_{|\mathbf{k}|<\mathfrak{C}}\right) \tfrac{1}{2e(\mathbf{k})} \int_{\mathbb{R}^3} \tfrac{d^3 q}{(2\pi)^3} \tfrac{1}{iq_0 - e(\mathbf{q}-\mathbf{s})} \tfrac{1}{iq_0 - e(\mathbf{q}-\mathbf{t})} \\
&= \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{k}}{(2\pi)^2} \left(f(\mathbf{k}/\mathfrak{C})^2 - \chi_{|\mathbf{k}|<\mathfrak{C}}\right) \tfrac{1}{2e(\mathbf{k})} W(0, |\mathbf{t}-\mathbf{s}|) \\
&= -\tfrac{m}{2\pi} \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{k}}{(2\pi)^2} \left(f(\mathbf{k}/\mathfrak{C})^2 - \chi_{|\mathbf{k}|<\mathfrak{C}}\right) \tfrac{1}{2e(\mathbf{k})} \\
&= -\tfrac{m^2}{2\pi} \int_{\mathbb{R}^2} \tfrac{d^2\mathbf{k}}{(2\pi)^2} \left(f(\mathbf{k})^2 - \chi_{|\mathbf{k}|<1}\right) \tfrac{1}{\mathbf{k}^2} + O(1/\mathfrak{C}^2)
\end{aligned}$$

In the second last equality we used Corollary II.2.

The proof of the bound involving G_{3b} is similar but easier. ■

Recall that, for $q_0, r \in \mathbb{C}$, $r \neq 0$

$$a(q_0, r) = \frac{r}{2} - \frac{im}{r} q_0$$

If $a(q_0, r) \notin [-1, 1]$ the quadratic equation

$$z^2 - 2a(q_0, r)z + 1 = 0$$

has two different roots whose product is one, but which are not complex conjugates of each other. The root with absolute value bigger than one has been denoted $\alpha(q_0, r)$.

Lemma II.10 *Let* $\mathbf{s}, \mathbf{t} \in \mathbb{R}^2$ *with* $|\mathbf{s}| = |\mathbf{t}| = k_F = 1$. *Denote by* 2θ *the angle between* $\mathbf{s}$

and $\mathbf{t}$. *Then for each* $r > 0$, $q_0 \neq 0$

$$\begin{aligned}
A(q_0, r, \theta) &= \frac{4m^2}{r} \frac{\alpha(r,q_0)+\alpha(r,q_0)^{-1}}{\alpha(r,q_0)-\alpha(r,q_0)^{-1}} \frac{1}{\alpha(r,q_0)^2+\alpha(r,q_0)^{-2}-2\cos(2\theta)} \\
&= \frac{2m^2}{r} \frac{1}{\alpha(r,q_0)-\alpha(r,q_0)^{-1}} \frac{a(r,q_0)}{a(r,q_0)^2-\cos^2\theta} \\
&= \frac{4m^2}{r} \frac{\alpha(r,q_0)+\alpha(r,q_0)^{-1}}{\alpha(r,q_0)-\alpha(r,q_0)^{-1}} \frac{1}{\sin 2\theta} \sum_{n\geq 1} \alpha(r,q_0)^{-2n} \sin(2n\theta)
\end{aligned}$$

Proof: Put $a = a(r,q_0)$, $\alpha = \alpha(r,q_0)$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way and write $q_0 = rz$ with $|z| = 1$. Without loss of generality we may assume that $\mathbf{s}$ and $\mathbf{t}$ are complex conjugate, so that

$$\mathbf{s} = e^{i\theta} \qquad \mathbf{t} = \overline{\mathbf{s}} = e^{-i\theta}$$

With this notation

$$\begin{aligned}
&\frac{1}{iq_0 - e(\mathbf{q}-\mathbf{s})} \frac{1}{iq_0 - e(\mathbf{q}-\mathbf{t})} \\
&\qquad = 4m^2 \frac{1}{2imq_0 - ((rz-\mathbf{s})(r\overline{z}-\overline{\mathbf{s}})-1)} \frac{1}{2imq_0 - ((rz-\overline{\mathbf{s}})(r\overline{z}-\mathbf{s})-1)} \\
&\qquad = \frac{4m^2}{r^2} \frac{1}{\overline{\mathbf{s}}z - 2a + \mathbf{s}\overline{z}} \frac{1}{\mathbf{s}z - 2a + \overline{\mathbf{s}z}} \\
&\qquad = \frac{4m^2}{r^2} \frac{1}{\overline{\mathbf{s}}z - 2a + \frac{1}{\overline{\mathbf{s}}z}} \frac{1}{\mathbf{s}z - 2a + \frac{1}{\mathbf{s}z}} \\
&\qquad = \frac{4m^2}{r^2} \frac{1}{\left(\overline{\mathbf{s}}z - \alpha\right)\left(1 - \frac{1}{\overline{\mathbf{s}}z\alpha}\right)} \frac{1}{\left(\mathbf{s}z - \alpha\right)\left(1 - \frac{1}{\mathbf{s}z\alpha}\right)} \\
&\qquad = \frac{4m^2}{r^2} \frac{1}{\left(z - \mathbf{s}\alpha\right)\left(z - \overline{\mathbf{s}}\alpha\right)\left(1 - \frac{1}{\mathbf{s}z\alpha}\right)\left(1 - \frac{1}{\overline{\mathbf{s}}z\alpha}\right)}
\end{aligned}$$

we apply the residue theorem in the region $\left\{ z \in \mathbb{C} \;\middle|\; |z| \geq 1 \right\}$ and get

$$\begin{aligned}
A(q_0, r, \theta) &= \frac{1}{2\pi} \int_{|\mathbf{q}|=r} \frac{1}{iq_0 - e(\mathbf{q}-\mathbf{s})} \frac{1}{iq_0 - e(\mathbf{q}-\mathbf{t})} \\
&= \frac{2m^2}{\pi r^2} \int_{|z|=1} \frac{r\,dz}{iz} \frac{1}{\left(z - \mathbf{s}\alpha\right)\left(z - \overline{\mathbf{s}}\alpha\right)\left(1 - \frac{1}{\mathbf{s}z\alpha}\right)\left(1 - \frac{1}{\overline{\mathbf{s}}z\alpha}\right)} \\
&= \frac{-4m^2}{r} \left[\frac{1}{\mathbf{s}\alpha\left(\mathbf{s}\alpha - \overline{\mathbf{s}}\alpha\right)\left(1 - \frac{1}{\mathbf{s}^2\alpha^2}\right)\left(1 - \frac{1}{\alpha^2}\right)} + \frac{1}{\overline{\mathbf{s}}\alpha\left(\overline{\mathbf{s}}\alpha - \mathbf{s}\alpha\right)\left(1 - \frac{1}{\alpha^2}\right)\left(1 - \frac{1}{\overline{\mathbf{s}}^2\alpha^2}\right)} \right]
\end{aligned}$$

$$= \frac{4m^2}{r} \frac{1}{\bar{\mathbf{s}} - \mathbf{s}} \frac{1}{\alpha - \alpha^{-1}} \left[\frac{1}{\mathbf{s}\alpha - \bar{\mathbf{s}}\alpha^{-1}} - \frac{1}{\bar{\mathbf{s}}\alpha - \mathbf{s}\alpha^{-1}} \right]$$
$$= \frac{4m^2}{r} \frac{\alpha + \alpha^{-1}}{\alpha - \alpha^{-1}} \frac{1}{\alpha^2 + \alpha^{-2} - (e^{i2\theta} + e^{-i2\theta})}$$

This proves the first formula. The second is an immediate consequence.

To prove the third, we use

$$\begin{aligned} \frac{4}{\pi} \int_0^{\pi/2} d\theta \, \frac{\sin(2\theta)\sin(2n\theta)}{\alpha^2 + \alpha^{-2} - 2\cos(2\theta)} &= -\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\theta \, \frac{\left(e^{2i\theta} - e^{-2i\theta}\right)\left(e^{2in\theta} - e^{-2in\theta}\right)}{\alpha^2 + \alpha^{-2} - \left(e^{2i\theta} + e^{-2i\theta}\right)} \\ &= \frac{1}{4\pi i} \int_{|z|=1} dz \, \frac{\left(z - \frac{1}{z}\right)\left(z^n - \frac{1}{z^n}\right)}{\left(z - \alpha^2\right)\left(z - \alpha^{-2}\right)} \\ &= \alpha^{-2n} \end{aligned}$$

■

Recall from Lemma II.9 that we wish to evaluate the integrals

$$\frac{1}{(2\pi)^2} \int_0^\infty dr \int_{-\infty}^\infty dq_0 \; A(q_0, r, \theta) R(q_0, r; \mathfrak{C})$$
$$\frac{1}{(2\pi)^2} \int_0^\infty dr \int_{-\infty}^\infty dq_0 \; A(q_0, r, \theta) W(q_0, r)$$

In Proposition II.1 and Lemma II.10 $A(q_0, r, \theta)$, $R(q_0, r; \mathfrak{C})$ and $W(q_0, r)$ have all been expressed as functions of $\alpha(q_0, r)$. We perform the change of variables

$$w = \alpha(q_0, r), \qquad \overline{w} = \alpha(-q_0, r)$$

Then

$$\begin{aligned} r &= \frac{1}{2}\left(w + \frac{1}{w} + \overline{w} + \frac{1}{\overline{w}}\right) = \frac{1}{2w}(w + \bar{w})\left(w + \frac{1}{\bar{w}}\right) \\ q_0 &= \frac{ir}{4m}\left(w + \frac{1}{w} - \overline{w} - \frac{1}{\overline{w}}\right) \end{aligned}$$

so that

$$\begin{aligned} dr &= \frac{1}{2w}\left(w - \frac{1}{w}\right) dw + \frac{1}{2\overline{w}}\left(\overline{w} - \frac{1}{\overline{w}}\right) d\overline{w} \\ dq_0 &= \frac{i}{4m}\left(w + \frac{1}{w} - \overline{w} - \frac{1}{\overline{w}}\right) dr + \frac{ir}{4m}\left(\frac{1}{w}\left(w - \frac{1}{w}\right) dw - \frac{1}{\overline{w}}\left(\overline{w} - \frac{1}{\overline{w}}\right) d\overline{w}\right) \\ dq_0 \wedge dr &= \frac{ir}{4m|w|^2}\left(w - \frac{1}{w}\right)\left(\overline{w} - \frac{1}{\overline{w}}\right) dw \wedge d\overline{w} \end{aligned}$$

Therefore, by Lemma II.10

$$A(q_0,r,\theta)\ dq_0\wedge dr = \frac{im}{|w|^2}\left(w-\frac{1}{w}\right)\left(\overline{w}-\frac{1}{\overline{w}}\right)\frac{w+\frac{1}{w}}{w-\frac{1}{w}}\ \frac{1}{w^2+w^{-2}-2\cos 2\theta}\ dw\wedge d\overline{w}$$
$$= im\ \frac{w^2+1}{w^4-2w^2\cos 2\theta+1}\left(1-\frac{1}{\overline{w}^2}\right)\ dw\wedge d\overline{w}$$

Similarly, by Proposition II.1

$$\begin{aligned} W(q_0,r) &= -\frac{m}{2\pi}+\frac{m}{4\pi r}\left(w-\frac{1}{w}+\bar w-\frac{1}{\bar w}\right)\\ &= -\frac{m}{2\pi}+\frac{m}{4\pi r}\frac{1}{w}(w+\bar w)\left(w-\frac{1}{\bar w}\right)\\ &= -\frac{m}{2\pi}+\frac{m}{2\pi}\frac{w-1/\bar w}{w+1/\bar w} = -\frac{m}{2\pi}+\frac{m}{2\pi}\frac{|w|^2-1}{|w|^2+1} \end{aligned}$$

Under the change of variables, the domain of integration $\left\{\ (q_0,r)\in \mathbb{R}^2\ \middle|\ r>0\ \right\}$ becomes

$$\begin{aligned} \Omega\ &=\ \left\{\ w\in\mathbb{C}\ \middle|\ |w|\ge 1,\ w+\overline{w}+w^{-1}+\overline{w}^{-1}>0\ \right\}\\ &=\ \left\{\ w\in\mathbb{C}\ \middle|\ |w|\ge 1,\ \mathrm{Re}\,w>0\ \right\} \end{aligned}$$

Neither $A(q_0,r,\theta)\,dq_0\wedge dr$ nor $A(q_0,r,\theta)W(q_0,r)\,dq_0\wedge dr$ are absolutely integrable on this domain. The same holds for $A(q_0,r,\theta)R(q_0,r,\mathfrak{C})\,dq_0\wedge dr$. Therefore we add and subtract counterterms that are later integrated separately.

Lemma II.11

a)

$$\int\left\{A(q_0,r,\theta)\left(W(q_0,r)+\tfrac{m}{2\pi}\right)-\frac{m^3}{2\pi r\left(a(q_0,r)^2-1\right)}\right\}dq_0\,dr = m^2(1-\ln 2)$$

b)

$$\int\left[A(q_0,r,\theta)-A(q_0,r,\pi/4)\right]\left(R(q_0,r,\mathfrak{C})+\tfrac{m}{2\pi}\ln r\right)dq_0\,dr = O(1/\mathfrak{C})$$

Proof: a) Since

$$\begin{aligned} \frac{dq_0\wedge dr}{r\left(a(q_0,r)^2-1\right)}\ &=\ \frac{i}{m|w|^2}\left(w-\frac{1}{w}\right)\left(\overline{w}-\frac{1}{\overline{w}}\right)\frac{1}{\left(w+\frac{1}{w}\right)^2-4}\ dw\wedge d\overline{w}\\ &=\ \frac{i}{m}\frac{1}{w^2-1}\left(1-\frac{1}{\overline{w}^2}\right)dw\wedge d\overline{w} \end{aligned}$$

the differential form

$$\omega = \left\{ A(q_0,r,\theta)\left(W(q_0,r)+\tfrac{m}{2\pi}\right) - \tfrac{m^3}{2\pi r\left(a(q_0,r)^2-1\right)} \right\} dq_0\wedge dr$$
$$= \frac{im^2}{2\pi}\left(\frac{|w|^2-1}{|w|^2+1}\,\frac{(w^2+1)}{w^4-2w^2\cos 2\theta+1} - \frac{1}{w^2-1}\right)\left(1-\frac{1}{\bar{w}^2}\right) dw\wedge d\overline{w}$$

is absolutely integrable on Ω.

Since ω is invariant under the involution $w\to -w$

$$\int \left\{ A(q_0,r,\theta)\left(W(q_0,r)+\tfrac{m}{2\pi}\right) - \tfrac{m^3}{2\pi r\left(a(q_0,r)^2-1\right)} \right\} dq_0\, dr = \int_\Omega \omega = \frac{1}{2}\int_{\left\{\, w\in\mathbb{C} \;\middle|\; |w|\geq 1 \,\right\}} \omega$$

Writing $w = \rho\zeta$ with $|\zeta|=1$ we get

$$\int_\Omega \omega = -\frac{im^2}{2\pi}\int_1^\infty \frac{d\rho}{\rho}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\,(\rho^2-\zeta^2)\left(\frac{\rho^2-1}{\rho^2+1}\,\frac{\rho^2\zeta^2+1}{\rho^4\zeta^4-2\rho^2\zeta^2\cos 2\theta+1} - \frac{1}{\rho^2\zeta^2-1}\right)$$

Observe that for $\rho>1$ all poles of the holomorphic differential form

$$\frac{d\zeta}{\zeta}(\rho^2-\zeta^2)\left(\frac{\rho^2-1}{\rho^2+1}\,\frac{\rho^2\zeta^2+1}{\rho^4\zeta^4-2\rho^2\zeta^2\cos 2\theta+1} - \frac{1}{\rho^2\zeta^2-1}\right)$$

lie inside the disk of radius $\rho^{-1}<1$, with the one exception of the pole at infinity. The residue of this form at infinity is

$$\frac{1}{\rho^2}\left(\frac{\rho^2-1}{\rho^2+1}-1\right) = \frac{-2}{\rho^2(\rho^2+1)}$$

Therefore

$$\int_\Omega \omega = 2m^2\int_1^\infty \frac{d\rho}{\rho^3}\,\frac{1}{\rho^2+1} = m^2(1-\ln 2)$$

b) Put

$$F(q_0,r) = \frac{m}{2\pi}\ln\left(r-\frac{2}{\alpha(q_0,r)}\right) + \frac{m}{4\pi}\ln\left(\frac{4\mathfrak{C}^2}{r^2-4imq_0-4}\right)$$

Then

$$R(q_0,r,\mathfrak{C})+\frac{m}{2\pi}\ln r = F(q_0,r)+\frac{m}{4\pi}\ln\left(1+\frac{r^2-4imq_0-4}{4\mathfrak{C}^2}\right)$$

Using

$$\left|\ln\left(1+\frac{r^2-4imq_0-4}{4\mathfrak{C}^2}\right)\right| \leq \text{const}\,\ln\left(1+\text{const}\,\frac{|w|^2}{\mathfrak{C}^2}\right) \qquad \text{for } |w|\geq 1$$
$$\left|[A(q_0,r,\theta)-A(q_0,r,\pi/4)]dq_0\wedge dr\right| \leq \text{const}\,\frac{1}{|w|^4}|dw\wedge d\bar{w}| \qquad \text{for } |w|\gg 1$$

and the fact that $[A(q_0,r,\theta)-A(q_0,r,\pi/4)]dq_0\wedge dr$ is L^1 on Ω, uniformly in θ, one sees that

$$\left\| \int_\Omega [A(q_0,r,\theta)-A(q_0,r,\pi/4)] \ln\left(1+\frac{r^2-4imq_0-4}{4\mathfrak{C}^2}\right) dq_0\wedge dr \right\|_\infty \le \text{const}\ \frac{\ln \mathfrak{C}}{\mathfrak{C}^2}$$

Observe that $F(q_0,r)$ is continuous outside $\{\ (r,0)\ \big|\ 0\le r\le 2\ \}$. Under the coordinate change $(q_0,r)\to(w,\bar w)$ this set is mapped to $\{\ w\in\mathbb{C}\ \big|\ |w|=1, \operatorname{Re} w\ge 0\ \}$, so it lies at the boundary of Ω. Since $F(q_0,r)$ has only logarithmic singularities, $[A(q_0,r,\theta)-A(q_0,r,\pi/4)]F(q_0,r)dq_0\wedge dr$ is absolutely integrable on Ω and its integral over Ω is continuous in θ. We show that this integral vanishes for $0<\theta<\pi/2$ by writing

$$\begin{aligned}\int_\Omega &[A(q_0,r,\theta)-A(q_0,r,\pi/4)]F(q_0,r)dq_0\wedge dr\\ &= -\int_\Omega \frac{2im(w^2+1)w^2\cos 2\theta}{(w^4+1)(w^4-2w^2\cos 2\theta+1)}\left(\bar w+\frac{1}{\bar w}\right)\frac{dF}{d\bar w}dw\wedge d\bar w\\ &\quad -\int_\Omega d\left[\frac{2im(w^2+1)w^2\cos 2\theta}{(w^4+1)(w^4-2w^2\cos 2\theta+1)}\left(\bar w+\frac{1}{\bar w}\right)F(q_0,r)dw\right]\end{aligned}$$

Observe that $\frac{dF}{d\bar w}$ is a rational function of w and $\bar w$. As in part a) it can be shown that the first term is zero.

Put, for $s\ge 1$

$$\begin{aligned} I_s &= \{\ w\in\mathbb{C}\ \big|\ \operatorname{Re} w=0,\ |\operatorname{Im} w|\ge s\ \}\\ U_s &= \{\ w\in\mathbb{C}\ \big|\ |w|=s,\ \operatorname{Re} w\ge 0\ \}\end{aligned}$$

By Stoke's Theorem, the second term is

$$-\lim_{s\searrow 1}\int_{I_s\cup U_s}\frac{2im(w^2+1)w^2\cos 2\theta}{(w^4+1)(w^4-2w^2\cos 2\theta+1)}\left(\bar w+\frac{1}{\bar w}\right)F(q_0,r)\ dw$$

We claim that $F(q_0,r)$ is an even function on I_1. Indeed, for $t\ge 1$ one has

$$\lim_{\substack{w\to\pm it\\ w\in\Omega}}\frac{m}{2\pi}\ln\Big(r-\frac{2}{\alpha(q_0,r)}\Big) = \frac{m}{2\pi}\ln\left(-\frac{2}{\pm it}\right) = -\frac{m}{2\pi}\ln|t|/2\ \pm\ \frac{m}{4}i$$

and as $w\in\Omega$ goes to $\pm it$ one has $\pm q_0\nearrow 0$, so that

$$\lim_{\substack{w\to\pm it\\ w\in\Omega}}\frac{m}{4\pi}\ln\left(\frac{4\mathfrak{C}^2}{r^2-4imq_0-4}\right) = \frac{m}{4\pi}\ln\left(-\mathfrak{C}^2\mp i0\right) = \frac{m}{4\pi}\ln\left(\mathfrak{C}^2\right)\mp\frac{m}{4}i$$

Consequently, the integrand is odd on I_1 and

$$\lim_{s\searrow 1}\int_{I_s}\frac{2im(w^2+1)w^2\cos 2\theta}{(w^4+1)(w^4-2w^2\cos 2\theta+1)}\left(\bar{w}+\frac{1}{\bar{w}}\right)F(q_0,r)\;dw=0$$

For w_0 with $|w_0|=1,\ \mathrm{Re}\,w_0>0,\ \pm\mathrm{Im}\,w_0>0$

$$\begin{aligned}\lim_{\substack{w\to w_0\\ w\in\Omega}}F(q_0,r)&=\frac{m}{2\pi}\ln\left(\frac{1}{2}\left(w_0-\frac{3}{w_0}+\overline{w_0}+\frac{1}{\overline{w_0}}\right)\right)+\frac{m}{4\pi}\ln\left(\frac{\mathfrak{C}^2}{(\mathrm{Re}\,w_0)^2-1}\mp i0\right)\\&=\left(\frac{m}{2\pi}\ln(2|\mathrm{Im}\,w_0|)\ \pm\frac{m}{4}i\right)+\left(\frac{m}{4\pi}\ln\left(\frac{\mathfrak{C}^2}{(\mathrm{Im}\,w_0)^2}\right)\mp\frac{m}{4}i\right)\\&=\frac{m}{2\pi}\ln 2\mathfrak{C}\end{aligned}$$

Since $(w^4+1)(w^4-2w^2\cos 2\theta+1)$ has simple zeroes

$$\begin{aligned}&\lim_{s\searrow 1}\int_{U_s}\frac{2im(w^2+1)w^2\cos 2\theta}{(w^4+1)(w^4-2w^2\cos 2\theta+1)}\left(\bar{w}+\frac{1}{\bar{w}}\right)F(q_0,r)\;dw\\&=\lim_{s\searrow 1}\int_{U_s}\frac{2im(w^2+1)w^2\cos 2\theta}{(w^4+1)(w^4-2w^2\cos 2\theta+1)}\left(w+\frac{1}{w}\right)\frac{m}{2\pi}\ln 2\mathfrak{C}\;dw\\&=\lim_{s\searrow 1}\frac{1}{2}\int_{|w|=s}\frac{2im(w^2+1)w^2\cos 2\theta}{(w^4+1)(w^4-2w^2\cos 2\theta+1)}\left(w+\frac{1}{w}\right)\frac{m}{2\pi}\ln 2\mathfrak{C}\;dw\\&=0\end{aligned}$$

by the residue theorem. ■

The harder counterterms of Lemma II.11 are treated in parts a) and b) of

Lemma II.12 *For* $0<2\theta\le\pi$

a)

$$\tfrac{1}{(2\pi)^2}\int_0^\infty dr\int_{-\infty}^\infty dq_0\;A(q_0,r,\theta)=-\tfrac{m}{2\pi}$$

b) There is a constant A_0 *such that*

$$\begin{aligned}\tfrac{1}{(2\pi)^2}\int_0^\infty dr\int_{-\infty}^\infty dq_0\;A(q_0,r,\theta)\ln r&=A_0+\tfrac{1}{\sin 2\theta}\sum_{n\ge 2}A_{n-1}\sin(2n\theta)\\&=A_0+\tfrac{3m}{8\pi}-\tfrac{m}{2\pi}\theta\tfrac{\cos\theta}{\sin\theta}\end{aligned}$$

where

$$A_{n-1}=\tfrac{(-1)^n m}{2\pi}\left[-\tfrac{1}{2n+2}+\tfrac{2}{2n}-\tfrac{1}{2n-2}\right]$$

Proof: a) Let $\mathbf{s}$ and $\mathbf{t}$ be of length k_F and separated by an angle 2θ. Then

$$\frac{1}{(2\pi)^2}\int_0^\infty dr\int_{-\infty}^\infty dq_0\ A(q_0,r,\theta)=\int_{\mathbb{R}^3}\frac{d^3q}{(2\pi)^3}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{s})}\frac{1}{iq_0-e(\mathbf{q}-\mathbf{t})}$$
$$=W(0,|\mathbf{s}-\mathbf{t}|)=-\frac{m}{2\pi}$$

b) By Lemma II.10

$$A(q_0,r,\theta)=\quad\frac{4m^2}{r}\frac{\alpha(r,q_0)+\alpha(r,q_0)^{-1}}{\alpha(r,q_0)-\alpha(r,q_0)^{-1}}\ \left[\alpha(r,q_0)^{-2}+\frac{1}{\sin 2\theta}\sum_{n\geq 2}\alpha(r,q_0)^{-2n}\sin(2n\theta)\right]$$

Evaluating the q_0 integral by contour integration methods one sees that

$$A_0=\frac{1}{(2\pi)^2}\int_0^\infty dr\int_{-\infty}^\infty dq_0\ \frac{4m^2}{r}\ \frac{\alpha(r,q_0)+\alpha(r,q_0)^{-1}}{\alpha(r,q_0)-\alpha(r,q_0)^{-1}}\ \alpha(r,q_0)^{-2}\ln r$$

is finite.

We now evaluate, for $n\geq 2$

$$A_{n-1}=\frac{1}{(2\pi)^2}\int_0^\infty dr\int_{-\infty}^\infty dq_0\ \frac{4m^2}{r}\ \frac{\alpha(r,q_0)+\alpha(r,q_0)^{-1}}{\alpha(r,q_0)-\alpha(r,q_0)^{-1}}\ \alpha(r,q_0)^{-2n}\ln r$$

Using the coordinates $w,\ \bar w$ as above,

$$\begin{aligned}A_{n-1}=&\frac{mi}{4\pi^2}\int_\Omega\frac{1}{|w|^2}\left(\bar w-\frac{1}{\bar w}\right)\left(w+\frac{1}{w}\right)\frac{1}{w^{2n}}\ln\left(\frac12(w+\bar w)\left(1+\frac{1}{|w|^2}\right)\right)dw\wedge d\bar w\\=&\frac{mi}{4\pi^2}\int_\Omega\left(1-\frac{1}{\bar w^2}\right)\left(1+\frac{1}{w^2}\right)\frac{1}{w^{2n}}\ln\left(\frac12(w+\bar w)\right)dw\wedge d\bar w\\&+\frac{mi}{4\pi^2}\int_\Omega\left(1-\frac{1}{\bar w^2}\right)\left(1+\frac{1}{w^2}\right)\frac{1}{w^{2n}}\ln\left(1+\frac{1}{|w|^2}\right)dw\wedge d\bar w\end{aligned}$$

Since

$$\begin{aligned}&\frac{mi}{4\pi^2}\int_\Omega\left(1-\frac{1}{\bar w^2}\right)\left(1+\frac{1}{w^2}\right)\frac{1}{w^{2n}}\ln\left(1+\frac{1}{|w|^2}\right)dw\wedge d\bar w\\&\quad=\frac{mi}{4\pi^2}\int_1^\infty d\rho\,\rho\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\left(1-\frac{\zeta^2}{\rho^2}\right)\left(1+\frac{1}{\rho^2\zeta^2}\right)\frac{1}{\rho^{2n}\zeta^{2n}}\ln\left(1+\frac{1}{|\rho|^2}\right)\\&\quad=0\end{aligned}$$

we have

$$\begin{aligned}A_{n-1}=&\frac{mi}{4\pi^2}\int_\Omega\left(1-\frac{1}{\bar w^2}\right)\left(1+\frac{1}{w^2}\right)\frac{1}{w^{2n}}\ln\left(\frac12(w+\bar w)\right)dw\wedge d\bar w\\=&-\frac{mi}{4\pi^2}\int_\Omega\left(1-\frac{1}{\bar w^2}\right)\ln\left(\frac12(w+\bar w)\right)dF(w,\bar w)\wedge d\bar w\end{aligned}$$

where

$$F(w,\bar w) = \frac{1}{2n+1}\left(\frac{1}{\bar w^{2n+1}} + \frac{1}{w^{2n+1}}\right) + \frac{1}{2n-1}\left(\frac{1}{\bar w^{2n-1}} + \frac{1}{w^{2n-1}}\right)$$

Observe that F vanishes on the imaginary axis. Using Stoke's Theorem

$$\begin{aligned}
A_{n-1} = & -\frac{mi}{4\pi^2}\int_\Omega d\left[\left(1-\frac{1}{\bar w^2}\right)\ln\left(\frac{1}{2}(w+\bar w)\right)F(w,\bar w)\,d\bar w\right] \\
& \qquad + \frac{mi}{4\pi^2}\int_\Omega \left(1-\frac{1}{\bar w^2}\right)\frac{1}{w+\bar w}F(w,\bar w)dw\wedge d\bar w \\
= & -\frac{mi}{4\pi^2}\int_{\substack{|w|=1\\ \mathrm{Re}\, w\geq 0}} \left(1-\frac{1}{\bar w^2}\right)\ln\left(\frac{1}{2}(w+\bar w)\right)F(w,\bar w)\,d\bar w \\
& \qquad - \frac{mi}{4\pi^2}\int_{\rho\geq 1} d\rho\;\rho\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\left(1-\frac{\zeta^2}{\rho^2}\right)\frac{1}{\rho\zeta+\rho\zeta^{-1}}F(\rho\zeta,\rho\zeta^{-1}) \\
= & \frac{mi}{4\pi^2}\int_{\substack{|w|=1\\ \mathrm{Re}\, w\geq 0}} (1-w^2)\ln\left(\frac{1}{2}(w+w^{-1})\right)F(w,w^{-1})\,\frac{dw}{w^2} \\
& + \frac{(-1)^n m}{2\pi}\int_{\rho\geq 1} d\rho\left[\frac{1}{(2n+1)}\frac{1}{\rho^{2n+3}} + \left(\frac{1}{2n+1}-\frac{1}{2n-1}\right)\frac{1}{\rho^{2n+1}} - \frac{1}{(2n-1)}\frac{1}{\rho^{2n-1}}\right] \\
= & \frac{mi}{4\pi^2}\int_{\substack{|w|=1\\ \mathrm{Re}\, w\geq 0}} \ln\left(\frac{1}{2}(w+w^{-1})\right)d\tilde F(w) \\
& + \frac{(-1)^n m}{2\pi}\left[\frac{1}{2n+1}\frac{1}{2n+2} + \left(\frac{1}{2n+1}-\frac{1}{2n-1}\right)\frac{1}{2n} - \frac{1}{2n-1}\frac{1}{2n-2}\right]
\end{aligned}$$

where

$$\begin{aligned}
\tilde F(w) = & -\frac{1}{(2n+1)(2n+2)}\left(w^{2n+2} + \frac{1}{w^{2n+2}} + (-1)^n 2\right) \\
& + \frac{1}{2n}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)\left(w^{2n} + \frac{1}{w^{2n}} - (-1)^n 2\right) \\
& + \frac{1}{(2n-1)(2n-2)}\left(w^{2n-2} + \frac{1}{w^{2n-2}} + (-1)^n 2\right)
\end{aligned}$$

By the Fundamental Theorem of Calculus, the first term

$$\begin{aligned}
\frac{mi}{4\pi^2}\int_{\substack{|w|=1\\ \mathrm{Re}\, w\geq 0}} \ln\left(\frac{1}{2}(w+w^{-1})\right)d\tilde F(w) & = -\frac{mi}{4\pi^2}\int_{\substack{|w|=1\\ \mathrm{Re}\, w\geq 0}} \tilde F(w)\frac{w^2-1}{w(w^2+1)}\;dw \\
& = -\frac{mi}{8\pi^2}\int_{|w|=1} \tilde F(w)\frac{w^2-1}{w(w^2+1)}\;dw \\
& = -\frac{mi}{8\pi^2}\int_{|w|=1/2} \tilde F(w)\frac{w^2-1}{w(w^2+1)}\;dw \\
& = 0
\end{aligned}$$

by the Residue Theorem. Because the sum over $n \geq 2$ is absolutely convergent it is straight forward to justify the interchange of the sum over n with the integrals. It is also straight forward to sum the Fourier series using

$$2\theta = -\sum_{n=1}^{\infty} (-1)^n \tfrac{2}{n} \sin(2n\theta)$$

■

Proof of Theorem II.7:

a) By Lemma II.9 and Lemma II.11b

$$\begin{aligned} G_{3a} =& \text{const}_f + \tfrac{1}{(2\pi)^2} \int A(q_0, r, \tfrac{\pi}{4}) \big(R(q_0, r, \mathfrak{C}) + \tfrac{m}{2\pi} \ln r \big) dq_0 dr \\ & - \tfrac{m}{(2\pi)^3} \int_0^\infty dr \int_{-\infty}^{\infty} dq_0 \; A(q_0, r, \theta) \ln r \end{aligned}$$

By Lemma II.12b,

$$\begin{aligned} \tfrac{m}{(2\pi)^3} \int_0^\infty dr \int_{-\infty}^{\infty} dq_0 \; A(q_0, r, \theta) \ln r &= \tfrac{m}{2\pi} A_0 + \tfrac{m}{2\pi} \sum_{n \geq 2} A_{n-1} \frac{\sin(2n\theta)}{\sin 2\theta} \\ &= \tfrac{m}{2\pi} A_0 + \tfrac{m}{\pi} \sum_{n \geq 2} A_{n-1} \left(\cos 2(n-1)\theta + \cos 2(n-3)\theta + \cdots + \begin{cases} \cos(2\theta) & n \text{ even} \\ \frac{1}{2} & n \text{ odd} \end{cases} \right) \\ &= \text{const}_0 + \tfrac{m}{\pi} \sum_{\ell=1}^{\infty} \cos(2\ell\theta) \sum_{k=0}^{\infty} A_{\ell+2k} \\ &= \text{const}_0 + \tfrac{1}{2} \sum_{\ell=1}^{\infty} (-1)^\ell a_\ell \cos(2\ell\theta) \end{aligned}$$

Putting

$$B_{\mathfrak{C}} = \text{const}_f - \text{const}_0 + \tfrac{1}{(2\pi)^2} \int A(q_0, r, \tfrac{\pi}{2}) \big(R(q_0, r, \mathfrak{C}) + \tfrac{m}{2\pi} \ln r \big) dq_0 dr$$

we get the desired result.

b) By Lemma II.9, followed by Lemma II.11a and Lemma II.12b

$$
\begin{aligned}
&\left\| G_{3b}(s,t) - \tfrac{m^2}{(2\pi)^2}(1-\ln 2) \right\|_\infty \\
&= \left\| \tfrac{1}{(2\pi)^2} \int_0^\infty dr \int_{-\infty}^\infty dq_0 \ A(q_0,r,\theta) W(q_0,r) - \tfrac{m^2}{(2\pi)^2}(1-\ln 2) \right\|_\infty + O(1/\mathfrak{C}) \\
&= \left\| -\tfrac{m}{(2\pi)^3} \int A(q_0,r,\theta) \ dq_0 dr + \tfrac{m^3}{(2\pi)^3} \int \tfrac{1}{r\left(a(q_0,r)^2-1\right)} \ dq_0 dr \right\|_\infty + O(1/\mathfrak{C}) \\
&= \left\| \tfrac{m^2}{(2\pi)^2} + \tfrac{m^3}{(2\pi)^3} \int \tfrac{1}{r\left(a(q_0,r)^2-1\right)} \ dq_0 dr \right\|_\infty + O(1/\mathfrak{C}) \\
&= O(1/\mathfrak{C})
\end{aligned}
$$

since

$$
\int_0^\infty \frac{dr}{r} \int_{-\infty}^\infty \frac{dq_0}{a(q_0,r)^2 - 1} \ = \ -\frac{2\pi}{m}
$$

Appendix. Restriction of the Bethe-Salpeter Equation to the Fermi Surface

We use the Theorem stated in §1 and standard properties of the two point function G and Bethe-Salpeter kernel K to show

Proposition A.1 *Let $\beta_0(\lambda)$ be the solution of*

$$1 = \frac{\alpha\lambda^3}{\beta_0} \sum_{t_0 \in \frac{\pi}{\beta_0}(2\mathbb{Z}+1)} \int \frac{d^2\mathbf{t}}{(2\pi)^2} \frac{f(\mathbf{t}/\mathfrak{C})^2}{t_0^2 + e(\mathbf{t})^2}$$

Then, for sufficiently small λ there exists a $\beta_c(\lambda)$ close to $\beta_0(\lambda)$ and a nonzero function $\chi(s_0, |\mathbf{s}|; \lambda)$ that solves the $\ell = 1$ equation (I.2)

$$\chi(s_0, |\mathbf{s}|) = -\tfrac{1}{\beta_c} \sum_{t_0 \in \frac{\pi}{\beta_c}(2\mathbb{Z}+1)} \int \tfrac{d^2\mathbf{t}}{(2\pi)^2} K_1(s_0, |\mathbf{s}|, t_0, |\mathbf{t}|; \beta_c) |G(t)|^2 \chi(t_0, |\mathbf{t}|)$$

in perturbation theory.

For the proof of this Proposition we use the following properties of G and K_1:

$$\sup_{\beta>1} |G(k)| \le \frac{\text{const}\, f(|\mathbf{k}|/\mathfrak{C})}{|ik_0 - e(\mathbf{k})|} \tag{A.1}$$

$$\sup_{\beta>1} \left| G(k) - \frac{f(|\mathbf{k}|/\mathfrak{C})}{ik_0 - e(\mathbf{k})} \right| \le \frac{\text{const}\, \lambda f(|\mathbf{k}|/\mathfrak{C})}{|ik_0 - e(\mathbf{k})|} \tag{A.2}$$

There exists an $\epsilon > 0$ such that, for all sufficiently small λ,

$$\sup_{\beta>1} \sup_{t} |K_1(k_0, x, t_0, |\mathbf{t}|; \beta) - K_1(\mathbf{f}, t_0, |\mathbf{t}|; \beta)| \le \text{const}\, \lambda^2 [k_0^2 + |x - k_F|^2]^{\epsilon/2} \tag{A.3}$$

$$|K_1(\mathbf{f}, \mathbf{f}; \beta) - \Lambda_1)| \le \text{const}\, \frac{\lambda^2}{\beta^\epsilon} \tag{A.4}$$

where $\mathbf{f} = (\frac{\pi}{\beta}, k_F)$. That these properties are valid to all orders of perturbation theory will be proven elsewhere.

The first property and (I.4) implies

$$\frac{1}{\beta} \sum_{k_0} \int d^2\mathbf{k} |G(k)|^2 \le \text{const} \ln \beta \tag{A.5}$$

$$\sup_{\beta>1} \frac{1}{\beta} \sum_{k_0} \int d^2\mathbf{k} |G(k)|^2 \left(k_0^2 + e(\mathbf{k})^2\right)^{\epsilon/2} \le \text{const}_{\epsilon} \tag{A.6}$$

For convenience we now suppress the dependence of K_1 on β.

Lemma A.2 *Put, for $\beta > 0$, $W_\beta = \left\{ (k_0, x) \;\middle|\; k_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1),\ 0 \le x \le 2\mathfrak{C} \right\}$. Then there exists an $\epsilon > 0$ such that, for all sufficiently small λ and $\beta > 1$,*

a) $$\sup_{(k_0,x)\in W_\beta} \frac{1}{\beta} \sum_{t_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \frac{d^2\mathbf{t}}{(2\pi)^2} \frac{|K_1(k_0, x, t_0, |\mathbf{t}|) - K_1(\mathbf{f}, t_0, |\mathbf{t}|)|}{[k_0^2 + |x - k_F|^2]^{\epsilon/2}} |G(t)|^2 \le \text{const}\ \lambda^2 \ln \beta$$

b) $$\sup_{(k_0,x)\in W_\beta} \frac{1}{\beta} \sum_{t_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \frac{d^2\mathbf{t}}{(2\pi)^2} \frac{|K_1(k_0, x, t_0, |\mathbf{t}|) - K_1(\mathbf{f}, t_0, |\mathbf{t}|)|}{[k_0^2 + |x - k_F|^2]^{\epsilon/2}} |G(t)|^2 [t_0^2 + e(t)^2]^{\epsilon/2} \le \text{const}\ \lambda^2 \le 1/2$$

c) $$\frac{1}{\beta} \sum_{t_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \frac{d^2\mathbf{t}}{(2\pi)^2} |K_1(\mathbf{f}, t_0, |\mathbf{t}|) - K_1(\mathbf{f}, \mathbf{f})| |G(t)|^2 \le \text{const}\ \lambda^2$$

Here all the constants const *are independent of β.*

Proof: By (A.3)

$$\frac{|K_1(k_0, x, t_0, |\mathbf{t}|) - K_1(\mathbf{f}, t_0, |\mathbf{t}|)|}{[k_0^2 + |x - k_F|^2]^{\epsilon/2}} \le \text{const}\ \lambda^2$$

So parts a) and b) follow immediately from (A.5) and (A.6). By (A.3) and the symmetry of K_1

$$|K_1(\mathbf{f}, t_0, |\mathbf{t}|) - K(\mathbf{f}, \mathbf{f})| \le \text{const}\ \lambda^2 [t_0^2 + e(\mathbf{t})^2]^{\epsilon/2}$$

Again, part c) follows from (A.6). ■

Proof of Proposition A.1: If $\tilde{\chi}$ obeys

$$1 + \tilde{\chi}(s_0, |\mathbf{s}|) = -\frac{1}{\beta} \sum_{t_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \frac{d^2\mathbf{t}}{(2\pi)^2} K_1(s_0, |\mathbf{s}|, t_0, |\mathbf{t}|) |G(t)|^2 [1 + \tilde{\chi}(t_0, |\mathbf{t}|)] \tag{A.7}$$

then $1 + \tilde{\chi}$ obeys (I.2). We look for a solution obeying $\tilde{\chi}(\mathbf{f}) = 0$. Restricting (A.7) to the Fermi surface gives

$$1 = -\frac{1}{\beta} \sum_{t_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \frac{d^2\mathbf{t}}{(2\pi)^2} K_1(\mathbf{f}, t_0, |\mathbf{t}|) |G(t)|^2 [1 + \tilde{\chi}(t_0, |\mathbf{t}|)] \tag{A8.a}$$

The difference between (A.7) and (A.8a) is

$$\tilde{\chi}(s_0, |\mathbf{s}|) = -\frac{1}{\beta} \sum_{t_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \frac{d^2\mathbf{t}}{(2\pi)^2} [K_1(s_0, |\mathbf{s}|, t_0, |\mathbf{t}|) - K_1(\mathbf{f}, t_0, |\mathbf{t}|)] |G(t)|^2 [1 + \tilde{\chi}(t_0, |\mathbf{t}|)] \tag{A.8b}$$

Move all the $\tilde{\chi}$'s in (A.8b) to the left hand side.

$$\begin{aligned}\tilde{\chi}(s_0,|\mathbf{s}|) + \tfrac{1}{\beta} \sum_{t_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \tfrac{d^2\mathbf{t}}{(2\pi)^2}[K_1(s_0,|\mathbf{s}|,t_0,|\mathbf{t}|) - K_1(\mathbf{f},t_0,|\mathbf{t}|)]|G(t)|^2\tilde{\chi}(t_0,|\mathbf{t}|) \\ = \quad -\tfrac{1}{\beta} \sum_{t_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \tfrac{d^2\mathbf{t}}{(2\pi)^2}[K_1(s_0,|\mathbf{s}|,t_0,|\mathbf{t}|) - K_1(\mathbf{f},t_0,|\mathbf{t}|)]|G(t)|^2\end{aligned} \tag{A.9}$$

Define the norm

$$\|\tilde{\chi}\|_{\beta,\epsilon} = \sup_{(k_0,x)\in\frac{\pi}{\beta}(2\mathbb{Z}+1)\times[0,2\mathfrak{C}]} \frac{1}{[k_0^2+|x-k_F|^2]^{\epsilon/2}}|\tilde{\chi}(k_0,x)|$$

and the associated Banach space

$$\mathcal{B}_{\beta,\epsilon} = \left\{\ \tilde{\chi} : \tfrac{\pi}{\beta}(2\mathbb{Z}+1)\times[0,2\mathfrak{C}] \to \mathbb{C}\ \middle|\ \|\tilde{\chi}\|_{\beta,\epsilon} < \infty\ \right\}$$

By Lemma A.2b, $[K_1(s_0,|\mathbf{s}|,t_0,|\mathbf{t}|) - K_1(\mathbf{f},t_0,|\mathbf{t}|)]|G(t)|^2$ is the kernel of an integral operator on $\mathcal{B}_{\beta,\epsilon}$ whose norm is bounded by one half for all sufficiently small λ and all $\beta >$ const. Furthermore, by Lemma A.2a, the right hand side of (A.9) is an element of $\mathcal{B}_{\beta,\epsilon}$ with norm at most const $\lambda^2\ln\beta$. Consequently, for each $\beta \geq$ const, (A.8a) has a unique solution $\tilde{\chi}_{\beta,\lambda}\in\mathcal{B}_{\beta,\epsilon}$ and this solution has norm at most const $\lambda^2\ln\beta$.

We now show that, for each sufficiently small λ, there is a $\beta(\lambda)$ such that $\tilde{\chi}_{\beta(\lambda),\lambda}$ obeys (A.8a). Indeed (A.8a) is equivalent to

$$\begin{aligned}1 = &-\tfrac{1}{\beta}\Lambda_1\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}|G(t)|^2 \\ &-\tfrac{1}{\beta}\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}K_1(\mathbf{f},t_0,\mathbf{t})|G(t)|^2\tilde{\chi}_{\beta,\lambda}(t) - \tfrac{1}{\beta}\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}[K_1(\mathbf{f},t_0,\mathbf{t})-\Lambda_1]|G(t)|^2 \\ =&\tfrac{\alpha\lambda^3}{\beta}\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}\tfrac{f(\mathbf{t}/\mathfrak{C})^2}{t_0^2+e(\mathbf{t})^2} \\ &-\tfrac{1}{\beta}(\Lambda_1+\alpha\lambda^3)\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}|G(t)|^2 + \tfrac{\alpha\lambda^3}{\beta}\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}\left[|G(t)|^2 - \tfrac{f(\mathbf{t}/\mathfrak{C})^2}{t_0^2+e(\mathbf{t})^2}\right] \\ &-\tfrac{1}{\beta}\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}K_1(\mathbf{f},t_0,\mathbf{t})|G(t)|^2\tilde{\chi}_{\beta,\lambda}(t) - \tfrac{1}{\beta}\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}[K_1(\mathbf{f},t_0,\mathbf{t})-\Lambda_1]|G(t)|^2\end{aligned} \tag{A.10}$$

As observed in (I.4),

$$\tfrac{\alpha\lambda^3}{\beta}\sum_{t_0}\int\tfrac{d^2\mathbf{t}}{(2\pi)^2}\tfrac{f(\mathbf{t}/\mathfrak{C})^2}{t_0^2+e(\mathbf{t})^2} = \tfrac{m}{2\pi}\alpha\lambda^3\big(\ln\beta + O(1)\big)$$

with const > 0. Similarly by the main Theorem, (I.4), (A.2) and (A.5)

$$\tfrac{1}{\beta}(\Lambda_1 + \alpha\lambda^3)\sum_{t_0}\int \tfrac{d^2\mathbf{t}}{(2\pi)^2}|G(t)|^2 = O(\lambda^4)\big(\ln\beta + O(1)\big)$$

$$\tfrac{\alpha\lambda^3}{\beta}\sum_{t_0}\int \tfrac{d^2\mathbf{t}}{(2\pi)^2}\left[|G(t)|^2 - \tfrac{f(\mathbf{t}/\mathfrak{C})^2}{t_0^2+e(\mathbf{t})^2}\right] = O(\lambda^4)\big(\ln\beta + O(1)\big)$$

Since $K_1 = O(\lambda^2)$, $\|\tilde{\chi}_{\beta,\lambda}\| = O(\lambda^2\ln\beta)$ and (A.5)

$$\tfrac{1}{\beta}\sum_{t_0}\int \tfrac{d^2\mathbf{t}}{(2\pi)^2}K_1(\mathbf{f},t_0,\mathbf{t})|G(t)|^2\tilde{\chi}_{\beta,\lambda}(t) = O(\lambda^4)\ln\beta$$

Finally, by Lemma A.2c and (A.4),

$$\begin{aligned}&\tfrac{1}{\beta}\sum_{t_0}\int \tfrac{d^2\mathbf{t}}{(2\pi)^2}[K_1(\mathbf{f},t_0,\mathbf{t}) - \Lambda_1]|G(t)|^2\\&\quad = \tfrac{1}{\beta}\sum_{t_0}\int \tfrac{d^2\mathbf{t}}{(2\pi)^2}[K_1(\mathbf{f},t_0,\mathbf{t}) - K_1(\mathbf{f},\mathbf{f})]G(t)|^2 + \tfrac{1}{\beta}\sum_{t_0}\int \tfrac{d^2\mathbf{t}}{(2\pi)^2}[K_1(\mathbf{f},\mathbf{f}) - \Lambda_1]|G(t)|^2\\&\quad = O(\lambda^2)\end{aligned}$$

So (A.10) is of the form

$$1 + O(\lambda^2) = \tfrac{m\alpha}{2\pi}\lambda^3\big(1+O(\lambda)\big)\ln\beta$$

Since both sides of (A.10) are continuous in β, it has a solution of the form

$$\ln\beta = \frac{2\pi}{m\alpha\lambda^3}\big(1+O(\lambda)\big)$$

■

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