Two Dimensional Many Fermion Systems as Vector Models

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Abstract We show that, in two space dimensions, the full many Fermion interaction has the structure of a vector model at every energy scale. The "colors" arise from a decomposition of the Fermi circle into sectors of length set by the energy scale. These sectors play an essential role in summing up the complete perturbation series. The three dimensional vertex is intermediate between vector and matrix models. In any number of dimensions the reduced interaction is a vector model.

PACS 05.30F, 74.20F

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* Research supported in part by the Natural Sciences and Engineering Research Council of Canada and the Schweizerischer Nationalfonds zur Forderung der wissenschaftlichen Forschung
† Research supported in part by the Forschungsinstitut für Mathematik, Zürich

§I Introduction

In [1] we considered a $d \geq 2$ dimensional many Fermion interaction whose $\ell = 0$ angular momentum sector is attractive and dominant. We showed that the effective vertex in the symmetry breaking energy regime $\big(k_0^2 + e(\mathbf{k})^2\big)^{1/2} \approx \Delta$, where Δ is the BCS gap and $e(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$, has the structure of an N component vector model with $N \approx \Delta^{-(d-1)}$. We also verified that bubbles are neutral in powers of $1/N$ and that all other Fermion loops are proportional to strictly positive powers of $1/N$. It was then argued that this observation justifies the usual one loop BCS gap equation.

Here we show, using the results of [2], that in two space dimensions the full interaction has the structure of a vector model at *all* energy scales. This structure was used in [2] to show that the sum all of perturbation theory down to energy scale $\approx \Delta$ converges for all bare coupling constants $|\lambda| \leq \mathrm{const}$ where const is a strictly positive constant *independent* of Δ. We will also draw attention to the difference between two and three dimensions.

As in [1] we consider the model formally characterized by the action

$$\mathcal{A}(\psi,\bar\psi) \;=\; -\int \!d\!\!\bar{}\,k\, \big(ik_0 e(\mathbf{k})\big)\bar\psi(k)\psi(k) - \mathcal{V}(\psi,\bar\psi) \tag{I.1}$$

in which $k = (k_0, \mathbf{k}) \in \mathbb{R}^{d+1}$ and $d\!\!\bar{}\,k = \frac{d^{d+1}k}{(2\pi)^{d+1}}$. The interaction is

$$\mathcal{V}(\psi,\bar\psi) = \tfrac{\lambda}{2} \int \prod_{i=1}^{4} d\!\!\bar{}\,k_i \;(2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4)\; \bar\psi(k_1)\psi(k_3)\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\,\bar\psi(k_2)\psi(k_4) \tag{I.2}$$

Here, $k' = (0, \frac{\mathbf{k}}{|\mathbf{k}|}\sqrt{2\mathbf{m}\mu})$ is the projection of k onto the Fermi surface. In the above expressions, the electron fields are vectors $\psi(k) = \begin{pmatrix}\psi_\uparrow(k)\\ \psi_\downarrow(k)\end{pmatrix}$ and $\bar\psi(k) = \big(\bar\psi_\uparrow(k), \bar\psi_\downarrow(k)\big)$ whose components $\psi_\sigma(k)$, $\bar\psi_\sigma(k)$ are the generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other. The generating functional for the associated connected Euclidean Green's functions is

$$\begin{aligned}\mathcal{S}(\phi,\bar\phi) \;&=\; \log \tfrac{1}{\mathcal{Z}} \int e^{\int d\!\!\bar{}\,k\,(\bar\phi\psi+\bar\psi\phi)} e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_\sigma(k)\, d\bar\psi_\sigma(k) \\ &=\; \log \tfrac{1}{\mathcal{Z}} \int e^{\int d\!\!\bar{}\,k\,(\bar\phi\psi+\bar\psi\phi)} e^{-\mathcal{V}(\psi,\bar\psi)}\, d\mu_C(\psi,\bar\psi)\end{aligned}$$

where $d\mu_C(\psi,\bar\psi)$ is the Grassmann Gaussian measure with mean zero and covariance

$$C(\xi_1,\xi_2) = \delta_{\sigma_1,\sigma_2} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i\langle p\,,\,\xi_1-\xi_2\rangle_-}}{ip_0 - e(\mathbf{p})} \qquad \langle p\,,\,\xi\rangle_- \;=\; -p_0 t + \mathbf{p}\cdot\mathbf{x} \tag{I.3}$$

§II Slices and Sectors

In order to systematically investigate the long range behavior of correlation functions at low temperature, it is natural to use a renormalization group analysis ([3],[4]) near the Fermi surface. To do this, we slice the covariance (free propagator) around its singularity on the Fermi sphere.

To analyze the ultraviolet and infrared behavior of a relativistic field theory, one defines a momentum k to be of scale j if $|k| \approx M^j$. Here, M is just a fixed constant that determines the scale units. As $j \to \infty$, the momentum k approachs the ultraviolet end of the model. As $j \to -\infty$, k approachs the infrared end of the model.

In non-relativistic solid state physics the natural scales consist of finer and finer shells around the Fermi surface. For each negative integer $j = 0, -1, -2, \ldots$ the j-th slice contains all momenta in a shell of thickness M^j a distance M^j from the singular locus $\left\{\ k \in \mathrm{I\!R}^{d+1} \ \middle|\ k_0 = 0,\ |\mathbf{k}| = \sqrt{2\mathbf{m}\mu}\ \right\}$. The propagator for the j-th slice is

$$C^j(\xi_1, \xi_2) = \delta_{\sigma_1,\sigma_2} \int đk \frac{e^{i\langle k,\xi_1-\xi_2\rangle_-}}{ik_0 - e(\mathbf{k})} 1_j\left(k_0^2 + e(\mathbf{k})^2\right) \tag{II.1}$$

where $1_j\left(k_0^2 + e(\mathbf{k})^2\right)$ is the characteristic function for the set $M^j \le |ik_0 - e(\mathbf{k})| < M^{j+1}$. For simplicity, we have introduced a sharp partition of unity even though a smooth one is required for a complete, technically correct analysis [3,II.1]. Summing over $j \le 0$, we obtain the full infrared propagator $C(\xi_1, \xi_2) = \sum_{j \le 0} C_j(\xi_1, \xi_2)$. The full Schwinger functions are obtained by assigning each line of each Feynman diagram a scale j and then summing over all such assignments.

Each single scale propagator (II.1) is supported in momentum space on a $d+1$ dimensional manifold with boundaries. The natural coordinates for this manifold are k_0, $\eta = e(\mathbf{k})$ and $\mathbf{k}' = \sqrt{2\mathbf{m}\mu}\frac{\mathbf{k}}{|\mathbf{k}|}$. In these coordinates the shell is $\left\{\ k \ \middle|\ M^j \le \sqrt{k_0^2 + \eta^2} \le \mathrm{const}\, M^j\ \right\}$ and is topologically $S^{d-1} \times S^1 \times [0,1]$. However the first factor, the Fermi sphere S^{d-1}, should be viewed as having a macroscopic radius of order 1 while the remaining factors $S^1 \times [0,1]$ should be viewed as having a small diameter of order M^j at scale j.

The fact that this manifold has two length scales, 1 and M^j, of radically different size reflects the basic anisotropy between frequency k_0 and momentum $\mathbf{k}$. It implies, in contrast to the field theory case, that the behavior of $C^j(\xi_1, \xi_2)$ at large $\xi_1 - \xi_2$ cannot be simply characterized as 'decay at rate M^{-j}'. Rather, C^j looks like

$$\left|C^j(\xi_1, \xi_2)\right| \le \text{const}\, M^j \big[1 + |\mathbf{x}_1 - \mathbf{x}_2|\big]^{(1-d)/2} \big[1 + M^j |\xi_1 - \xi_2|\big]^{-N}$$

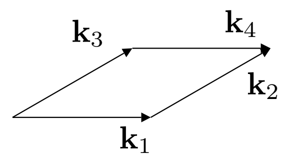
when a smooth cutoff function is used.

To obtain regions in momentum space all of whose dimensions are of order M^j, it is natural to further divide the j^{th} shell, through a partition of unity, into $N_j = M^{-(d-1)j}$ pieces, each having longest and shortest diameters of order M^j. We call each piece an isotropic sector. These sectors function as colors in a many component model. This decomposition is intimately related to Haldane's innovative, nonperturbative investigation of the Fermi surface. Related ideas appear in [5].

We now discuss an important question. Every interaction vertex couples four fields, each of which has a sector index. How many 4-tuples of sector indices are coupled?

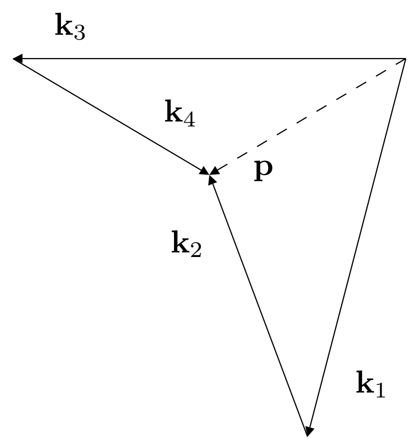
The momenta $k_1,\ k_2,\ k_3$ and k_4 of the four fields coupled by the vertex (I.2) are constrained by conservation of momentum. If all of these momenta are of scale $j \ll 0$, then all of their 0^{th} components are very close to zero and all of their lengths are very close to $\sqrt{2\mathbf{m}\mu}$. Generically, these are the only constraints relating $k_1,\ k_2,\ k_3$ and k_4.

In two space dimensions four momenta $\mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k}_4$ of very nearly the same length which obey $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = 0$ form an approximate parallelogram.



The directions of two sides of a parallelogram may be chosen arbitrarily. Once they have been chosen the directions of the remaining two sides are completely determined. Thus roughly speaking, in two space dimensions, the sector indices of two momenta may be chosen arbitrarily and once they have been chosen the sector indices of the remaining two momenta are fixed. As there are M^{-j} different possible values of the sector index in two space dimensions, the interaction (I.2) couples roughly M^{2j} four-tuples of sector indices at scale j in two space dimensions.

In $d > 2$ space dimensions, the sectors of $\mathbf{k}_1$ and $\mathbf{k}_2$ can again be chosen arbitrarily. There are $M^{-2(d-1)j}$ possibilities. Once $\mathbf{k}_1$ and $\mathbf{k}_2$ are chosen, the sum $\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ is fixed. But the triangle with sides $\mathbf{k}_3$, $\mathbf{k}_4$ and $\mathbf{p}$ is still free to rotate about the fixed $\mathbf{p}$.



The set of possible orientations of this triangle is S^{d-2}. Crudely speaking this accounts for another $M^{-(d-2)j}$ possible sector choices. This calculation suggests that (I.2) couples on the order of $M^{-(3d-4)j}$ four-tuples of sectors.

The following lemma, which is proven in [2], gives the precise result that is motivated by the argument above. During a rigorous construction it is necessary to work in finite volume. Then, momentum is not exactly conserved. That is, $|\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4| \le mM^j$ for some $m > 0$. The lemma also takes this into account.

Lemma. *Let $m \ge 1$. The number of four-tuples $\{S_1, S_2, S_3, S_4\}$ of sectors of scale j for which there exist $\mathbf{k}_i \in \mathbb{R}^d$, $i = 1, 2, 3, 4$ satisfying $\mathbf{k}'_i \in S_i$, $|\mathbf{k}_i - \mathbf{k}'_i| \le \mathrm{const}\, M^j$ and $|\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4| \le mM^j$ is bounded by* $\mathrm{const}\, m^d M^{(-3d+4)j}(1 + |j|\delta_{d,2})$.

Thus, in two dimensions, the number of four-tuples of sectors coupled at scale j, with approximate conservation of momentum, is about N_j^2 where $N_j = M^{-j}$ is the number of sectors. There is a logarithmic correction, $|j|$, that is generated when the "parallelogram" almost collapses to a line. This logarithm can be accommodated constructively. See §III.

Recall that a vector model $(\vec{\phi} \cdot \vec{\phi})^2$ with N colours couples N^2 four-tuples of colours. The color index of one ϕ in a dot product determines the index of the other ϕ. The Lemma and the discussion preceding it show that the full two dimensional many Fermion vertex at scale j conserves sector indices (up to corrections lower order in N_j) just as an N_j-vector model conserves color. Thus, the two dimensional vertex has the structure of a vector model. We expect that this remark applies in other two dimensional solid state physics problems, for example, localization of a single two dimensional electron in a weak random potential.

In three dimensions, the full vertex does not conserve sector index. Now, in addition to the usual sector indices of $\mathbf{k}_1$ and $\mathbf{k}_2$, the angle between the triangles $\mathbf{k}_1\mathbf{k}_2\mathbf{p}$ and $\mathbf{k}_3\mathbf{k}_4\mathbf{p}$ may also be chosen freely. The vertex is "twisted". By the Lemma, the number of coupled sector four-tuples is M^{-5j}. Since, the total number of sectors in three dimensions is $N_j = M^{-2j}$, we have $M^{-5j} = N_j^{5/2}$. This count is intermediate between that for a vector model and that for a matrix model. We have already seen that the n component vector model $(\vec{\phi}\cdot\vec{\phi})^2$ couples N^2 four-tuples. In a matrix model such as $Tr\Phi^4$ where Φ is an $N \times N$ matrix, the number of choices is N^4.

The conclusion of the last paragraph applies to generic vertices. In any dimension, there are special interactions which have sector index conservation laws of vector type. For example, the BCS interaction for s-wave Cooper pairs

$$-\lambda \int_{|q|<\text{const}\, M^j} đq\, đt\, đs\ \bar{\psi}_\uparrow(t+\tfrac{q}{2})\bar{\psi}_\downarrow(-t+\tfrac{q}{2})\psi_\downarrow(-s+\tfrac{q}{2})\psi_\uparrow(s+\tfrac{q}{2})$$

or, indeed, the reduced interaction

$$\sum_{\alpha,\beta\in\{\uparrow,\downarrow\}} \lambda \int_{|q|<\text{const}\, M^j} đq\, đt\, đs\ \ \langle \mathbf{t},-\mathbf{t}|V|\mathbf{s},-\mathbf{s}\rangle\, \bar{\psi}_\alpha(t+\tfrac{q}{2})\bar{\psi}_\beta(-t+\tfrac{q}{2})\psi_\beta(-s+\tfrac{q}{2})\psi_\alpha(s+\tfrac{q}{2})$$

The vector structure of the s-wave vertex is discussed in [1]. More generally, in three dimensions, any vertex that forces the four momenta to lie in a (possibly vertex dependent) plane has the structure of a vector model.

§III Sectors and the Convergence of Many Fermion Perturbation Series

In this section we compare the size of a single graph of order n with the size of the sum of all graphs of order n. A complete discussion requires a full renormalization group analysis at all scales. Here, to concentrate on the effects of large order perturbation theory, rather than the behaviour of the flow arising from low orders of perturbation theory, we fix a scale j .

The Schwinger functions

$$S_{2p}(\xi_1, \xi_2, ..., \xi_{2p-1}, \xi_{2p}) = \frac{1}{Z} \int \psi(\xi_1)\bar{\psi}(\xi_2)...\psi(\xi_{2p-1})\bar{\psi}(\xi_{2p}) e^{-\lambda \mathbf{V}} d\mu_{C^j}(\psi, \bar{\psi}) \qquad \text{(III.1)}$$

for a theory at scale j are obtained by integrating against the Grassmann Gaussian measure $d\mu_{C^j}$ whose propagator C^j is given by (II.1). Here, $\lambda \mathbf{V} = \mathcal{V}$. These Schwinger are analytic in λ in a neighbourhood of $\lambda = 0$. Does the radius of analyticity, r_j, remain bounded away from zero as $j \to -\infty$? Control of r_j is an essential ingredient in any nonperturbative program to prove that the renormalization group flow converges.

We have

Theorem. [2] *In two space dimensions* $r_j \geq \text{const} > 0$

So, for a weakly coupled two dimensional model with a circular Fermi surface, there are (as expected) no exotic effects due to high orders of perturbation theory. Presently, there is no analogue of the Theorem in three dimensions. Our best estimate, to date, is $r_j \geq \text{const}\, M^{j/2}$.

To explain the role of sectors in the proof of the Theorem we must explain how power counting estimates for a single graph of given order (perturbative power counting) differs from power counting estimates for the *sum* of all graphs of that order (constructive power counting). Let $G_{n,p}$ be a connected graph of scale j and order n with $2p$ external lines. The number l of lines and the number L of independent loops in this graph are given by $l = 2n - p$ and $L = l - n + 1$. Since, the propagator $\frac{1_j(p_0^2 + e(\mathbf{p})^2)}{ip_0 - e(\mathbf{p})}$ of scale j is bounded by M^{-j} and supported on a set of volume M^{2j}, the perturbative power counting bound, ignoring unimportant constants, for $G_{n,p}$ is

$$M^{-jl} M^{2jL} = M^{(2-p)j}$$

The first factor arises from taking the supremum $\frac{1_j(p_0^2 + e(\mathbf{p})^2)}{ip_0 - e(\mathbf{p})}$ on each line. The second factor is the total volume of integration for the L momentum integrals. This perturbative power counting is independent of dimension and is typical of strictly renormalizable theories like ϕ_4^4.

The same bound can also be derived in position space as follows. Write C_j as the sum of $M^{-(d-1)j}$ single sector propagators and smooth off the characteristic function 1_j. Each single sector propagator obeys the position space bound

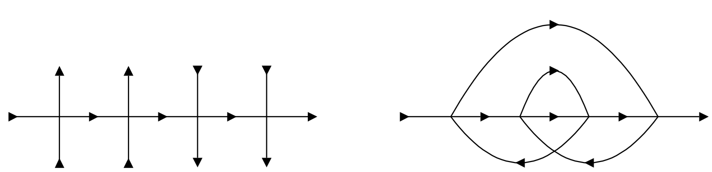
$$\begin{aligned}|C^{j,s}(\xi_1,\xi_2)| &= \delta_{\sigma_1,\sigma_2}\left|\int \frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{i\langle p,\xi_1-\xi_2\rangle_-}}{ip_0-e(\mathbf{p})}1_j(p_0^2+e(\mathbf{p})^2)g_s(\mathbf{p})\right| \\ &\le \text{const}\, M^{(d+1)j}M^{-j}[1+M^j|\xi_1-\xi_2|]^{-100}\end{aligned} \tag{III.2}$$

where 100 is a generic large number. The first factor is the volume of integration. The second is the supremum of the integrand. The third is derived through multiple integration by parts. Thus, when $G_{n,p}$ is evaluated in position space, there is one factor of M^{dj} per line. Furthermore, the position of one vertex is held fixed and the positions of the remaining $n-1$ vertices are integrated over $\mathbb{R}^{d+1}$. Since each such integral is controlled using a function that decays at rate M^j each such integral yields $M^{-(d+1)j}$. Finally the sector index of each propagator must be summed over. But, by conservation of momentum, there really are only L independent sector index sums. So the bound is

$$M^{ldj}M^{-(n-1)(d+1)j}M^{-L(d-1)j} = M^{j(2-p)}$$

Ideally, the sum of all n^{th} order graphs contributing to (III.1) would also be bounded by $\text{const}^{\,n}M^{j(2-p)}$. However, the argument given above cannot be directly applied to the sum of all graphs because they do not share a fixed loop structure.

Up to now, constructive methods to sum up graphs [2,6] have more or less proceeded along the following lines. The sum over connected graphs is divided into two phases. The first is the sum over all trees. The second is a sum over graphs having the given tree as a spanning tree. The second phase is implemented by choosing “Wick contractions” to add loops to the spanning tree.



a linear spanning tree   a Wick contraction scheme

The choice of tree, by Cayley’s theorem, is bounded by n^{n-2} and is paid for (up to an unimportant e^n) by the symmetry factor $1/n!$ for n vertices that arises from the expansion of the exponential $e^{-\lambda \mathbf{V}}$. Once the tree is known, one can integrate over the positions of vertices at a cost of $M^{-(n-1)(d+1)j}$. The number of possible sector index assignments to the lines of the tree lines depends somewhat on the exact branching structure of the tree. It can be as large as $M^{-(n-1)(d-1)j}$. This happens in, for example, the typical case of a linear tree. Combining this, as before, with the prefactor M^{ldj} of (III.2) for all the lines we get $M^{(2-p)dj}$, independent of n. However it remains to bound the sum over possible "Wick contractions" for the loop lines.

In fermionic theories, this sum has the form of a determinant. To bound this determinant without expanding it fully (which would lead to the usual divergence of perturbation theory) one must exploit the Pauli exclusion principle. It states that there is, roughly speaking, only one $\bar{\psi}$ and one ψ per unit volume in phase space. The decay of the propagator in (III.2) damps contractions between ψ's and $\bar{\psi}$'s that are widely separated (in units of M^{-j}) in position space. So we may pretend that there is only one ψ and one $\bar{\psi}$ in each sector of momentum space. (The technical details filling in this argument are given in [2].) Since fields may only contract to other fields in the same sector (or neighbouring sectors), once sector attributions are made for each uncontracted field hooked to the spanning tree the pattern of Wick contractions is essentially determined.

It only remains to count the number of possible sector index assignments to the ψ's and $\bar{\psi}$'s that are not in the spanning tree. At this point the key difference between two and three dimensions appears. Consider the typical case in which the spanning tree is a linear tree. Since sector indices have already been assigned to the lines of the spanning tree, we know the sector indices of two out of the four fields at each vertex. In two dimensions, the factorized form of the vertex fixes the sector indices of the remaining two fields up to the logarithmic correction of the Lemma. In three dimensions there is no logarithm , but the "twist" costs an M^{-j} per vertex. This means that a naive lower bound for the radius of convergence r_j is M^j in three dimensions and $1/|\log j|$ in two dimensions.

We can actually get a better bound on r_j by using "anisotropic" sectors whose dimensions tangent to the Fermi surface are of order $M^{j/2}$ and whose remaining dimensions are still of order M^j. Using Pauli's principle in corresponding anisotropic unit volume cells of phase space one obtains that the radius of convergence r_j is at least $M^{j/2}$ in three dimensions and $O(1)$ in two dimensions.

The general conclusion of this section, is that sectors are unnecessary for bounding a single graph, but are essential for summing up the full perturbation series.

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