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Hardy and Paley inequalities for fully-odd Vilenkin Systems

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Abstract. We consider Vilenkin systems for which only the identity function has even order. There is an unconventional, symmetric way to enumerate such systems so that the indices for complex conjugates are negatives of each other. We show for this enumeration that if all the negatively-indexed coefficients of an integrable function vanish, then the other coefficients satisfy the counterparts of the classical inequalities of Hardy and Paley for H^1-functions.

This follows easily from previous work if the fully-odd Vilenkin system is multiplicative and of bounded type. Our proof is new in that context; for other fully-odd systems, our results are also new. In all cases, they are equivalent to similar inequalities for the conventional enumeration of the fully-odd system for the H^1-space defined via a conjugate function.

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1. Introduction

Symmetric enumeration of certain Vilenkin systems was used in work [1, §5] on integrability of Vilenkin series of unbounded type. There, this enumeration had the advantage that the sequence of Fejér kernels, defined as averages of *symmetric* Dirichlet kernels, is bounded in L^1-norm, while this is not true for any standard enumeration. Moreover, there are results in [1, §5] for the unconventional enumeration where it is not yet known whether the standard counterpart holds.

In the rest of this section, we repeat the description of symmetric enumeration given in [1], state our main results, and connect them with previous work. We prove them in the next three sections. We use the term *fully-odd Vilenkin system* for any orthonormal system of functions that can be constructed in the following way.

Choose a nonatomic probability space Ω, like the interval $[0, 1)$, and a sequence $(p_r)_{r=0}^{\infty}$ of *odd* prime numbers. Let Γ_0 be the singleton set containing the constant function 1. Let ϕ_1 be a function taking each value in the set of p_1-th roots of unity with probability $1/p_1$. Let Γ_1 be the set of all functions ϕ_1^n with $0 \le n < p_1$.

For each positive integer r, let $m_r = \prod_{s=1}^{r} p_s$. When $r > 1$, assume that a set Γ_{r-1} of m_{r-1} functions on Ω has been specified. Then select three functions α_r, β_r, and γ_r with the following properties:

(i) α_r belongs to Γ_{r-1};
(ii) β_r is a p_r-th root of α_r that is constant on each set where α_r is constant;
(iii) γ_r takes each value in the set of p_r-th roots of unity with probability $1/p_r$;
(iv) γ_r is independent of the functions in the set Γ_{r-1}.

Let ϕ_r be the product $\gamma_r\beta_r$. Then $\phi_r^{p_r} = \alpha_r \in \Gamma_{r-1}$. Let Γ_r be the set of all functions obtained by multiplying the functions in Γ_{r-1} by the powers ϕ_r^n with $0 \le n < p_r$.

Continue this for all r. The system of functions constructed in this way is said [12] to be of *bounded type* if the sequence (p_r) is bounded. When $\alpha_r = 1$ for all r, the system is said [13] to be of *multiplicative type*. Some authors include the latter property in their definition of Vilenkin system, and some do not, making comparison of results in the area tricky. In studies of multiplicative systems, the numbers p_r are often only required to be positive integers, rather than being prime as above. The methods we use in this paper work as long as these numbers are odd.

In the standard enumeration of the system, the functions in Γ_r are indexed by the set of integers in the interval $[0, m_r)$. Assume that the set Γ_{r-1} has already been enumerated as $\{\chi_n\}$ with $0 \le n < m_{r-1}$. Each integer n in the interval $[m_{r-1}, m_r)$ has a unique representation as $n = jm_{r-1} + k$ with $1 \le j < p_r$ and $0 \le k < m_{r-1}$; then let χ_n be the product $(\phi_r)^j\chi_k$.

Our assumption that the prime numbers p_r are all odd makes all the products m_r odd too; then $n_r \equiv (m_r - 1)/2$ is an integer. In the symmetric enumeration, the set Γ_r is mapped to the set of integers in the interval $[-n_r, n_r]$. Each integer in this interval has a unique representation as a sum of products $q_s m_{s-1}$, where $1 \le s \le r$ and the integers q_s satisfy $-p_s/2 < q_s < p_s/2$. Moreover [1, §5], each function in Γ_r has a unique representation as a product of finitely-many powers $\phi_s^{q_s}$ with the same restrictions on s and q_s. Call this factorization of the function *minimal* and use the same terminology for the corresponding additive representation of the integer n.

We use the different notation $(\psi_n)_{n=-\infty}^{\infty}$ for the functions in the Vilenkin system when we enumerate them *symmetrically* by letting

$$\psi_n = \prod_s \phi_s^{q_s} \qquad \text{if} \qquad n = \sum_s q_s m_{s-1}$$

in a minimal way. In both the sum and the product there are only finitely-many nontrivial terms.

The following facts are easy to verify and will be useful later. If a function belongs to the system, so does its complex conjugate. Moreover, $\overline{\psi_n} = \psi_{-n}$; when $k > 0$, use $\tilde{k}$ to denote the index for which $\overline{\chi_k} = \chi_{\tilde{k}}$. If $\psi_n = \chi_k$, then $|n| \le k$, with equality if and only if the integers q_s in the representation of n are all nonnegative. If the system is of bounded type, then there is a constant C so that $k \le C|n|$ whenever $\chi_k = \psi_n$. There is no such constant C if the system is not of bounded type, but $\min\{k, \tilde{k}\} \le 2|n|$ in any case.

Denote the probability measure on Ω by $d\omega$. Given a function f in $L^1(d\omega)$ and an integer n, let

$$\hat{f}(n) = \int_\Omega f(\omega)\overline{\psi_n(\omega)}\, d\omega.$$

We also regard $\hat{f}$ as a function on the set $\Gamma = \{\psi_n\}_{n=-\infty}^{\infty}$, and then use the notation $\hat{f}(\psi_n)$ rather than $\hat{f}(n)$.

Some results about coefficients of functions in classical Hardy spaces can be proved using the group structure of the index set Z as the main tool. We show here that enough of this structure transfers to symmetrically enumerated Vilenkin systems to prove the counterparts of two well known facts for the trigonometric system. Recall [9] that a *Paley sequence* is a sequence (λ_j) of positive integers with the property that there is a constant K so that each interval $(2^s, 2^{s+1}]$ contains at most K terms in the sequence; in particular, this is true if there is a constant $C > 1$ so that $\lambda_{j+1} \ge C\lambda_j$ for all j.

Theorem 1. *Let $f \in L^1(d\omega)$ and $\hat{f}(\psi_n) = 0$ for all $n < 0$. Then*

$$\sum_{n>0} \frac{\left|\hat{f}(\psi_n)\right|}{n} < \infty. \tag{1.1}$$

Moreover, if (λ_j) is a Paley sequence, then

$$\sum_{j=1}^{\infty} \left|\hat{f}(\psi_{\lambda_j})\right|^2 < \infty. \tag{1.2}$$

To connect this with other work, we note first that there is an obvious candidate here for the rôle of conjugate function. Given a function f in L^1, we ask if there is a function $\tilde{f}$ in L^1, so that

$$\int_\Omega \tilde{f}(\omega)\overline{\psi_n(\omega)}\, d\omega = -i \cdot \operatorname{sgn}(n)\hat{f}(\psi_n) \qquad \text{for all } n.$$

An equivalent way to state our theorem is that the inequalities hold for any integrable function f that has an integrable conjugate function. For fully-odd systems of multiplicative type, that conjugate function differs only by a constant factor $\pm i$ from the one introduced by P. Simon [10]; our description of it only appears different because of our unconventional enumeration of the fully-odd Vilenkin system.

Corollary 2. *If an integrable function f has an integrable conjugate, then its coefficients $(\hat{f}(\chi_k))_{k=0}^{\infty}$ with respect to the standard enumeration satisfy the condition that*

$$\sum_{k>0} \frac{\left|\hat{f}(\chi_k)\right|}{k} < \infty, \tag{1.3}$$

and they also satisfy the condition that

$$\sum_{j=1}^{\infty} \left|\hat{f}(\chi_{\lambda_j})\right|^2 < \infty \tag{1.4}$$

for all Paley sequences (λ_j).

Various kinds of H^1-spaces have been considered for various Vilenkin systems, and Hardy inequalities (1.3) have been shown [5] to hold for some of these H^1-spaces and to fail for others. When the fully-odd Vilenkin system is multiplicative and of bounded type, the H^1-space defined via the conjugate function coincides with [14] those defined via martingales. Inequality (1.3) is known [2] for the martingale H^1-spaces for such systems. Inequality (1.1) then follows, because indices k and n that enumerate the same function all satisfy a common inequality of the form $k \leq C|n|$ when the system is of bounded type.

The same equivalences between types of H^1-spaces makes inequality (1.4) already known for multiplicative systems of bounded type; inequality (1.2) then follows for such systems. As with the Hardy inequality, our proof does not use this transference.

When the fully-odd system is *not* of bounded type, the theorem and corollary are new. Finding an independent proof of the corollary would provide another proof of the theorem for the following reaons. If f satisfies the hypothesis of the corollary, then so does $\overline{f}$. Hence the versions of inequalities (1.3) and (1.4) with $\hat{f}(\chi_k)$ replaced by $\hat{f}(\chi_{\tilde{k}})$ also hold. Inequality (1.1) then follows because $2|n| \geq \min\{k, \tilde{k}\}$ when $\psi_n = \chi_k$. Inequality (1.2) follows because, if (λ_j) is a Paley sequence and $\psi_{\lambda_j} = \chi_{k_j}$ for all j, then (k_j) is a Paley sequence.

2. Hardy inequalties for systems of bounded type.

To prove the first part of Theorem 1 we show that there is a constant C so that

$$\sum_{n=1}^{n_r} \frac{|\hat{f}(n)|}{n} \leq C \left\| f - \hat{f}(0) \right\|_1 \tag{2.1}$$

for all nonnegative integers r and all functions f satisfying the hypotheses of the theorem. We will see later that the same large enough constant C can be used for all fully-odd systems, but we will work in this section on systems of bounded type.

We use a variant [4] of the method introduced in one of the proofs [7] of the Littlewood conjecture about L^1-norms of exponential sums. For a small, positive constant c and each nonnegative integer r, we construct a function g_r so that $\widehat{g_r}(n) = 0$ for all $n > n_r$, while

$$\left| \widehat{g_r}(n) - c \frac{\operatorname{sgn}\left[\hat{f}(n)\right]}{n} \right| \leq \frac{c}{2n} \text{ when } 1 \leq n \leq n_r. \tag{2.2}$$

Moreover, g_r is a linear combination of finitely-many of the functions ψ_n, and $\|g_r\|_\infty \leq 1$. Then the absolute value of the integral of $[f - \hat{f}(0)] \cdot \overline{g_r}$ is bounded above by $\|f - \hat{f}(0)\|_1$, but the real part of that integral is bounded below by

$$\frac{c}{2} \sum_{n=1}^{n_r} \frac{|\hat{f}(n)|}{n}.$$

This yields inequality (2.1) with $C = 2/c$.

For a choice of the positive constant c that we will specify soon, and each positive integer r, we let

$$F_r = c \sum_{n_{r-1} < n \leq n_r} \frac{\operatorname{sgn}\left[\hat{f}(n)\right]}{n} \psi_n, \tag{2.3}$$

with the convention that $n_0 = 0$. Among other things, we need to have that

$$\|F_r\|_\infty \leq \frac{1}{2} \qquad \text{for all } r. \tag{2.4}$$

Now $\|F_1\|_\infty \leq c n_1$, and

$$\|F_r\|_\infty \leq c \ln\left(\frac{n_r}{n_{r-1}}\right) = c \ln(p_r)$$

when $r \geq 2$. Since the system is of bounded type, inequality (2.4) holds for all sufficiently small positive values of c. We then let

$$g_0 = 0 \qquad \text{and} \qquad g_r = F_r + \left[1 - 4|F_r|^2\right] g_{r-1} - \overline{F_r}\,[g_{r-1}]^2 \text{ for all } r > 0. \tag{2.5}$$

Condition (2.4) and induction on r imply [8] that

$$\|g_r\|_\infty \le 1 \qquad \text{for all } r. \tag{2.6}$$

Let

$$d_j = -4|F_j|^2 g_{j-1} - \overline{F_j}[g_{j-1}]^2 \qquad \text{for all } j > 0,$$

and let

$$e_r = \sum_{j=1}^{r} d_j \qquad \text{for all } r.$$

Following [7], we call the quantities d_j and e_r *detriments*. We then have that

$$g_r = e_r + \sum_{j=1}^{r} F_j. \tag{2.7}$$

The Vilenkin coefficients of the sum of the functions F_j above vanishes outside the interval $[1, n_r]$ and coincides with $c \cdot \mathrm{sgn}[\hat{f}(n)]/n$ on that interval. This reduces the verification of the desired conditions on the coefficients of g_r to checking that $\widehat{e_r}(n) = 0$ when $n > n_r$ and that $|\widehat{e_r}(n)| \le c/2n$ when $1 \le n \le n_r$.

We claim that $\widehat{g_j}$ and $\widehat{d_j}$ both vanish outside the interval $[-n_j, n_j]$ that indexes the set Γ_j. This is certainly true when $j = 1$, because $g_1 = F_1$ and $d_1 = 0$; assume that our claim is true when $j = r-1$, and consider the terms in line (2.5) that add to form g_r. The coefficients of F_r vanish outside $(n_{r-1}, n_r]$, and those of g_{r-1} vanish outside $[-n_{r-1}, n_{r-1}]$ by our inductive hypothesis. Hence the coefficients of $F_r + g_{r-1}$ vanish outside $[-n_r, n_r]$.

Turning to the parts of d_r, we use the fact that the functions in Γ_r form a group under multiplication. By the inductive hypothesis, g_{r-1} is a linear combination of functions in Γ_{r-1}, which is a subgroup of Γ_r; hence $[g_{r-1}]^2$ is also a linear combination of functions in Γ_{r-1}, and these functions all belong to Γ_r. Clearly, F_r is a linear combination of functions in Γ_r, and so is $\overline{F_r}$. Then the products $\overline{F_r}F_r g_{r-1}$ and $\overline{F_r}[g_{r-1}]^2$ are also linear combinations of functions in Γ_r. This confirms our claim that $\widehat{d_j}$ vanishes outside the interval $[-n_j, n_j]$. So does

$$\widehat{g_j} = \widehat{F_j} + \widehat{g_{j-1}} + \widehat{d_j}.$$

In particular, $\widehat{e_r}(n) = 0$ for all $n > n_r$. We estimate $\widehat{e_r}(n)$ when $1 \le n \le n_r$ by estimating the corresponding coefficients of d_j. We first consider those coefficients for $-\overline{F_j}[g_{j-1}]^2$. We saw above that the function g_{j-1} and its square are linear combinations of functions in the set Γ_{j-1}; any function in this set is a minimal product of powers $\phi_s^{q_s}$ with $q_s = 0$ when $s \ge j$. On the other hand, F_j is a combination of such products with $q_j > 0$ and $q_s = 0$ when $s > j$; then $\overline{F_j}$ is a similar combination of such products with $q_j < 0$. Since Γ_{j-1} is a group, multiplying $\overline{F_j}$ by $-[g_{j-1}]^2$ still gives a linear combination of such products with $q_j < 0$ and with $q_s = 0$ when $s > j$. Any such product is indexed as ψ_n for some negative integer n.

This makes the coefficients of $-\overline{F_j}[g_{j-1}]^2$ vanish on the set of nonnegative integers, and reduces our task to estimating the coefficients of $-4|F_j|^2 g_{j-1}$. If $j = 1$, then $g_{j-1} = 0$, and all these coefficients vanish. Suppose that $j > 1$. Since $\|g_{j-1}\|_\infty \leq 1$, each of the coefficients of $-4|F_j|^2 g_{j-1}$ is bounded in modulus by the L^1-norm of $4|F_j|^2$, and hence by the square of the L^2-norm of $2F_j$. That square is equal to the sum of the squares of the coefficients of $2F_j$, that is to

$$\sum_{n_{j-1}<n\leq n_j} \left(\frac{2c}{n}\right)^2 < \frac{4c^2}{n_{j-1}}. \tag{2.8}$$

This is an upper bound for the modulus of any coefficient of d_j, and these coefficients vanish to the right of n_j.

Given a positive integer n, choose j so that $n_{j-1} < n \leq n_j$. If $j > 1$, then

$$|\widehat{e_r}(n)| \leq \sum_{s\geq j} \left|\widehat{d_s}(n)\right| \leq \sum_{s\geq j} \frac{4c^2}{n_{s-1}}. \tag{2.9}$$

Since the primes p_s used to define the Vilenkin system are all odd, $n_s \geq 3n_{s-1}$ for all $s > 1$. Hence the series on the right in line (2.9) can be majorized by the geometric series

$$\sum_{s\geq j} \frac{4c^2}{n_{j-1}} \left(\frac{1}{3}\right)^{s-j} = \frac{16c^2}{3n_{j-1}}.$$

Our current assumption that the system is of bounded type means that there is a constant K so that $n_j/n_{j-1} \leq K$ for all $j > 1$. Combining these estimates gives that

$$|\hat{e}_r(n)| \leq \frac{16Kc^2}{3n} \quad \text{for all } n > n_1. \tag{2.10}$$

If $j = 1$ instead, then $0 < n \leq n_1$. Since $d_1 = 0$,

$$|\widehat{e_r}(n)| \leq \sum_{s>1} \left|\widehat{d_s}(n)\right| \leq \sum_{s>1} \frac{4c^2}{n_{s-1}},$$

and the right side here is again majorized by $16Kc^2/(3n)$. That majorant is bounded above by $c/2n$, as desired, if $c \leq 3/(32K)$. Any such choice of c will also make $\|F_r\|_\infty \leq 1/2$ for all r. This completes our proof of the Hardy inequality for systems of bounded type.

3. Hardy inequalities for systems of unbounded type.

When the system is not of bounded type, inequalities (2.4) and (2.10) can fail if the functions F_r are defined as in the previous section. To deal with this, we use a suitable increasing sequence $(t_i)_{i=0}^{\infty}$ of nonnegative integers to split some of the intervals $[n_{r-1}, n_r)$. Our procedure is similar to one used in a dual setting in [11] and [5]. We include the numbers n_r, as a subsequence (t_{i_r}) say of (t_i), and if $p_r = 3$, then n_{r-1} and n_r are consecutive terms in (t_i), that is $i_r = i_{r-1} + 1$. When $p_r > 3$, we choose the succesive numbers t_i that lie in the interval $[n_{r-1}, n_r)$ so that the number of integers in $[-n_{r-1}, t_i]$ doubles each time i increases, until we reach the point where $[-n_{r-1}, t_i]$ contains at least 1/4 of the integers in $[-n_{r-1}, n_r]$, and then we let $t_{i+1} = n_r = t_{i_r}$.

Another way to describe this is to note first that $[-n_{r-1}, n_{r-1}]$, which is the same as $[-n_{r-1}, t_{i_{r-1}}]$, contains exactly $m_r = 2n_r + 1$ integers. Then $(t_{i_{r-1}}, t_{i_{r-1}+1}]$ contains the next m_r integers, $(t_{i_{r-1}+1}, t_{i_{r-1}+2}]$ contains the next $2m_r$ integers after that, and the lengths of the intervals $(t_s, t_{s+1}]$ continue to double, except that the length of the final interval $(t_{i_r-1}, t_{i_r}]$ does not have to be equal to a power of 2 times m_r. That length is still divisible, however, by m_r, and is larger than the length of the rest of $(n_{r-1}, n_r]$. One reason for this divisibility property is that each of the sets $(t_s, t_{s+1}] \cup [-t_{s+1}, -t_s)$, where $n_{r-1} \le t_s < n_r$, enumerates a union of finitely-many cosets of Γ_{r-1} in Γ_r.

We now work with the revised functions

$$F_j = c \sum_{t_{j-1} < n \le t_j} \frac{\operatorname{sgn}\left[\hat{f}(n)\right]}{n} \psi_n, \tag{3.1}$$

with $t_0 \equiv 0$. Since the numbers $\ln(t_j/t_{j-1})$ with $j \ge 2$ form a bounded sequence, we recover the conclusion that

$$\|F_j\|_\infty \le \frac{1}{2} \qquad \text{for all } j \tag{3.2}$$

if the positive constant c is sufficiently small.

Let T be the set of positive integers j for which $t_j = n_r$ for some r, that is $j = i_r$; call these values of j *terminal* indices. Let E be the set of even positive integers that are not terminal, and let O be the set of odd positive integers that are not terminal. Let $F_j^T = F_j$ if j is terminal, and let $F_j^T = 0$ otherwise. Define functions F_j^E and F_j^O similarly. Then let

$$g_0^T = 0 \quad \text{and} \quad g_j^T = F_j^T + \left(1 - 4\left|F_j^T\right|^2\right) g_{j-1}^T - \overline{F_j^T}\left[g_{j-1}^T\right]^2 \text{ for all } j \ge 1. \tag{3.3}$$

Define functions g_j^E and g_j^O similarly; then g_j^T, g_j^E, and g_j^O all belong to the unit ball of L^∞. Let

$$g_j = g_j^T + g_j^E + g_j^O. \tag{3.4}$$

Then $\|g_j\|_\infty \le 3$ for all j.

Inequality (2.1) follows with $C = 6/c$ if condition (2.2) holds. Verifying that condition again reduces to locating and estimating the nonzero coefficients of detriments. Let

$$d_j^T = -4\left|F_j^T\right|^2 g_{j-1}^T - \overline{F_j^T}\left[g_{j-1}^T\right]^2 \qquad \text{for all } j > 0,$$

and define d_j^E and d_j^O similarly. Let

$$d_j = d_j^T + d_j^E + d_j^O \quad \text{for all } j, \qquad \text{and let } e_r = \sum_{j=1}^{r} d_j \quad \text{for all } r.$$

For each value of j, two of the functions F_j^T, F_j^E, and F_j^O are trivial, and the other one is equal to F_j. It follows that formula (2.7) still holds.

The detriment d_j^T vanishes if F_j^T does, and similarly for d_j^E and d_j^O. So, for each value of j, at most one of these detriments can have nonzero coefficients. Assume for the moment that these coefficients vanish to the right of t_j. Also assume that the nonzero coefficients of $-F_j^T[g_{j-1}^T]^2$ all have negative indices, and that the same is true when the superscript T is replaced by E or O.

Then matters reduce to applying ℓ^2-estimates to the coefficients of products like $-4|F_j^T|^2 g_{j-1}^T$. When $j > 1$ the modulus of any coefficient of the product is at most $4c^2/t_{j-1}$. When $n \in (t_{j-1}, t_j]$ and $j > 1$, this leads to the inequality

$$|\widehat{e_r}(n)| \le \sum_{s \ge j}\left|\widehat{d_s}(n)\right| \le \sum_{s \ge j}\frac{4c^2}{t_{s-1}}.$$

Since $t_s \ge 2t_{s-1}$ for all $s > 1$, the series on the right above can again be majorized by a geometric series, which converges this time to $8c^2/t_{j-1}$. Since $t_j \le 4t_{j-1}$ for all $j > 1$, it follows that

$$|\widehat{e_r}(n)| \le \frac{32c^2}{n} \qquad \text{for all } n \ge n_1.$$

Again, the fact that $d_1 = 0$ makes it easy to also obtain this estimate when $0 < n \le n_1$. The right side above is majorized by $c/2n$ if $c \le 1/64$. Then inequality (2.1) holds with

$$C = \frac{6}{c} = 384$$

for all fully-odd Vilenkin systems.

Finally, we verify the assumptions made above about the locations of the nonzero coefficients of the detriments and the products $-F_j^T[g_{j-1}^T]^2$, $-F_j^E[g_{j-1}^E]^2$, and $-F_j^O[g_{j-1}^O]^2$. We begin by checking that the argument in the previous section applies to the first of these products, which vanishes unless $j \in T$. Recall that the set T is enumerated as $\{i_r\}_{r=1}^{\infty}$, and that $t_{i_r} = m_r$. Then

$$g_{i_r}^T = F_{i_r} + \left(1 - 4\left|F_{i_r}\right|^2\right) g_{i_{r-1}}^T - \overline{F_{i_r}}\left[g_{i_{r-1}}^T\right]^2$$

when $r > 1$. The coefficients of F_{i_r} vanish outside the corona Γ_r/Γ_{r-1}. It follows by induction on r that the coefficients of $g^T_{i_r}$ vanish outside the subgroup Γ_r; in particular, these coefficients vanish to the right of $m_r = t_{i_r}$, and so do the coefficients of the corresponding detriment. It also follows that the nonzero coefficients of $-\overline{F^T_{i_r}}[g^T_{i_{r-1}}]^2$ all have negative indices.

Now consider g^E_j, but note that the same reasoning applies to g^O_j. Enumerate the set E, in increasing order, as $\{j_k\}_{k=1}^K$; here K is finite only if $p_r = 3$ for most values of r. Then

$$g^E_{j_k} = F_{j_k} + \left(1 - 4\,|F_{j_k}|^2\right) g^E_{j_{k-1}} - \overline{F_{j_k}}\left[g^E_{j_{k-1}}\right]^2$$

when $k > 1$. Let $r(j)$ be the smallest integer for which $n_{r(j)} \geq t_j$. As in the previous case, the coefficients of g^E_j and d^E_j vanish outside the subgroup $\Gamma_{r(j)}$.

By itself, this does not yield the desired upper bound of t_j on the supports of these coefficients, because $t_j < n_{r(j)}$ when j is not terminal. These supports are easier to analyse when j_k is *initial*, that is $r(j') < r(j_k)$ whenever $j' < j_k$ and $j' \in E$. Then the coefficients of $g_{j_{k-1}}$ vanish outside the subgroup $\Gamma_{r(j_k)-1}$, while the coefficients of $F^E_{j_k}$ vanish outside the corona $\Gamma_{r(j_k)}/\Gamma_{r(j_k)-1}$. So the nonzero coefficients of $-\overline{F^E_{j_k}}[g_{j_k-1}]^2$ all have negative indices when j_k is initial.

Consider the coefficients of $F^E_{j_k}$ in that case. If they differ from 0 at ψ_n, then ψ_n is a product of powers $\phi_s^{q_s}$ with $0 < q_{r(j_k)} < p_{r(j_k)}/2$ and $q_s = 0$ when $s > r(j_k)$. Moreover, it must actually be true here that

$$0 < q_{r(j_k)} < \frac{p_{r(j_k)}}{4},$$

because any larger value of $q_{r(j_k)}$ would force the index n into the second half of the interval $(n_{r(j_k)-1}, n_{r(j_k)}]$, and that half is covered by $(t_{s-1}, t_s]$ for the smallest terminal index s that exceeds j_k; being terminal, that index s does not belong to the set E.

This makes it easier to study the indices of nonzero terms in the product $\overline{F^E_{j_k}}F^E_{j_k}$. Suppose that two functions, ψ and ψ' say, in the Vilenkin system are factored minimally as products of powers $\phi_s^{q_s}$ and $\phi_s^{q'_s}$ with $q_s = 0$ and $q'_s = 0$ for all $s > r(j_k)$. Then their product $\psi'' = \psi\psi'$ also has a minimal factorization with indices $q''_s = 0$ for all $s > r(j_k)$. Moreover,

$$q''_{r(j_k)} = q_{r(j_k)} + q'_{r(j_k)} \tag{3.5}$$

provided that the right side above lies in the interval $(-p_r(j_k)/2, p_{r(j_k)}/2)$. In particular, this happens if

$$|q_{r(j_k)}| + |q'_{r(j_k)}| < \frac{p_{r(j_k)}}{2}. \tag{3.6}$$

Condition (3.6) is satisfied by any pair of nonzero terms in the sum giving $F^E_{j_k}$ when j_k is initial. Let $Q(j_k)$ be the largest value of $q_{r(j_k)}$ in the minimal factorizations of functions represented by integers in the interval $(t_{j_{k-1}}, t_{j_k}]$. Then the coefficients of $\overline{F^E_{j_k}}F^E_{j_k}$ can only differ from 0 at functions with minimal factorizations with indices q_s that vanish for all $s > r(j_k)$ and for which $|q_{r(j_k)}| < Q(j_k)$. This is also true for the coefficients of the product $|F^E_{j_k}|^2 g_{j_k-1}$, because g_{j_k-1} is a sum of terms with $q_s = 0$ for all $s \geq r(j_k)$ when j_k is initial. Then the coefficients of $|F^E_{j_k}|^2 g_{j_k-1}$ vanish to the right of t_{j_k}, as required. So do the coefficients of $g^E_{j_k}$ and $d^E_{j_k}$.

We now suppose that j_k belongs to E but is *not* initial. To get similar conclusions about coefficients, we consider the set of even integers $j' \leq j_k$ for which $r(j') = r(j_k)$. We show by induction on j' that the coefficients of $g^E_{j'}$ must vanish except at functions that have a minimal representation with $q_s = 0$ for all $s > r(j_k)$ and with

$$-2Q(j') \leq q_{r(j_k)} \leq Q(j'). \tag{3.7}$$

We note that $2Q(j') < p_{r(j_k)}/2$ here, because these indices j' are *not* terminal.

There is an initial even integer, j'_0 say, for which $r(j'_0) = r(j_k)$. By the previous analysis, the nonzero terms in the expansion of $g^E_{j'_0}$ all have minimal factorizations with

$$-Q(j'_0) \leq q_{r(j_k)} \leq Q(j'_0).$$

Hence the inductive assumption (3.7) is more than satisfied when $j' = j'_0$. Suppose it is true when $j' = h$, and pass to the next case, if there is one, where $j' = h+2$ and $j' \leq j_k$. Since $h+2$ is not terminal, $Q(h+2) < p_{r(j_k)}/2$. Recall that the lengths of the nonterminal subintervals $(t_{s-1}, t_s]$ in $(n_{r(j_k)-1}, n_{r(j_k)}]$ double as s increases; it follows that the integers in the first such subinterval $(t_{s-1}, t_s]$ are all minimally represented with $q_{r(j_k)} = 1$, that $q_{r(j_k)}$ takes the values 2 and 3 for all the integers in the next such subinterval, and so on. Therefore $Q(h)$ has the form $2^m - 1$ for some integer m, and then $Q(h+2) = 2^{m+2} - 1$; in particular, $Q(h) < Q(h+2)/4$.

Consider the nonzero terms in the expansion of the functions that add to form g^E_{h+2} from g^E_h and F^E_h. By hypothesis, those terms in the expansion of g^E_h satisfy condition (3.7) at the level $j' = h$, and hence at the level $j' = h+2$. The function F^E_{h+2} clearly satisfies condition (3.7) at that level. By the inductive assumption and the fact that $Q(h) < p_{r(j_k)}/4$, condition (3.6) is satisfied with j_k replaced by h by all the nonzero terms in the expansion of g^E_h, so that those terms in $[g^E_h]^2$ all have

$$-4Q(h) \leq q_{r(j_k)} \leq 2Q(h).$$

Since $4Q(h) + Q(h+2) < 2Q(h+2) < p_{r(j_k)}/2$, the nonzero terms in $\overline{F^E_{h+2}}[g^E_h]^2$ all have

$$-4Q(h) - Q(h+2) \leq q_{r(j_k)} \leq 2Q(h) - Q(h+2),$$

and hence

$$-2Q(h+2) < q_{r(j_k)} < -\frac{Q(h+2)}{2} < 0. \tag{3.8}$$

These terms satisfy condition (3.7) with $j' = h + 2$, and are indexed by negative integers, as required. Finally, the doubling process ensures that the nonzero terms in F_{h+2}^E all have

$$\frac{Q(h+2)}{2} < q_{r(j_k)} \leq Q(h+2).$$

Then the nonzero terms in $\overline{F_{h+2}^E} F_{h+2}^E$ satisfy

$$-\frac{Q(h+2)}{2} < q_{r(j_k)} < \frac{Q(h+2)}{2}.$$

Multiplying by $g_{h+1}^E = g_h^E$ only yields terms with

$$-2Q(h) - \frac{Q(h+2)}{2} < q_{r(j_k)} < Q(h) + \frac{Q(h+2)}{2},$$

and then

$$-Q(h+2) < q_{r(j_k)} < Q(h+2).$$

Since each part of g_{h+2}^E satisfies condition (3.7) with $j' = h + 2$, so does g_{h+2}^E. This completes the proof of the Hardy inequality.

4. Related results and methods.

It suffices to prove the Paley inequality (1.2) in the case where each of the intervals $(t_i, t_{i+1}]$ used in Section 3 contains at most one term of the sequence (λ_j). In that case, it is enough to show that there is a constant C so that

$$\sum_{\lambda_j \leq n_r} \left|\hat{f}(\lambda_j)\right|^2 \leq C\left(\|f\|_1\right)^2 \tag{4.1}$$

for all r. By the same duality that leads from condition (2.2) to (2.1), it suffices to show that for each sequence $(v(j))_{j=1}^{\infty}$ of numbers with $\|v\|_2 \leq 1/2$ and each positive integer r there is a function g_r satisfying the following conditions. First, $\widehat{g_r}(n) = 0$ for all $n > n_r$. Next,

$$|\widehat{g_r}(\lambda_j) - v(\lambda_j)| \leq \frac{1}{2}|v(\lambda_j)| \qquad \text{when } \lambda_j \leq n_r, \tag{4.2}$$

and $\widehat{g_r}(n) = 0$ for all other integers n in the interval $[1, n_r]$. Finally, g_r is again a linear combination of finitely-many of the functions ψ_n, and $\|g_r\|_\infty \leq 3$.

To this end, modify the construction used in the previous section by defining F_i to be 0 when there is no term of the sequence (λ_j) in the interval $(t_{i-1}, t_i]$ and to be $v(j)\psi_j$ when j is the unique index for which $\lambda_j \in (t_{i-1}, t_i]$. Then $\|F_i\|_\infty \leq 1/2$ for all i, and the construction in the previous section produces a function g_r with $\|g_r\|_\infty \leq 3$ as required.

The analysis of supports of coefficients still applies. It shows that g_r satisfies the conditions specified above, except possibly for (4.2). The part of the detriment with coefficients in the interval $[1, n_r]$ still comes entirely from the terms $-4|F_i^E|^2 g_{i-1}^E$ and their counterparts with the superscript E replaced by O or T. These coefficients of the detriment can be found exactly using the fact that now $|F_i|^2$ is either 0 or equal to $|v(j)|^2$ for the appropriate index j. It follows that if $\lambda_j \in E$ and $\lambda_j \leq n_r$, then

$$\widehat{g_r^E}(\lambda_j) = v(\lambda_j) \prod_{\lambda_s \in E \cap (\lambda_j, n_r]} \left(1 - |v(\lambda_s)|^2\right). \tag{4.3}$$

It also follows that $\widehat{g_r^E}(\lambda_j) = 0$ if $\lambda_j \notin E$. Similar conclusions hold when the superscript E is replaced by O or T. Inequality (4.2) then follows because

$$\prod_{j=1}^{\infty}\left(1 - |v(\lambda_j)|^2\right) \geq \exp\left[-\sum_{j=1}^{\infty}|v(\lambda_j)|^2\right] \geq e^{-1/4}.$$

Functions g_r with the properties needed to prove the Paley inequality can also be obtained by applying [3, Theorem 4] three times, in each case replacing the variables z_n there by the succesive functions ψ_i as the index i increases through one of the sets E, O, or T. In fact, that method yields a function g_r with condition (4.2) replaced by the sharper requirement that $\widehat{g_r}(\lambda_j) = v(\lambda_j)$ when $\lambda_j \leq n_r$. The method used above can also be adjusted to produce a function satisfying this sharper version of (4.2).

To compare inequality (1.2) and inequality (1.4), define a bijection σ between the set of nonnegative integers and the set of integers by letting $\sigma(k) = n$ if $\chi_k = \psi_n$. It is easy to check that if $(\lambda_j)_{j=1}^{\infty}$ is a Paley sequence, then so is $(\sigma^{-1}(\lambda_j))_{j=1}^{\infty}$. On the other hand, σ maps each Paley sequence to a sequence whose range is a union of the ranges of a Paley sequence and the negative of a Paley sequence. So inequality (1.2) holds for all Paley sequences and all integrable functions with integrable conjugates if and only if the same is true for inequality (1.4).

Our results and methods apply directly to the H^1-space defined using a conjugate function. When the fully-odd system is multiplicative and of bounded type, that H^1-space coincides [14] with the one defined using martingales; it seems likely that the same is true when the fully-odd system is of bounded type but not multiplicative. When the system is fully-odd but not of bounded type, the conclusions in Theorem 1 do not follow from the assumption that f belongs to the martingale H^1-space. For the first conclusion and systems that are of multiplicative but unbounded type this is in [5]; in all cases, one can use the uniform-boundedness principle and the fact that, when the system is not of bounded type, there are martingale differences with L^1-norm equal to 1 for which the left sides of inequalities (1.1) and (1.2) are as large as one likes. So the martingale version of H^1 differs from the one defined using a conjugate function when the system is of unbounded type.

Finally, there are H^1-spaces defined via atoms as in [11] and [5]. The conclusions of Theorem 1 hold for functions f in these spaces, because these conclusions follow in a uniform way for atoms, just as they do on the unit circle. See [5] for a discussion of some cases of this. It is shown in [11] that the atomic H^1-space is included in the one defined by conjugation when the system of multiplicative type, even when it is not of bounded type. It seems likely that the same is true when the system is not multiplicative. Many variants of Hardy and Paley inequalities are considered in [15, Chapter 6] for a large variety of H^1-spaces defined using martingales or atoms.

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