Integrability of multiple series

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Abstract

We show that if the coefficients in a multiple trigonometric series tend to 0 and if their mixed differences are small enough, then the series represents an integrable function provided that the coefficients in the complex form of the series also satisfy a symmetry condition. Multiple cosine series automatically satisfy the symmetry condition. We also show that if the coefficients in a multiple Walsh series tend to 0 and if their mixed differences are small enough, then the multiple Walsh series represents an integrable function.

1 Introduction

The condition that we impose on the sizes of mixed differences is strictly weaker than those that have been considered previously for multiple series. Our use of a symmetry condition for multiple trigonometric series also seems to be new. The single-variable versions of our results are known [4, 5, 2, 1]. Our general plan is the same as in [2] and [1], but the presence of the extra variables leads to new complications.

In this section we present some notation and state our integrability theorems. In Section 2, we outline a proof our result on trigonometric series. We complete that proof in Sections 3 and 4. We prove our theorem for Walsh series in Section 5. In Section 6, we compare our hypotheses with earlier ones. Finally, in Section 7, we relate our conditions to other restrictions, on sums of mixed differences, that also imply integrability.

We use the multiindex notation as in [15]. The symbol K denotes an integer that is greater than 1, and lower-case letters near the beginning of the Greek alphabet denote lists, like $(\alpha_k)_{k=1}^K$, of K integers. The symbols Z and Z_+ denote the sets of integers and nonnegative integers respectively. We write $\beta \geq 0$ if $\beta \in Z_+^K$, and $\gamma \geq \alpha$ if $\gamma - \alpha \geq 0$. We write $\beta > 0$ if $\beta_k > 0$ for all k; this is a stronger restriction than merely requiring that $\beta \geq 0$ and $\beta \neq 0$.

We mostly use lower-case letters near the end of the Roman alphabet to denote lists, like $(t_k)_{k=1}^K$ of numbers in the real interval $[-1/2, 1/2)$. One exception to this is that we use the symbol z to denote the mapping

$$t \mapsto z(t) \equiv (z_k(t))_{k=1}^K \equiv (e^{2\pi i t_k})_{k=1}^K$$

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of the set $X \equiv [-1/2, 1/2)^K$ onto the set T^K of lists of complex numbers of modulus 1. By convention,

$$z^\alpha(t) = \prod_{k=1}^{K} z_k(t)^{\alpha_k}.$$

The symbol m denotes Lebesgue measure on X, and L^p denotes the usual space of equivalence classes of measurable functions on (X, m). The functions z^α form a complete orthonormal system in $L^2(X, m)$. Every function f in L^1 has Fourier coefficients given by

$$\hat{f}(\alpha) = \int_X f(t) z^{-\alpha}(t) dm(t).$$

We say that a series, in the complex form

$$\sum_{\alpha \in Z^K} c(\alpha) z^\alpha, \tag{1.1}$$

represents an integrable function, or simply that the series is *integrable*, if there is an f in L^1 so that $c(\alpha) = \hat{f}(\alpha)$ for all α in Z^K.

By Riemann-Lebesgue, $c(\alpha)$ then tends to 0. There are many ways to see that this is not sufficient for integrability. Since conditions on the smoothness of a distribution corresponds to conditions on the size of the transform of the distribution, it is reasonable to expect that if the coefficients in a series tend to 0 and have small enough differences, then the series will be integrable.

We work with mixtures of differences in all directions. For each index k with $1 \le k \le K$, let

$$\Delta_k c(\alpha) = c(\alpha) - c(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1}, \ldots, \alpha_K),$$

be the usual forward difference with respect to the k-th component. The operators Δ_k commute; let $\Delta = \prod_{k=1}^{K} \Delta_k$. When $K = 2$ for instance,

$$\Delta c(3, 5) = c(4, 6) - c(4, 5) - c(3, 6) + c(3, 5).$$

The condition that we impose on the sizes of the quantities $\Delta c(\alpha)$ involves amalgams [8] of ℓ^1-norms and ℓ^2-norms. For each positive integer m, let $J(m)$ be the set of integers in the interval $[-2^{m-1}, 2^{m-1})$; it should be clear from the context whether we intend m to be an integer or Lebesgue measure. Given a multiindex β in Z_+^K, let $J(\beta)$ be the cartesian product of the sets $J(\beta_k)$. Also let 2^β be the multiindex with components 2^{β_k}. Given two multiindices α and γ, denote the multiindex with components $\alpha_k \gamma_k$ by $\alpha\gamma$. For each β in Z_+^K, the sets $J(\beta) + \gamma 2^\beta$ with γ in Z^K are disjoint and cover Z^K.

Given a function d on Z^K, let $\|d\|_{1,2,2^\beta}$ be the quantity obtained by combining norms as follows. First compute the ℓ^1-norm of the restriction of d to each set $J(\beta) + \gamma 2^\beta$. This norm depends on the choice of γ, and hence defines a function on Z^K. Then compute the ℓ^2-norm of that function. Suppose for instance that $K = 2$ and $\beta = (3, 4)$. Then

$$\|d\|_{1,2,2^\beta} = \left\{ \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left[\sum_{m=i2^3-2^2}^{i2^3+2^2-1} \sum_{n=j2^4-2^3}^{j2^4+2^3-1} |d(m, n)| \right]^2 \right\}^{1/2}. \tag{1.2}$$

Call a set $J(\beta)+\gamma 2^\beta$ a *middle translate* if some component of γ is equal to 0; this is equivalent to the set $J(\beta)+\gamma 2^\beta$ having a member with some component equal to 0. Let $\|d\|'_{1,2,2^\beta}$ denote the quantity obtained by proceeding as in the definition of $\|d\|_{1,2,2^\beta}$ but omitting the middle translates. In (1.2), for instance, this means omitting the terms in the sums where one or both of the indices i and j are equal to 0. Then let

$$\|c\|_\Delta \equiv \sum_{\beta \geq 0} \|\Delta c\|'_{1,2,2^\beta}. \tag{1.3}$$

Definition 1.1 *Call a function, c say, on Z^K* regular *if it tends to* 0 *and has the property that* $\|c\|_\Delta < \infty$.

The quantity $\|\Delta c\|'_{1,2,2^\beta}$ is only affected by the sizes of the difference Δc outside the union of the middle translates corresponding to 2^β. For each multiindex α there are only finitely many values of β for which α lies outside this union of middle translates. Because of this, there are nontrivial regular sequences, including radial ones of the form $g(\sum_{k\in I}(\alpha_k)^2)$, where g is any function on $[0,\infty)$ that tends to 0 and has derivatives that tend to 0 rapidly enough.

We begin our description of the symmetry condition that we use by recalling two formulations of it for single-variable series. It is shown in [2] that a series

$$\sum_{n=-\infty}^{\infty} c(n)z^n \tag{1.4}$$

with regular coefficients is integrable if and only if

$$\sum_{m=0}^{\infty} |c(2^m) - c(-2^m)| < \infty. \tag{1.5}$$

In this case, regularity implies that the sequence $(c(n))$ has bounded variation, and then condition (1.5) is equivalent to other conditions like

$$\sum_{n=1}^{\infty} \frac{|c(n) - c(-n)|}{n} < \infty. \tag{1.6}$$

For multiple series, our symmetry condition is a a family of regularity conditions on various new sequences obtained from the original sequence by combining some components in an antisymmetric way. If c is *fully even* in the sense that $c(\alpha)$ is not affected if α_k is replaced by $-\alpha_k$ for any k, then these new sequences are all 0, and c automatically satisfies the symmetry condition. Given integers n and k with k in the set $I \equiv \{1,\ldots,K\}$, consider the restriction of the function c to the set of multiindices α with $\alpha_k = n$. Identify this function in the obvious way with a function, $c_{(k,n)}$ say, on Z^{K-1}, and use the complement of $\{k\}$ in I to index Z^{K-1} in this case. Then form the difference

$$\sigma_{(k,n)}c = c_{(k,n)} - c_{(k,-n)}.$$

When $K=2$, for instance the function c can be represented by a doubly infinite matrix, and $\sigma_{(1,n)}c$ is then the row vector obtained by subtracting row $-n$ of the matrix from row n.

In contrast, $\sigma_{(2,n)}c$ is then the column vector equal to the difference between columns n and $-n$ in the matrix.

Products of these operators $\sigma_{(k,n)}$ with distinct indices k are well defined, and the operators commute. When $K = 2$, applying both of the operators $\sigma_{(1,3)}$ and $\sigma_{(2,5)}$ to c, in either order, yields the constant

$$c(3,5) - c(-3,5) - c(3,-5) + c(-3,-5). \tag{1.7}$$

Given a nonempty subset S of I and a multiindex γ in Z^S, let

$$\sigma_{[S,\gamma]}c \equiv \left[\prod_{k\in S} \sigma_{(k,\gamma_k)}\right] c. \tag{1.8}$$

Use $|S|$ to denote the cardinality of S; then $\sigma_{[S,\gamma]}c$ has $K - |S|$ components. When $|S| < K$, define the functional $\|\cdot\|_\Delta$ on sequences on $Z^{K-|S|}$ as before, except for replacing K by $K-|S|$. If $|S| = K$, then $\sigma_{[S,\gamma]}c$ is a constant, as in (1.7); in this case, define $\|\sigma_{[S,\gamma]}c\|_\Delta$ to be the absolute value of $\sigma_{[S,\gamma]}c$.

Definition 1.2 *Call a function, c say, on Z^K* sufficiently symmetric *if*

$$\|c\|_\Sigma \equiv \sum_{|S|>0} \sum_{\beta\in Z_+^S} \|\sigma_{[S,2^\beta]}c\|_\Delta < \infty. \tag{1.9}$$

In Section 4, we will explain why this is equivalent, for regular sequences c, to either of the following conditions. First, we can then replace 2^β by $(q_k(\beta_k))_{k\in S}$, where each sequence $(q_k(j))_{j=0}^\infty$ takes its values in the set of positive integers and has the property that there are constants C_1 and C_2, with $C_1 > 1$ so that

$$C_1 q_k(j) \le q_k(j+1) \le C_2 q_k(j) \tag{1.10}$$

for all j. Second, we can use all the multiindices that have all positive components, rather than just using those of the form 2^β. Indeed, let P denote the set of positive integers, and given γ in P^S, let $|[\gamma]| = \prod_{k\in S} |\gamma_k|$. We then require that

$$\sum_{|S|>0} \sum_{\gamma\in P^S} \frac{\|\sigma_{[S,\gamma]}c\|_\Delta}{|[\gamma]|} < \infty. \tag{1.11}$$

It is also the case that $\|c\|_\Delta < \infty$ if and only if

$$\sum_{\gamma>0} \frac{\|\Delta c\|'_{1,2,\gamma}}{|[\gamma]|} < \infty; \tag{1.12}$$

here one must also make appropriate modifications to the definitions of the sets $J(\beta)$.

Theorem 1.1 *If the coefficients in the complex form of a multiple trigonometric series form a sequence that is regular and sufficiently symmetric, then the series represents an integrable function.*

In particular, regularity implies integrability for series with fully even coefficients. These are the series (1.1) that can be rewritten as multiple cosine series in the form

$$\sum_{\alpha \geq 0} a(\alpha) \cos(\alpha_1 t_1) \ldots \cos(\alpha_K t_K) \tag{1.13}$$

Let $N(\alpha)$ be the number of components of α that differ from 0. Then $a(\alpha) = 2^{N(\alpha)} c(\alpha)$ for all $\alpha \geq 0$. In this situation, define $\|a\|'_{1,2,2^\beta}$ to be the quantity obtained by proceeding as before except for only using the translates $J(\beta) + \gamma 2^\beta$ with $\gamma > 0$, because these translates are the ones that are included in Z_+^K, the domain of a. Then define $\|a\|_\Delta$ as in (1.3). Say that a is *weakly regular* if it tends to 0 and $\|a\|_\Delta < \infty$. This only restricts the sizes of the differences Δa in the *interior* of Z_+^K, that is on the set of multiindices in Z_+^K with no component equal to 0.

Since $a(\alpha) = 2^K c(\alpha)$ on this set of interior multiindices, it follows that $\|a\|_\Delta < \infty$ if $\|c\|_\Delta < \infty$. Given a subset S of I, let c_S be the restriction of the function c to the set of multiindices α in Z^K with $\alpha_k = 0$ for all k in S; regard c_S as a function on $Z^{K-|S|}$. We will see in Section 4 that regularity of c on Z^K implies that all the functions $c_{(k,n)}$ are regular on Z^{K-1}. Iterating this shows that all the functions c_S must then be regular on their domains.

Definition 1.3 *Call a function, a say, on Z_+^K* regular *if a is weakly regular on Z_+^K and each of the functions a_S is weakly regular on its domain.*

As the discussion before the definition suggests, it turns out that a is regular on Z_+^K if and only if there is a fully even, regular function c on Z^K with $a(\alpha) = 2^{N(\alpha)} c(\alpha)$ for all α in Z_+^K. So the case of Theorem 1.1 where the coefficients are fully even is equivalent to the following statement.

Theorem 1.2 *If the coefficients in a multiple cosine series form a regular sequence, then the series represents an integrable function.*

Multiple sine series

$$\sum_{\alpha > 0} b(\alpha) \sin(\alpha_1 t_1) \ldots \sin(\alpha_K t_K) \tag{1.14}$$

have complex forms (1.1) in which the coefficients $(c(\alpha))$ are *fully odd* in the sense that they are odd in each component. In particular, $c(\alpha) = 0$ whenever α has some component equal to 0, and $b(\alpha) = (2i)^K c(\alpha)$ for all α in P^K. Sufficient symmetry of the sequence c corresponds to a combination of conditions on the restrictions of the sequence b to certain subsets of P^K. Given an integer k with $1 \leq k \leq K$ and a positive integer n, define a new sequence $b_{(k,n)}$ much as before, except that its domain is now identified with P^{K-1} indexed by the complement of $\{k\}$ in I, rather than being identified with Z^{K-1}. Let $\rho_{(k,n)}$ be the operator that maps the sequence b to the sequence $b_{(k,n)}$, and let

$$\rho_{[S,\gamma]} b \equiv \left[\prod_{k \in S} \rho_{(k,\gamma_k)} \right] b. \tag{1.15}$$

whenever S is a subset of I and $\gamma \in P^S$. Define quantities $\|\rho_{[S,\gamma]} b\|_\Delta$ by restricting all calculations to $P^{K-|S|}$.

Definition 1.4 *Call a sequence* $(b(\alpha))_{\alpha\in P^K}$ strongly regular *if it is weakly regular, and*

$$\|b\|_\Sigma \equiv \sum_{|S|>0} \sum_{\beta\in Z_+^S} \|\rho_{[S,2^\beta]}b\|_\Delta < \infty. \tag{1.16}$$

Again, it is equivalent to require that the weakly regular sequence b have the property that

$$\sum_{|S|>0} \sum_{\gamma\in P^S} \frac{\|\rho_{[S,\gamma]}b\|_\Delta}{|[\gamma]|} < \infty. \tag{1.17}$$

The case of Theorem 1.1 where the coefficients are fully odd is equivalent to the following statement.

Theorem 1.3 *If the coefficients in a multiple sine series form a sequence that is strongly regular, then the series represents an integrable function.*

The counterparts of Theorems 1.2 and 1.3 for single-variable series were proved in [4], [5], and [2]. For such sine series with regular coefficients the symmetry condition is also necessary [5, 2] for integrability. In Section 7, we will explain why we doubt that this extends to multiple series. In Section 4, we will complete the subdivision of Theorem 1.1 into special cases by dealing with multiple trigonometric series having real forms with terms

$$d(\alpha)\left[\prod_{k\in S}\cos(2\pi\alpha_k t_k)\right]\left[\prod_{k\notin S}\sin(2\pi\alpha_k t_k)\right] \tag{1.18}$$

where S is a fixed proper subset of I.

Another complete orthonormal system on (X, m) consists of tensor products of Walsh functions. The single-variable Walsh system is often indexed by Z_+, in the Paley ordering [16]. This leads to an indexing of the set of tensor products of Walsh functions as $(w_\alpha)_{\alpha\geq 0}$.

Theorem 1.4 *If the coefficients in a multiple Walsh series form a regular sequence, then the series represents an integrable function.*

Our proof of this is simpler than our proof of Theorem 1.1. Since many of the same issues arise in both cases, the reader who is familiar with Walsh series may wish to turn next to Section 5 to see how we deal with these issues for multiple Walsh series, and then read the proof, in Sections 2–4, for multiple trigonometric series.

2 Outline of the proof for multiple trigonometric series

In this section, we offer a formal argument that reduces the proof of Theorem 1.1 to a sequence of lemmas that we will prove in Section 3. We will justify the formal argument in Section 4.

We follow the plan that we used [2] for single-variable series, but the extra variables here make some steps more complicated, and we have to deal with combinations that do not arise in the case of single-variable series. Given a series in the complex form (1.1) with regular coefficients, we multiply it formally by

$$h(t) = \prod_{k=1}^{K} [1 - e^{-2\pi i t_k}]. \tag{2.1}$$

This yields the series

$$\sum_{\alpha \in Z^K} \Delta c(\alpha) z^\alpha(t). \tag{2.2}$$

We will show in Section 4 that

$$\|\Delta c\|_1 \leq C \|c\|_\Delta. \tag{2.3}$$

Thus the series (2.2) converges absolutely, to $f(t)$ say. So, at least formally, integrability of the series (1.1) is equivalent to integrability of the function F that maps t to $f(t)/h(t)$.

When $t_k \in [-1/2, 1/2)$, the absolute-value of $[1 - e^{-2\pi i t_k}]$ lies between $4|t_k|$ and $2\pi|t_k|$. Hence integrability of the series (1.1) is formally equivalent to that of the function f with respect to the measure $w(t)\,dt$ where

$$w(t) = \frac{1}{\prod_{k=1}^{K} |t_k|}. \tag{2.4}$$

To deal with this weighted integrability question, we split the domain into pieces where $w(t)$ is nearly constant. For each nonnegative integer m, let $E(m)$ be the set of real numbers with absolute values lying in the interval $(1/2^{m+2}, 1/2^{m+1}]$. Given a multiindex β in Z_+^K, let $E(\beta)$ be the cartesian product of the sets $E(\beta_k)$.

We follow the standard convention concerning notation for averages of integrable functions over measurable sets with positive measure. If G is such a function on X and E is such a subset of X, let

$$G_E = \frac{1}{m(E)} \int_E G\,dm$$

Denote $\sum_{k=1}^{K} |\beta_k|$ by $|\beta|$. The set $E(\beta)$ has measure $2^{-|\beta|-K}$. On it, the values taken by $w(t)$ all lie between $2^{|\beta|+K}$ and $2^{|\beta|+2K}$. So the function f is integrable with respect to the weight w if and only if

$$\sum_{\beta \geq 0} |f|_{E(\beta)} < \infty. \tag{2.5}$$

Most of the analysis in this section will focus on this condition and variants of it.

Given a multiindex β in Z_+^K and a subset R of I, let $W(R, \beta)$ be the set of all multiindices α for which $\alpha_k \in [-2^{\beta_k}, 2^{\beta_k})$ if $k \in R$, and $\alpha_k \notin [-2^{\beta_k}, 2^{\beta_k})$ if $k \notin R$. Then let $f_{\beta,R}(t)$ be the sum of the terms in the series (2.2) for which $\alpha \in W(R, \beta)$. Since the series (2.2) converges absolutely, so do the partial series $f_{\beta,R}(t)$, and

$$f(t) = \sum_{R \subset I} f_{\beta,R}(t). \tag{2.6}$$

This splits f into 2^K pieces tailored to β. We also denote $f_{\beta,I}$ by s_β, and $f_{\beta,\emptyset}$ by T_β, and we call these pieces the *inner* and *outer* sums respectively. They are the only pieces that arise when $K = 1$.

The Fourier coefficients of T_β are equal to the coefficients in the series defining T_β because that series converges absolutely. So $\widehat{T_\beta}$ vanishes on the union of the middle translates of $J(\beta)$, and on its support $\widehat{T_\beta}$ coincides with Δc. Hence

$$\|\widehat{T_\beta}\|_{1,2,2^\beta} \leq \|\Delta c\|'_{1,2,2^\beta}. \tag{2.7}$$

This allows us to estimate certain L^2-averages.

Lemma 2.1 *There is a constant C so that, for any integrable function G and any multiindex β in Z_+^K,*

$$\left\{\left[|G|^2\right]_{E(\beta)}\right\}^{1/2} \leq C\|\hat{G}\|_{1,2,2^\beta}. \tag{2.8}$$

We will prove this in Section 3. We note here that

$$|G|_{E(\beta)} \leq \left\{\left[|G|^2\right]_{E(\beta)}\right\}^{1/2},$$

by Cauchy-Schwarz. Combining this with inequalities (2.8) and (2.7) yields that

$$\sum_{\beta\geq 0} |T_\beta|_{E(\beta)} \leq C\|c\|_\Delta. \tag{2.9}$$

We work with modifications of the functions $f_{\beta,R}$ when $|R|>0$. Given such a set R and the function f, we form a new function f^R by setting $t_k=0$ for all k in R. When $K=3$ for instance,

$$f^{\{1,3\}}(t) = f(0,t_2,0).$$

In contrast to our policy for the functions c_S, we initially regard f^R as a function on the interval $[-1/2,1/2)^K$, although the value of f^R at t does not change if we change the values of t_k with k in R. We let $f^\emptyset = f$. We then form the combination

$$f^{(R)} \equiv \sum_{S\subset R} (-1)^{|S|} f^S. \tag{2.10}$$

When $K=2$,

$$f^{(\{1,2\})}(t) = f(t_1,t_2) - f(0,t_2) - f(t_1,0) + f(0,0).$$

We use notation like $f^R_{\beta,R}$ for what we should, strictly speaking, denote by $(f_{\beta,R})^R$. Given a set E, we denote its indicator function by 1_E.

Lemma 2.2 *There is a constant C so that*

$$\sum_{\beta\geq 0} \left\|s_\beta^{(I)} \cdot 1_{E(\beta)}\right\|_\infty \leq C\|\Delta c\|_1. \tag{2.11}$$

It follows that

$$\sum_{\beta\geq 0} \left|s_\beta^{(I)}\right|_{E(\beta)} \leq C\|\Delta c\|_1. \tag{2.12}$$

There are estimates for mixtures of suprema and L^2-norms for functions derived from the intermediate pieces $f_{\beta,R}$, and these lead to estimates for L^2-averages.

Lemma 2.3 *There is a constant C so that*

$$\sum_{\beta \geq 0} \left\{ \left[\left| f_{\beta,R}^{(R)} \right|^2 \right]_{E(\beta)} \right\}^{1/2} \leq C \|c\|_\Delta \tag{2.13}$$

for all proper subsets R of I.

Therefore,

$$\sum_{0<|R|<K} \sum_{\beta \geq 0} \left| f_{\beta,R}^{(R)} \right|_{E(\beta)} \leq 2^K C \|c\|_\Delta. \tag{2.14}$$

Combining inequalities (2.9), (2.12), (2.3), and (2.14) yields that if c is regular then

$$\sum_{\beta \geq 0} \sum_{R \subset I} \left| f_{\beta,R}^{(R)} \right|_{E(\beta)} \leq C' \|c\|_\Delta. \tag{2.15}$$

As in the discussion around (2.5), this means that the function $\tilde{f}$ that is defined to coincide with $\sum_{R \subset I} f_{\beta,R}^{(R)}$ on each of the sets $E(\beta)$ is integrable with respect to the weight w. Thus the issue of integrability for a series with regular coefficients reduces to the question of weighted integrability for the difference $f - \tilde{f}$.

We will see that this difference is essentially a sum of functions each of which depends locally on fewer than K variables. So the regularity hypothesis in Theorem 1.1 allows us to reduce the integrability question to similar questions about functions of fewer variables. The symmetry hypothesis will allow us to show that these functions are integrable with respect to appropriate weights.

The decomposition (2.6) came from splitting $\hat{f}$ into pieces supported by disjoint subsets of Z^K. In the analysis that follows, it seems better to work with sets that overlap, and then use the inclusion-exclusion principle. Given a multiindex $\beta \geq 0$ and a nonempty subset R of I, let $Q(R, \beta)$ be the set of multiindices α for which $\alpha_k \in [-2^{\beta_k}, 2^{\beta_k})$ for all k in R. Let $f_{Q(R,\beta)}(t)$ be the sum of the terms in the series (2.2) for which $\alpha \in Q(R, \beta)$. In contrast to the definition of $f_{\beta,R}$, there is no restriction here on the indices α_k for which $k \notin R$. By the inclusion-exclusion principle,

$$f_{\beta,R} = \sum_{U \supset R} (-1)^{|U|-|R|} f_{Q(U,\beta)}.$$

So

$$\sum_{R \subset I} f_{\beta,R}^{(R)} = \sum_{R \subset I} \sum_{S \subset R} (-1)^{|S|} \sum_{U \supset R} (-1)^{|U|-|R|} f_{Q(U,\beta)}^S.$$

On the set $E(\beta)$, the function $\tilde{f}$ coincides with this expansion, and the function f coincides with the sum of the terms in it with $S = \emptyset$. To expand the difference $\tilde{f} - f$, remove the terms where $S = \emptyset$, and change the order to summation to get

$$\sum_{U \subset I} (-1)^{|U|} \sum_{\emptyset \neq S \subset U} (-1)^{|S|} f_{Q(U,\beta)}^S \sum_{S \subset R \subset U} (-1)^{-|R|}.$$

The inner sum vanishes unless $S = U$, so that on the set $E(\beta)$

$$\tilde{f} - f = \sum_{\emptyset \neq S \subset I} (-1)^{|S|} f_{Q(S,\beta)}^S. \tag{2.16}$$

To show that $\tilde{f} - f$ is integrable with respect to the weight w, we show that

$$\sum_{\beta \geq 0} \left| f^S_{Q(S,\beta)} \right|_{E(\beta)} < \infty \tag{2.17}$$

whenever S is a nonempty subset of I.

By symmetry, it is sufficient to do this when S has the form $\{M+1, \ldots, K\}$ for some integer $M < K$. Then the quantity $f^S_{Q(S,\beta)}(t)$ does not depend on the components t_k with $k > M$, because it is computed by setting these components equal to 0. On the other hand, only the components β_k with $k > M$ matter in defining the set $Q(S,\beta)$ and the function $f^S_{Q(S,\beta)}$. To exploit these patterns, we write $\beta = (\mu, \nu)$, where $\mu \in Z^M_+$ and $\nu \in Z^{K-M}_+$, and we split t in a similar way as (u, v), with u in $[-1/2, 1/2)^M$ and v in $[-1/2, 1/2)^{K-M}$. We write $u = Pt$ and $\nu = R\beta$ in these cases.

Now define a function $h^{(\nu)}$ on $[-1/2, 1/2)^M$ by letting

$$h^{(\nu)}(u) = f^S_{Q(S,(0,\nu))}((u,0));$$

as noted above, this is also equal to $f^S_{Q(S,\beta)}(t)$ whenever $Pt = u$ and $R\beta = \nu$. Moreover, applying this definition to the absolutely convergent series (2.2) yields that

$$h^{(\nu)}(u) = \sum_{\lambda \in Z^M} \tau_{[S,2^\nu]}(\lambda) z^\lambda(u), \tag{2.18}$$

where

$$\tau_{[S,2^\nu]} c(\lambda) \equiv \sum_{-2^\nu \leq \eta < 2^\nu} \Delta c(\lambda, \eta). \tag{2.19}$$

and $z^\lambda(u) \equiv \prod_{k=1}^M z_k(u_k)^{\lambda_k}$. Denote the set $\{1, \ldots, M\}$ by U, and factor the operator Δ as

$$\left[\prod_{k \in U} \Delta_k\right] \left[\prod_{k \in S} \Delta_k\right] = \Delta_U \Delta_S,$$

say. Use this in (2.19) to get that

$$\tau_{[S,2^\nu]} c(\lambda) = \Delta_U \sum_{-2^\nu \leq \eta < 2^\nu} \Delta_S c(\lambda, \eta). \tag{2.20}$$

Because the way that the terms in it are mixed differences, the sum in (2.20) collapses to $\sigma_{[S,2^\nu]}(\lambda)$. Therefore

$$h^{(\nu)}(u) = \sum_{\lambda \in Z^M} \Delta_U \left\{\sigma_{[S,2^\nu]} c\right\}(\lambda) z^\lambda(u). \tag{2.21}$$

This series is related to

$$\sum_{\lambda \in Z_M} \sigma_{[S,2^\nu]} c(\lambda) z^\lambda(u) \tag{2.22}$$

in the same way that (2.2) is related to (1.1). The hypothesis that the sequence c is sufficiently symmetric ensures that the coefficients in (2.22) are regular. It also guarantees that they are sufficiently symmetric since applying further antisymmetrizing operators $\sigma_{(k,n)}$ with $k \le M$ to these coefficients just yields sequences $\sigma_{[V,\gamma]}c$ for sets V that strictly include S.

Because $f^S_{Q(S,\beta)}(t)$ does not depend on the last $K-M$ components of t, and because $f^S_{Q(S,\beta)}$ projects to $h^{(\nu)}$ and $E(\beta)$ projects to $E(\mu)$, the averages $|f^S_{Q(S,\beta)}|_{E(\beta)}$ and $|h^{(\nu)}|_{E(\mu)}$ are equal. So

$$\sum_{\beta \ge 0} \left| f^S_{Q(S,\beta)} \right|_{E(\beta)} = \sum_{\nu \ge 0} \sum_{\mu \ge 0} |h^\nu|_{E(\mu)}. \tag{2.23}$$

We will prove inequality (2.5) by showing that there is an absolute constant C_K so that

$$\sum_{\beta \ge 0} |f|_{E(\beta)} \le C_K \left(\|c\|_\Delta + \|c\|_\Sigma \right), \tag{2.24}$$

whenever the function f and the sequence c are related as specified at the beginning of this section. If we had already proved inequality (2.5) in some other way, then it would follow from the closed-graph theorem that there must be a constant C_K with this property.

We verify (2.24) by induction on K. The case where $K = 1$ is proved, by some of the methods in this paper, in [2]. Assume that the counterparts of the inequality holds in all dimensions less than K. Applying this assumption to the series (2.22) yields that the inner sum on the right side of (2.23) is bounded above by

$$C_{K-|S|} \left(\|\sigma_{[S,2^\nu]}c\|_\Delta + \|\sigma_{[S,2^\nu]}c\|_\Sigma \right).$$

So the left side of (2.23) is bounded above by

$$C_{K-|S|} \sum_{\nu \ge 0} \left(\|\sigma_{[S,2^\nu]}c\|_\Delta + \|\sigma_{[S,2^\nu]}c\|_\Sigma \right). \tag{2.25}$$

By the comments about the regularity and sufficient symmetry of the sequence of coefficients in (2.22), the quantity (2.25) is bounded above by $C'\|c\|_\Sigma$. Combining these bounds for all nonempty subsets S of I, and using inequality (2.15) yields inequality (2.24).

3 Proofs of the lemmas

We postpone the proof of Lemma 2.1 until the end of this section, because it uses the same ideas as the proof of its single-variable counterpart in [2], and we do not need these ideas to prove the other lemmas. We will reuse the ideas in the proof of Lemma 2.2, and we begin with it.

Given a point t in the set X, let D_t be the set of all points, r say, in X for which r_k lies between 0 and t_k for all k. Then $s^{(I)}_\beta(t)$ is equal to the integral over the set D_t of the mixed partial derivative obtained by differentiating s_β once with respect to each component. On this set, the absolute value of this mixed partial derivative is bounded above by

$$\sum_{\alpha \in W(I,\beta)} |\Delta c(\alpha)| \prod_{k=1}^{K} |2\pi\alpha_k|.$$

Since the set D_t has measure less than $2^{-|\beta|-K}$ for all t in $E(\beta)$,

$$\left\| s_\beta^{(I)} \cdot 1_{E(\beta)} \right\|_\infty \leq (2\pi)^K 2^{-|\beta|-K} \sum_{\alpha \in W(I,\beta)} |\Delta c(\alpha)|\, |[\alpha]|. \tag{3.1}$$

It follows that

$$\sum_{\beta \geq 0} \left\| s_\beta^{(I)} \cdot 1_{E(\beta)} \right\|_\infty \leq \pi^K \sum_{\alpha \in Z^K} \left[\sum_{\alpha \in W(I,\beta)} 2^{-|\beta|} \right] |[\alpha]|\, |\Delta c(\alpha)|. \tag{3.2}$$

Fix α in Z^K. Among the multiindices β in Z_+^K with the property that $\alpha \in W(I,\beta)$ there is a unique one, $\beta(\alpha)$ say, for which $2^{|\beta|}$ is minimal. Then $|[\alpha]| \leq 2^{|\beta(\alpha)|}$, and $\alpha \in W(I,\gamma)$ if and only if $\gamma \geq \beta(\alpha)$. So in (3.2) the term $|\Delta c(\alpha)|$ is multiplied by no more than

$$2^{|\beta(\alpha)|} \sum_{\gamma \geq \beta(\alpha)} 2^{-|\gamma|} = \sum_{\gamma \geq 0} 2^{-|\gamma|}. \tag{3.3}$$

The sum on the right is equal to $(\sum_{j=0}^\infty 2^{-j})^K = 2^K$. Hence the right side of (3.2) is majorized by $(2\pi)^K \|\Delta c\|_1$. This completes the proof of Lemma 2.2.

We prove Lemma 2.3 by combining the methods in the proof of Lemma 2.2 with the validity of Lemma 2.1 for series in fewer variables. Suppose again that the set S has the form $\{M+1, \ \ldots, \ K\}$ for some integer $M < K$. Again split multiindices β in Z_+^K as $(\mu, \ \nu)$ and points t in X as (u, v). Recall that the function $f_{\beta,S}$ was defined via a set $W(S,\beta)$ of multiindices in Z_+^K. Given α in $W(S,\beta)$, split it as (ζ, η). Then $\zeta \in W(\emptyset, \mu)$, and $\eta \in W(S, \ \nu)$.

This allows us to rewrite the series for $f_{\beta,S}^{(S)}(u,v)$ as

$$\sum_{\zeta \in W(\emptyset,\mu)} \left\{ \sum_{\eta \in W(S,\nu)} \Delta c(\zeta,\eta) [z^\eta]^{(S)}(v) \right\} z^\zeta(u). \tag{3.4}$$

For fixed v, this becomes an outer sum, $T_\mu^{(v,\nu)}(u)$ say, for a series in M variables. Lemma 2.1 applies to this sum, and yields useful estimates provided that we can control the sizes of the coefficients of $T_\mu^{(v,\nu)}$. We only need such estimates when $(u,v) \in E(\beta)$, and then the methods in the proof of Lemma 2.2 show that

$$\left| \sum_{\eta \in W(S,\nu)} \Delta c(\zeta,\eta) [z^\eta]^{(S)}(v) \right| \leq \sum_{\eta \in W(S,\nu)} |\Delta c(\zeta,\eta)| 2^{-|\nu|} \pi^{|S|} |[\eta]|. \tag{3.5}$$

Note that this upper bound is uniform in v.

In the sum on the right in (3.5), there is actually no contribution from the terms where some component of η vanishes, because then $|[\eta]| = 0$. Every other index η in the sum belongs to some *principal translate* of the form $J(\delta) + \theta 2^\delta$, where δ and θ belong to Z_+^S and Z^S respectively, with $\delta < \nu$ and each component of θ equal to ± 1. Consider the part

$$d_{\delta,\theta}^{(\nu)}(\zeta) \equiv \pi^{|S|} 2^{-|\nu|} \sum_{\eta \in [J(\delta)+\theta 2^\delta]} |\Delta c(\zeta,\eta)| |[\eta]|, \tag{3.6}$$

of the sum (3.5), corresponding to one such principal translate. For the indices η in (3.6), the factor $|[\eta]|$ is at most $2^{|\delta|+|S|}$. So the ℓ^1-norm of $d^{(\nu)}_{\delta,\theta}$ on the set $J(2^\mu)+\xi 2^\mu$ is bounded above by

$$\pi^{|S|}2^{-|\nu-\delta|+|S|}\sum_{\zeta\in[J(2^\mu)+\xi 2^\mu]}\ \sum_{\eta\in[J(\delta)+\theta 2^\delta]}|\Delta c(\zeta,\eta)|\,, \tag{3.7}$$

which is equal to $(2\pi)^{|S|}2^{-|\nu-\delta|}$ times the ℓ^1-norm of the restriction of the sequence Δc on the set $J((\mu,\ \delta))+(\xi,\ \theta)2^{(\mu,\ \delta)}$.

If no component of ξ vanishes, then this set is not a middle translate of $J((\mu,\delta))$, and it is included in the computation of $\|\Delta c\|_{1,2,2^{(\mu,\delta)}}$. Squaring the estimate (3.7) and adding over all indices ξ in Z^M with no vanishing components yields that

$$\|d^{(\nu)}_{\delta,\theta}\|'_{1,2,2^\mu}\le(2\pi)^{|S|}2^{-|\nu-\delta|}\|\Delta c\|'_{1,2,2^{(\mu,\delta)}}.$$

Form another sequence $d^{(\nu)}_\delta$ by adding together all the sequences $d^{(\nu)}_{\delta,\theta}$ with each component of θ equal to ±1. Then

$$\|d^{(\nu)}_{\delta}\|'_{1,2,2^\mu}\le 2^{|S|}(2\pi)^{|S|}2^{-|\nu-\delta|}\|\Delta c\|'_{1,2,2^{(\mu,\delta)}}.$$

Summing these estimates over all choices of δ with $\delta\le\nu$ provides the upper bound

$$(4\pi)^{|S|}\sum_{0\le\delta\le\nu}2^{-|\nu-\delta|}\|\Delta c\|'_{1,2,2^{(\mu,\delta)}}$$

for the value of the functional $\|\cdot\|'_{1,2,2^\mu}$ applied to the coefficients of $T^{(v,\nu)}_\mu$. Since these coefficients vanish on the middle translates of $J(\mu)$, we can use $\|\cdot\|'_{1,2,2^\mu}$ and $\|\cdot\|_{1,2,2^\mu}$ interchangeably here.

By Lemma 2.1 in M variables,

$$\left\{\left[\left|T^{(v,\nu)}_\mu\right|^2\right]_{E(\mu)}\right\}^{1/2}\le C_M(4\pi)^{|S|}\sum_{0\le\delta\le\nu}2^{-|\nu-\delta|}\|\Delta c\|'_{1,2,2^{(\mu,\delta)}}.$$

This holds uniformly in v, and so also gives an upper bound for

$$\left\{\left[\left|f^{(S)}_{\beta,S}\right|^2\right]_{E(\beta)}\right\}^{1/2}.$$

Adding these bounds for all $\beta=(\mu,\nu)\ge0$ gives no more than

$$C_M(4\pi)^{|S|}\sum_{\mu\ge0}\sum_{\delta\ge0}\|\Delta c\|'_{1,2,2^{(\mu,\delta)}}\sum_{\nu\ge\delta}2^{-|\nu-\delta|}. \tag{3.8}$$

As in the proof of Lemma 2.2, the inner sum in (3.8) is equal to $2^{|S|}$, so that the triple sum (3.8) is equal to

$$C_M(8\pi)^{|S|}\|c\|_\Delta.$$

This completes the proof of Lemma 2.3.

We have two proofs of Lemma 2.1, one using known estimates for Fourier transforms on R^K, and one that is more self-contained. We begin both proofs by rewriting the lemma in the form stating that

$$\|G \cdot 1_{E(\beta)}\|_2 \leq Cm(E(\beta))^{1/2}\|\hat{G}\|_{1,2,2^\beta} \tag{3.9}$$

for all integrable functions G on the set X. By duality, this is equivalent to the assertion that

$$\|\hat{H}\|_{\infty,2,2^\beta} \leq Cm(E(\beta))^{1/2}\|H\|_2 \tag{3.10}$$

for all integrable functions H that vanish off the set $E(\beta)$. The functional $\|\cdot\|_{\infty,2,2^\beta}$ is computed like $\|\cdot\|_{1,2,2^\beta}$ except that that the first step is to take ℓ^∞-norms over various translates of the set $J(\beta)$ rather ℓ^1-norms.

Extend H to all of R^K by making it identically 0 off X. Then (3.10) follows from the validity of the corresponding estimate for $\|\hat{H}\|_{\infty,2,2^\beta}$ for all integrable functions H on R^K with the property that H vanishes off $E(\beta)$. Now the symbol $\hat{}$ denotes the Fourier transform on R^K, and $\|\hat{H}\|_{\infty,2,2^\beta}$ is computed using a cover of R^K by disjoint translates of the set $\prod_{k=1}^K[-2^{\beta_k-1}, 2^{\beta_k-1})$. Let $\bar{\gamma}$ be the multiindex with all components equal to 1. Changing variables appropriately reduces matters to showing that

$$\|\hat{H}\|_{\infty,2,\bar{\gamma}} \leq C\|H\|_2 \tag{3.11}$$

for all integrable functions H on R^K that vanish off the set X. This inequality is known [6], and has been proved in various ways [8] as a cornerstone for the Hausdorff-Young theorem for amalgams.

To prove inequality (3.9) more directly, we split the set $E(\beta)$ as a union of 2^K disjoint translates of the set, $D(\beta)$ say, consisting of all points t in X with $-2^{\beta_k-3} \leq t_k < 2^{\beta_k-3}$. Since translating a function does not change the absolute values of its Fourier coefficients, it suffices to prove a version of inequality (3.9) with $E(\beta)$ replaced by $D(\beta)$. Consider the function

$$g = 2^{-|\beta|}\sum_{\alpha\in J(\beta)} z^\alpha.$$

Then the real part of $g(t)$ is bounded below by $(1/\sqrt{2})^K$ for all t in $D(\beta)$. So

$$\|G \cdot 1_{D(\beta)}\|_2 \leq 2^{K/2}\|G \cdot g\|_2 = 2^{K/2}\|\hat{G} * \hat{g}\|_2,$$

where $*$ denotes convolution on Z^K.

Now split $\hat{G}$ into pieces, P_γ say, supported by the various sets $J(\beta) + \gamma 2^\beta$ that are considered in computing $\|\hat{G}\|_{1,2,2^\beta}$. By Young's inequality for convolution,

$$\|\hat{g} * P_\gamma\|_2 \leq \|\hat{g}\|_2\|P_\gamma\|_1 = 2^{-|\beta|/2}\|P_\gamma\|_1.$$

It may happen that the various functions $\hat{g} * P_\gamma$ have disjoint supports. In that special case,

$$(\|\hat{g} * \hat{G}\|_2)^2 = \sum_{\gamma\in Z^K}(\|\hat{g} * P_\gamma\|_2)^2 \leq \sum_{\gamma\in Z^K} 2^{-|\beta|}(\|P_\gamma\|_1)^2. \tag{3.12}$$

The right side of (3.12) is just $2^{-|\beta|}(\|\hat{G}\|_{1,2,2^\beta})^2$. So

$$\|G \cdot 1_{D(\beta)}\|_2 \leq 2^{(K-|\beta|)/2}\|\hat{G}\|_{1,2,2^\beta} \tag{3.13}$$

in this case. This matches the pattern in (3.9) because

$$m(D(\beta)) = 2^{-2K-|\beta|}.$$

The functions $\hat{g} * P_\gamma$ will have disjoint support if $P_\gamma \equiv 0$ for all choices of γ outside a particular coset of the subgroup $(2Z)^K$ in Z^K. In general, just split $\hat{G}$ into 2^K parts, for each of which the corresponding functions P_γ have this vanishing property outside one of the cosets of $(2Z)^K$, and apply inequality (3.13) to each part of G.

4 Conditions on individual differences

We show first that if c is regular on Z^K, then $\Delta c \in \ell^1$. The ℓ^1-norm of the restriction of Δc to each principal translate of $J(\beta)$ is bounded above by $\|\Delta c\|'_{1,2,2^\beta}$. By the Schwarz inequality, the ℓ^1-norm of the restriction of Δc to the union of all the principal translates of $J(\beta)$ is at most $2^{K/2}\|\Delta c\|'_{1,2,2^\beta}$. As β varies, these translates cover the set of interior multiindices in Z^K. So we get the upper bound $2^{K/2}\|c\|_\Delta$ for the ℓ^1-norm of the restriction of Δc to the set of interior multiindices.

The fact that c tends to 0 then enables us to estimate the ℓ^1-norm of Δc on the rest of Z^K. When $K = 2$ for instance, we have that

$$\sum_{i=-\infty}^{\infty} \Delta c(i,j) = 0 \tag{4.1}$$

for all j, and similarly for sums with i fixed and j varying. Hence,

$$|\Delta c(0,j)| \leq \sum_{i \neq 0} |\Delta c(i,j)|.$$

It then follows that

$$\sum_{j \neq 0} |\Delta c(0,j)| \leq \sum_{i \neq 0}\sum_{j \neq 0} |\Delta c(i,j)| \leq 2^{2/2}\|c\|_\Delta,$$

and similarly for $\sum_{i \neq 0} |\Delta c(i,0)|$. Finally,

$$|\Delta c(0,0)| \leq \sum_{i \neq 0} |\Delta c(i,0)|.$$

Next we show that the functions c_S must all be regular. Again let $K = 2$ for simplicity. Since $\Delta c \in \ell^1$, the series (4.1) converges absolutely; moreover,

$$c(0,j) - c(0,j+1) = \sum_{i=-\infty}^{-1} \Delta c(i,j) \tag{4.2}$$

for all j. Given a nonnegative integer m, let $d_m(j)$ be the part of this sum where

$$i \in [-2^m - 2^{m-1}, -2^m + 2^{m-1}),$$

and estimate as in the proof of Lemma 2.3. Let r be a nonnegative integer and s be an integer. Then

$$\sum_{j=s2^r-2^{r-1}}^{s2^r+2^{r-1}-1} |d_m(j)| \leq \sum_{j=s2^r-2^{r-1}}^{s2^r+2^{r-1}-1} \sum_{i=-2^m-2^{m-1}}^{-2^m+2^{m-1}-1} |\Delta c(i,j)|.$$

Square both sides of this inequality, and add as the index s runs through the set of nonzero integers to get that

$$\|d_m\|'_{1,2,2^r} \leq \|\Delta c\|'_{1,2,2^{(m,r)}}.$$

Therefore

$$\sum_{r\geq 0} \|d_m\|'_{1,2,2^r} \leq \sum_{r\geq 0} \|\Delta c\|'_{1,2,2^{(m,r)}}.$$

Finally,

$$\|c_{\{1\}}\|_\Delta \leq \sum_{m\geq 0}\sum_{r\geq 0} \|d_m\|'_{1,2,2^r} \leq \sum_{m\geq 0}\sum_{r\geq 0} \|\Delta c\|'_{1,2,2^{(m,r)}} = \|c\|_\Delta.$$

Similar arguments show that all slices of a regular sequence are regular, and that slices of weakly regular sequences on Z_+^K are weakly regular.

It may appear at first sight that only symmetry is needed to verify that multiple cosine series with regular coefficients have complex form with regular coefficients. There is an asymmetry in the definition of Δc, however, and this is one reason for assuming in Theorem 1.2 that various slices a_S are weakly regular. For a fully even sequence c on Z^2 for instance,

$$\Delta c(-1,n) = c(0,n) - c(1,n). \tag{4.3}$$

If the sequence c is related to the sequence a on Z_+^2 as in the discussion leading up to the statement of Theorem 1.2, then the contributions of the differences (4.3) to $\|c\|_\Delta$ can be estimated using the weak regularity of the slices $a_{1,0}$ and $a_{1,1}$.

Given a sequence d on Z^K, also denote the quantity $\sum_{\beta\geq 0} \|d\|'_{1,2,2^\beta}$ by $\|d\|'$. Our analysis actually shows that

$$\|c_{\{1\}}\|_\Delta \leq \|(\Delta c)\cdot 1_H\|', \tag{4.4}$$

where H is the left half of Z^2 consisting of all multiindices with negative first components. In the same way, if $0 < m < n$ and $S(m,n)$ is the strip in Z^2 where $m \leq \alpha_1 < n$, then

$$\|c_{\{n\}} - c_{\{m\}}\|_\Delta \leq \|(\Delta c)\cdot 1_{S(m,n)}\|', \tag{4.5}$$

and similar estimates hold for pairs of negative integers m and n.

Call a sequence $(q(j))_{j=0}^\infty$ of positive integers a *Paley* sequence if it can be split into finitely-many subsequences each of which satisfies the counterpart of condition (1.10) with constants C_2 and $C_1 > 1$. Another characterization of Paley sequences is the fact [14] that for each of them there is a constant C so that the sequence has at most C terms in each interval $[2^m, 2^{m+1})$. It follows from (4.5) that if c is regular on Z^2, then

$$\sum_{j=0}^\infty \|\sigma_{[1,q(j)]}c\|_\Delta < \infty \tag{4.6}$$

for one such sequence q if and only (4.6) is true for all Paley sequences q. Similar statements hold on Z^K when $K > 2$. So we can safely replace the powers of 2 in the definition of sufficient symmetry by Paley sequences.

Given a regular sequence c, one way to construct a Paley sequence is to let each $q(j)$ be an integer n in the interval $[2^j, 2^{j+1})$ chosen so that $\|\sigma_{[1,n]}c\|_\Delta$ is minimal compared to the values of this functional for other choices of n in $[2^j, 2^{j+1})$. An alternative is to select the largest value of $\|\sigma_{[1,n]}c\|_\Delta$ in each dyadic interval. Condition (4.6) holds for both of these choices of q or for neither of them. It follows that such a regular sequence c is sufficiently symmetric if and only if condition (1.11) is satisfied.

Another consequence of this analysis is that sufficient symmetry is not affected by translation, although $\|\cdot\|_\Sigma$ is. It is also true that every translate of a regular sequence is regular, although the functional $\|\cdot\|_\Delta$ is not invariant under translation.

We now justify the formal calculations that we used in Section 2 to reduce the integrability of (1.1) to the weighted integrability of f. Assume that the sequence c is regular and sufficiently symmetric. Define $f(t)$ to be the sum of the absolutely convergent series (2.2). We have shown that the hypotheses on c imply the integrability of $f(t)$ with respect to $w(t)dt$. So the function $F : t \mapsto f(t)/h(t)$ is integrable.

Writing $f(t) = F(t)h(t)$ and computing the integrals giving the Fourier coefficients of f shows that $\hat{f}(\alpha) = \Delta\hat{F}(\alpha)$ for all α. But also $\hat{f} = \Delta c$ because of the absolute convergence of the series defining f. Thus the sequence $\hat{F} - c$ tends to 0, and $\Delta[\hat{F} - c] \equiv 0$. It is easy to see that if a sequence on Z tends to 0 and the first difference of the sequence is identically 0, then the sequence must be constant, and hence identically 0. Iterating this observation shows that $\hat{F} - c \equiv 0$ here. So the series (1.1) does indeed represent the integrable function F.

Now consider mixed sine and cosine series with terms as in (1.18), that is with cosines occurring for indices α_k for k's in a fixed subset S of I, and sines occurring for the other components of α. The coefficients $d(\alpha)$ in such a series only need to be defined initially on the part of Z_+^K where $\alpha_k > 0$ for all k in the complement U of S in I. We adopt the convention that $d \equiv 0$ on the rest of Z_+^K. Then the only subsets R of I for which the slices d_R can be nontrivial are those with $R \subset S$.

The complex form of such a mixed series will have coefficients, $c(\alpha)$ say, that are even as functions of the components α_k with k in S, and odd as functions of the other components. These coefficients will form a regular sequence if and only if the coefficients $d(\alpha)$, extended to all of Z_+^K as above, form a regular sequence on Z_+^K. On the other hand, sufficient symmetry of c is equivalent to the condition that

$$\sum_{\emptyset \neq R \subset U} \sum_{\beta \in Z_+^R} \|\rho_{[R,2^\beta]}d\|_\Delta < \infty. \tag{4.7}$$

When c is weakly regular, this is equivalent to requiring that

$$\sum_{\emptyset \neq R \subset U} \sum_{\gamma \in P^R} \frac{\|\rho_{[R,\gamma]}d\|_\Delta}{|[\gamma]|} < \infty. \tag{4.8}$$

So for a mixed series with terms (1.18) to be integrable it suffices that the coefficients satisfy three types of conditions. First, they should form a weakly regular sequence; this restricts the sizes of the mixed differences on P^K. Second, the slices d_R with $\emptyset \neq R \subset S$ should also be weakly regular; this restricts the sizes of lower-order differences on the significant part of $Z_+^K \setminus P^K$. Third, the slices $\rho_{[R,\gamma]}d$ with $\emptyset \neq R \subset U$ should satisfy conditions (4.7) and (4.8).

5 Walsh series

In order to allow people to read this section before reading the proof for trigonometric series, we specify some of our notational conventions again. We also modify some notation to suit the case of multiple Walsh series. Given a multiindex β in Z_+^K, let $I(\beta)$ be the set of multiindices α in Z^K for which $0 \leq \alpha < 2^\beta$. Denote the indicator function of a given set E by 1_E, and let $\bar{\gamma}$ be the multiindex with all components equal to 1. It is easy to check that a sequence a on Z_+^K is weakly regular if and only if a tends to 0 and

$$\sum_{\beta\geq 0}\left[\sum_{\gamma\geq\bar{\gamma}}\left\{\left\|(\Delta a)\cdot 1_{I(\beta)+\gamma 2^\beta}\right\|_1\right\}^2\right]^{1/2} < \infty. \tag{5.1}$$

For this section only, we redefine $\|a\|_\Delta$ to be the left side above, and $\|a\|'_{1,2,2^\beta}$ to be the quantity that is summed on β in (5.1); we also redefine the quantities $\|a_S\|_\Delta$ for various subsets S of I in the same way.

The Walsh functions are often regarded as being initially defined on the interval $[0, 1)$, but also being extended periodically to the real line. This allows us to shift the set X, with no effect on integrability, so that, in this section only, the symbol X denotes the product $[0, 1)^K$. In this context, given a nonnegative integer m, let $E(m)$ be the interval $(1/2^{m+1}, 1/2^m]$, and given β in Z_+^K, let $E(\beta)$ be the cartesian product of the sets $E(\beta_k)$.

We suppose at first that the series

$$\sum_{\alpha\geq 0} a(\alpha)w_\alpha(t) \tag{5.2}$$

converges absolutely, to $F(t)$ say, and we estimate $\|F\|_1$ in terms of the various quantities $\|a_S\|_\Delta$. Clearly,

$$\|F\|_1 = \sum_{\beta\geq 0}\|F\cdot 1_{E(\beta)}\|_1. \tag{5.3}$$

Since the series (5.2) converges absolutely, we can freely regroup the terms in it in any convenient way. Given a multiindex β in Z_+^K, let $\hat{\beta} = \beta + \bar{\gamma}$, and then split (5.2) into pieces of the form

$$\sum_{\alpha\in I(\hat{\beta})+\gamma 2^{\hat{\beta}}} a(\alpha)w_\alpha(t),$$

where $\gamma \geq 0$. Factor this piece as

$$w_{\gamma 2^{\hat{\beta}}}(t)\sum_{\alpha\in I(\hat{\beta})} a(\alpha+\gamma 2^{\hat{\beta}})w_\alpha(t). \tag{5.4}$$

On the set $E(\beta)$, each of the functions w_α in (5.4) is equal to 1 if the number of components of α with $\alpha_k \geq 2^{\beta_k}$ is even, and to -1 otherwise. Denote this constant value of w_α by $\varepsilon_\beta(\alpha)$, and let

$$A(\beta,\gamma) \equiv \sum_{\alpha\in I(\hat{\beta})} \varepsilon_\beta(\alpha)a(\alpha+\gamma 2^\beta). \tag{5.5}$$

Then the quantity (5.4) coincides with $A(\beta,\gamma)w_{\gamma 2^{\hat{\beta}}}(t)$ at all points t in the set $E(\beta)$, and hence

$$F\cdot 1_{E(\beta)} = (1_{E(\beta)})\cdot \sum_{\gamma\geq 0} A(\beta,\gamma)w_{\gamma 2^{\hat{\beta}}}.$$

Let

$$F_\beta = \sum_{\gamma\geq 0} A(\beta,\gamma)w_\gamma. \tag{5.6}$$

By the analysis above, $F(t) = F_\beta(2^{\hat{\beta}}t)$ at all points t in $E(\beta)$. Inserting this in (5.3), and changing variables yields the formula

$$\|F\|_1 = \sum_{\beta\geq 0} 2^{-|\hat{\beta}|}\|F_\beta\|_1. \tag{5.7}$$

We claim that

$$|A(\beta,\gamma)| \leq 2^{|\beta|} \sum_{\alpha\in I(\hat{\beta})+\gamma 2^{\hat{\beta}}} |\Delta a(\alpha)|. \tag{5.8}$$

To verify this, first rewrite the sum defining $A(\beta,\gamma)$ as

$$\sum_{\alpha\in I(\beta)} \sum_{0\leq\delta\leq\bar{\gamma}} (-1)^{|\delta|}a(\gamma 2^{\hat{\beta}}+\alpha+\delta 2^\beta).$$

Then observe that the inner sum above is also equal to

$$\sum_{\alpha\leq\eta<\alpha+2^\beta} \Delta a(\gamma 2^{\hat{\beta}}+\eta),$$

and that the absolute value of this sum is bounded above by

$$\sum_{\eta\in I(\hat{\beta})+\gamma 2^{\hat{\beta}}} |\Delta a(\eta)|.$$

The series defining F_β converges absolutely. For each subset S of the index set I, let $F_{\beta,S}$ be the sum of the terms in (5.6) for which $\gamma_k = 0$ for all k in S and $\gamma_k > 0$ for all other values of k. In particular,

$$F_{\beta,\emptyset} = \sum_{\gamma\geq\bar{\gamma}} A(\beta,\gamma)w_\gamma.$$

By inequality (5.8), the ℓ^2-norm of the sequence of coefficients in this series is majorized by $2^{|\beta|}\|a\|'_{1,2,2^{\hat{\beta}}}$. So this is also an upper bound for $\|F_{\beta,\emptyset}\|_2$, and hence for $\|F_{\beta,\emptyset}\|_1$ too. It follows that

$$\sum_{\beta\geq 0} 2^{-|\hat{\beta}|}\|F_{\beta,\emptyset}\|_1 \leq 2^{-K}\|a\|_\Delta. \tag{5.9}$$

By equation (5.7),

$$\|F\|_1 \leq \sum_{S\subset I}\sum_{\beta\geq 0} 2^{-|\hat{\beta}|}\|F_{\beta,S}\|_1. \tag{5.10}$$

Inequality (5.9) controls the part of this sum where S is empty.

For the other choices of S, it is useful to have a variant of inequality (5.8). The proof of that estimate really shows that

$$|A(\beta,\gamma)| \le \sum_{\alpha\in I(\beta)} \sum_{\alpha\le\eta<\alpha+2^{\beta}} |\Delta a(\gamma 2^{\hat\beta}+\eta)|. \tag{5.11}$$

Given a multiindex η in Z_+^K denote the product of its components by $|[\eta]|$. The multiindices η appearing in (5.11) all belong to the set $I(\hat\beta)$. For any such η, the number of multiindices α for which $0\le\alpha\le\eta$ is $|[\hat\eta]|$. Reversing the order of summation in (5.11) yields that

$$|A(\beta,\gamma)| \le \sum_{\eta\in I(\hat\beta)} |[\hat\eta]|\,|\Delta a(\gamma 2^{\hat\beta}+\eta)|. \tag{5.12}$$

The "series" for $F_{\beta,I}$ really just contains the constant term $A(\beta,0)$. So

$$\sum_{\beta\ge 0} 2^{-|\hat\beta|}\|F_{\beta,I}\|_1 = \sum_{\beta\ge 0} 2^{-|\hat\beta|}|A(\beta,0)|.$$

Inserting inequality (5.12) on the right side of this equation reveals that the left side is majorized by

$$\sum_{\eta\ge 0}\left\{\sum_{2^{\hat\beta}>\eta} |[\hat\eta]|2^{-|\hat\beta|}\right\}|\Delta a(\eta)|$$

Another way to state the relation between η and $\hat\beta$ here is that $2^{\hat\beta}\ge\hat\eta$. For each choice of η, there is a minimal multiindex, $\beta(\eta)$ say, with this property, and the quantity in the curly brackets is bounded above by

$$2^{|\widehat{\beta(\eta)}|}\sum_{\beta\ge\beta(\eta)} 2^{-|\hat\beta|} = \sum_{\delta\ge 0} 2^{-|\delta|}.$$

The right side here is equal to $[\sum_{m=0}^{\infty} 2^{-m}]^K = 2^K$, and it follows that

$$\sum_{\beta\ge 0} 2^{-|\hat\beta|}\|F_{\beta,I}\|_1 \le 2^K\|\Delta a\|_1.$$

Let

$$\|a\|_\star \equiv \|a\|_\Delta + \sum_{|S|>0}\|a_S\|_\Delta,$$

with the convention that $\|a_I\|_\Delta = |a(0)|$. The methods used in Section 4 show that if a is regular on Z_+^K, then $\|\Delta a\|_1 \le C\|a\|_\star$. Therefore

$$\sum_{\beta\ge 0} 2^{-|\hat\beta|}\|F_{\beta,I}\|_1 \le C'\|a\|_\star.$$

We estimate the quantities $\sum_{\beta\ge 0} 2^{-|\hat\beta|}\|F_{\beta,S}\|_1$ with $0<|S|<K$ by combining the methods used for the corresponding sums when $S=\emptyset$ and $S=I$. For simplicity, we present the idea in the special case where $K=2$ and $S=\{1\}$. Then

$$F_{\beta,S}(t) = \sum_{j=1}^{\infty} A(\beta,(0,j))z_2(t_2)^j.$$

The L^1-norm of $F_{\beta,S}$ is bounded above by the ℓ^2-norm of the sequence $A(\beta,(0,\cdot))$. To control this sequence norm, we construct various auxiliary sequences by combining some of the values of $|\Delta a|$ in the strip where $0 \le \alpha_1 < 2^{\beta_1+1}$. For each integer i in the interval $[0,\beta_1]$ define a sequence $(d^{(i)}_{\beta_2}(j))_{j=1}^{\infty}$ by letting

$$d^{(i)}_{\beta_2}(j) = \sum_{\alpha=(2^i, j2^{\beta_2})}^{(2^{i+1}-1,(j+1)2^{\beta_2}-1)} |\Delta a(\alpha)|.$$

Also let

$$d^{(-1)}_{\beta_2}(j) = \sum_{j2^{\beta_2}\le n<(j+1)2^{\beta_2}} |\Delta a(0,n)|.$$

By inequality (5.11),

$$\|A(\beta,(0,\cdot))\|_2 \le \sum_{i=-1}^{\beta_1} 2^{i+1}2^{\beta_2+1}\left\|d^{(i)}_{\beta_2}\right\|_2. \tag{5.13}$$

Because of the way that these sequences were chosen,

$$\|d^{(i)}_{\beta_1}\|_2 \le \|\Delta a\|'_{1,2,2^{(i,\beta_2)}},$$

when $i \ge 0$. Insert this in the part of the right side of (5.13) where $i \ge 0$, multiply by $2^{-|\beta|-2}$, and add over all choices of β in Z^K_+. The outcome is the combination

$$\sum_{\beta_2\ge 0}\sum_{i\ge 0}\left[\sum_{\beta_1\ge i} 2^{i-\beta_1}\right]\|\Delta a\|'_{1,2,2^{(i,\beta_2)}}, \tag{5.14}$$

which is equal to $2\|a\|_\Delta$ because the sum inside the square parentheses is equal to 2. The quantity (5.14) is the main part of an upper bound for $\sum_{\beta\ge 0} 2^{-|\hat{\beta}|}\|F_{\beta,S}\|_1$. Now consider the corresponding contributions from the part of the right side of (5.13) where $i=-1$. They add up to

$$\sum_{\beta_2\ge 0}\left[\sum_{\beta_1\ge 0} 2^{-\beta_1}\right]\|\Delta a(0,\cdot)\|'_{1,2,2^{\beta_2}}, \tag{5.15}$$

where the functionals $\|\cdot\|'_{1,2,2^j}$ are the single-variable versions that apply to functions on Z_+. The entries in the sequence $\Delta a(0,\cdot)$ are mixed differences of the form

$$[a(0,j)-a(0,j+1)]-[a(1,j)-a(1,j+1)].$$

With the convention that $\|d\|' \equiv \sum_{j\ge 0}\|d\|'_{1,2,2^j}$, expression (5.15) simplifies to $2\|\Delta a(0,\cdot)\|'$. The hypotheses in Theorem 1.4 partly concern expressions like $\|a_{\{1\}}\|_\Delta = \|\Delta_2 a(0,\cdot)\|'$ that involve single differences. The methods in Section 4 also show that $\|\Delta_2 a(1,\cdot)\|' \le C\|a\|_\Delta$. Therefore

$$\sum_{\beta\ge 0} 2^{-|\hat{\beta}|}\|F_{\beta,S}\|_1 \le C'\|a\|_\star.$$

The sum of these expressions over all subsets S of I majorizes $\|F\|_1$.

Finally, we remove the hypothesis that $a \in \ell^1$. Split the function a into pieces $a^{(1)}$ and $b^{(1)}$ as follows. For an integer r_1 to be specified later, let $d^{(1)}$ be the function on Z^K_+ that vanishes on the set I_1 of multiindices, α say, with $\alpha_k \le r_1$ for some k, and that coincides with Δa on the rest of Z_+^K. It is shown at the beginning of Section 4 that regularity implies that $\Delta a \in \ell^1$. For all α in Z_+^K, let $a^{(1)}(\alpha)$ be the sum of the absolutely convergent series

$$\sum_{\gamma \ge \alpha} d^{(1)}(\gamma).$$

Then $\Delta a^{(1)}(\alpha) = d^{(1)}(\alpha)$ for all α. Let $b^{(1)} = a - a^{(1)}$; then $b^{(1)}$ vanishes off the finite set I_1. The analysis given earlier applies to the Walsh polynomial

$$F^{(1)} \equiv \sum_{\alpha \ge 0} b^{(1)}(\alpha) z^\alpha,$$

and shows that $\|F^{(1)}\|_1 \le C''\|b^{(1)}\|_\star \le C''\|a\|_\star$.

For all sufficiently large values of r_1 it is the case that $\|a^{(1)}\|_\star \le (1/2)\|a\|_\star$. Make such a choice of r_1, and then split $a^{(1)}$ into pieces $a^{(2)}$ and $b^{(2)}$ much as above, in such a way that $\|a^{(2)}\|_\star$‘ $\le (1/2)\|a^{(1)}\|_\star$, while $b^{(2)}$ has finite support and $\|b^{(2)}\|_\star \le \|a^{(1)}\|_\star$. Let $F^{(2)}$ be the Walsh polynomial with coefficients $b^{(2)}$, and continue the process. The series $\sum_{n=1}^\infty \|F^{(n)}\|_1$ converges, and the completeness of L^1 then guarantees that there is an integrable function F with Walsh coefficients given by

$$\hat{F}(\alpha) = \sum_{n=1}^{\infty} b^{(n)}(\alpha)$$

for all α in Z_+^K. On the other hand, this series also converges to $a(\alpha)$. This completes the proof of Theorem 1.4.

6 Earlier work

The papers [4] and [2] contain summaries of earlier work on integrability of single-variable trigonometric series. In previous work on double cosine series and double sine series, the weakest integrability criteria on first-order mixed differences seem to be those used by F. Moricz in [11]. For other types of multiple trigonometric series, the weakest earlier criteria seem to be those presented in [17] by S.A. Telyakovskiĭ. These papers also contain discussions of other work on integrability for multiple trigonometric series. The paper [12], by Moricz and F. Schipp, seems to be the only earlier publication on integrability of multiple Walsh series. We now show how our criteria follow from the weakest conditions used previously.

The main part of Moricz's first integrability criterion for double cosine series, with coefficients tending to 0, is that there be some index $p > 1$ for which

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 2^{i+j} \left[2^{-i-j} \sum_{m=2^i}^{2^{i+1}-1} \sum_{n=2^j}^{2^{j+1}-1} |\Delta a(m,n)|^p \right]^{1/p} < \infty. \tag{6.1}$$

Moricz also requires here that

$$\sum_{i=0}^{\infty} 2^i \left[2^{-i} \sum_{m=2^i}^{2^{i+1}-1} |\Delta a(m,0)|^p \right]^{1/p} < \infty \tag{6.2}$$

for some $p > 1$, and similarly for the corresponding expression with i and m equal to 0 and j and n varying. These conditions are natural analogues of ones introduced by G.A. Fomin [7] in work on integrability of single-variable trigonometric series.

The hypotheses in Theorem 1.2 follow from those used by Moricz. In fact, the validity of condition (6.1) for some $p > 1$ implies that $\|a\|_\Delta < \infty$. For weakly regular sequences a, the validity of condition (6.2) for some $p > 1$ implies weak regularity of the slice $a_{\{2\}}$, and the validity of the transpose of condition (6.2) for some $p > 1$ implies weak regularity of $a_{\{1\}}$. The converse implications are false.

To verify these claims, proceed as in [4], [1], and [2]. Note first that

$$\left[2^{-i-j}\sum_{m=2^i}^{2^{i+1}-1}\sum_{n=2^j}^{2^{j+1}-1}|\Delta a(m,n)|^p\right]^{1/p}$$

is just the ℓ^p-average of Δa over the set $J(i,j)+(1,1)2^{(i,j)}$. It follows from Hölder's inequality that if condition (6.1) is satisfied for some value of the index p, then the condition also holds for all smaller values of p. The same is true for condition (6.2) and its transpose. Suppose without loss of generality that $1 < p \leq 2$ in all three conditions.

Rewrite (6.1) in the form asserting that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}2^{(i+j)/p'}\left[\sum_{m=2^i}^{2^{i+1}-1}\sum_{n=2^j}^{2^{j+1}-1}|\Delta a(m,n)|^p\right]^{1/p} < \infty, \tag{6.3}$$

where p' is the index conjugate to p. Consider the seemingly stronger condition that

$$\sum_{(i,j)\geq 0}2^{(i+j)/p'}\left[\sum_{(m,n)\geq(2^i,2^j)}|\Delta a(m,n)|^p\right]^{1/p} < \infty. \tag{6.4}$$

Because $p > 1$, the factors $2^{(i+j)/p'}$ grow geometrically with (i,j), and the same kind of estimation and reversal of orders of summation as in the proof of Lemma 2.2 shows that conditions (6.4) and (6.3) are equivalent in these cases.

As in our section on Walsh series, let $I(i,j)$ be the set of multiindices α with the property that $0 \leq \alpha < 2^{(i,j)}$. Consider the quantity that is summed on i and j in (6.4), and rewrite it as

$$\left[\sum_{(r,s)\geq(1,1)}\left\{(2^{(i+j)/p'}\left\|(\Delta a)\cdot 1_{I(i,j)+(r,s)2^{(i,j)}}\right\|_p\right\}^p\right]^{1/p}. \tag{6.5}$$

Regard the expression inside the curly brackets as 2^{i+j} times the ℓ^p-average of Δa over the set $I(i,\ j)\ +\ (r,\ s)2^{(i,j)}$. Since $p > 1$, this average majorizes the corresponding ℓ^1-average, so that its product with 2^{i+j} majorizes the ℓ^1-norm of Δa on the set $I(i,j)\ +\ (r,s)2^{(i,j)}$. Thus (6.5) is no smaller than the ℓ^p combination of these ℓ^1-norms. Since $p \leq 2$ this combination does not increase if we replace p by 2. Summing the resulting ℓ^2 combinations of ℓ^1-norms then gives that

$$\sum_{\beta\geq 0}\left[\sum_{\gamma\geq(1,1)}\left\{\|(\Delta a)\cdot 1_{I(\beta+\gamma 2^\beta)}\|_1\right\}^2\right]^{1/2} < \infty. \tag{6.6}$$

As in Section 5, it is easy to check that this condition holds if and only if $\|a\|_\Delta$ is finite.

Similarly, condition (6.2) implies that the function

$$p : m \mapsto a(m,0) - a(m,1)$$

is weakly regular on Z_+. As noted earlier, weak regularity of Δa on Z_+^2 implies that the slice $a_{(2,1)} : m \mapsto a(m,1)$ is also weakly regular on Z_+. Then the sum of this slice and the function p must be weakly regular, and it is equal to the slice $a_{\{2\}}$ mapping m to $a(m,0)$.

To see that the various converse implications are false, we use the fact that for each sequence d in $\ell^1(Z_+^2)$, the sequence a given by

$$a : (m,n) \mapsto \sum_{(i,j)\geq(m,n)} d(i,j) \tag{6.7}$$

tends to 0, and has the property that $\Delta a = d$. Suppose that these differences d are lacunary in the special sense that they vanish except on the set of pairs of the form 2^β, where $\beta \geq 0$. Let $g(\beta) = d(2^\beta)$ for all such multiindices β. Then a is weakly regular if and only if

$$\sum_{(m,n)\geq 0} \left[\sum_{(i,j)\geq(m,n)} |g(i,j)|^2 \right]^{1/2} < \infty. \tag{6.8}$$

On the other hand, the corresponding sequence a satisfies satisfies condition (6.1) if and only if

$$\sum_{(i,j)\geq 0} 2^{(i+j)/p'} |g(i,j)| < \infty. \tag{6.9}$$

Examples like the one where $g(i,j) = (i+1)^{-2}(j+1)^{-2}$ show that, when $p > 1$, this is a strictly stronger restriction that (6.8). These examples are also regular, but fail to satisfy condition (6.2) and its transpose when $p > 1$.

Condition (6.8) is the counterpart of a single-variable condition that arose in the study [18] of integrability of lacunary sine series with respect to dt/t. Moricz [10] has considered the corresponding problem for lacunary double sine series, and has shown, under reasonable auxiliary hypotheses, that condition (6.8) is necessary and sufficient for the integrability of such series with respect to $w(t)dm(t)$. This is related to the instance of our Theorem 1.2 where the sequence Δa has special lacunary form considered above.

When $p = 1$, condition (6.1) becomes the condition that $\|\Delta a\|_1 < \infty$, while condition (6.3) becomes the condition that

$$\sum_{\beta\geq 0} \sum_{\alpha\geq 2^\beta} |\Delta a(\alpha)| < \infty. \tag{6.10}$$

This is stronger than merely requiring that $\Delta a \in \ell^1$; in fact, as in [7], reversing the order of summation in condition (6.10) shows that it is equivalent to the requirement that

$$\sum_{\alpha\geq(1,1)} |\Delta a(\alpha)| \log(1+\alpha_1)\log(1+\alpha_2) < \infty.$$

The combination of this with similar counterparts of the endpoint case, where $p = 1$, of condition (6.2) and its transpose, is equivalent to the requirement that

$$\sum_{\alpha \geq 0} |\Delta a(\alpha)| \log(2 + \alpha_1) \log(2 + \alpha_2) < \infty, \tag{6.11}$$

which is Moricz's second sufficient condition for integrability of double cosine series with coefficients a that tend to 0. As in the cases where $p > 1$, condition (6.11) implies our regularity condition, via conditions like (6.10), and the same examples show that the converse is false.

In [12] Moricz and Schipp show that a double Walsh series is integrable if its coefficients tend to 0 and there is some index $p > 1$ for which the coefficients satisfy condition (6.1), condition (6.2), and the transpose of condition (6.2). As in the case of cosine series, such a Walsh series also satisfies the hypotheses of Theorem 1.4, but there are double Walsh series satisfying these hypotheses but not satisfying the conditions used in [12].

For double sine series with coefficients $(b(\alpha))_{\alpha > 0}$ that tend to 0, Moricz shows that if

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 2^{i+j} \left[2^{-i-j} \sum_{m=2^i}^{2^{i+1}-1} \sum_{n=2^j}^{2^{j+1}-1} m^p n^p \Delta b(m,n)|^p \right]^{1/p} < \infty. \tag{6.12}$$

for some $p > 1$, then the sine series is integrable. This condition clearly implies (6.1), and hence that b is weakly regular. It also implies strong regularity of b.

To verify this, fix an index $p > 1$, and choose a constant C that majorizes the left side of (6.12). Then let

$$R(i) = \left\{ \alpha \in P^2 : \alpha_1 \geq 2^i \right\},$$

and use the fact that

$$\|b_{(1,2^i)}\|_\Delta \leq \|\Delta b \cdot 1_{R(i)}\|'. \tag{6.13}$$

By our analysis of condition (6.1), the right side above is majorized by $C' 2^{-i} C$, and therefore

$$\sum_{i \geq 0} \|b_{(1,2^i)}\|_\Delta \leq 2C'C.$$

Similar estimates holds for sums of the quantities $\|b_{(2,2^j)}\|_\Delta$ and $|\rho_{[2^i,2^j]} b|$. Moreover, it is clear from this analysis that the extra factors $m^p n^p$ in (6.12) can be replaced by much smaller factors like $[\log(1+m)\log(1+n)]^{2p}$, and the resulting variant of (6.12) will still imply strong regularity, if the coefficients tend to 0, and hence integrability.

It is also shown in [11] that the condition that

$$\sum_{(m,n) > 0} \log(m+1) \log(n+1) |\Delta b(m,n)| < \infty \tag{6.14}$$

is sufficient for integrability of a double sine series with coefficients $b(\alpha)$ that tend to 0. To account for this by our methods, note first that, as in the case of (6.10), condition (6.14) implies weak regularity of b. It follows from (6.13) that

$$\|b_{(1,2^i)}\|_\Delta \leq C' \sum_{m \geq 2^i} \sum_{j \geq 0} \sum_{n \geq 2^j} |\Delta b(m,n)|.$$

Summing this inequality on i leads to condition (6.10) with the sequence a replaced by b. Hence the sequence b is strongly regular. Again the converse implication is false.

Cosine series, sine series, and mixed series in two and more variables are all considered by Telyakovskiĭ in the paper [17]. To compare the conditions used there with ours, we translate those conditions into our notation. We also permute the variables so that the sine factors correspond to the initial M variables. Denote the coefficient sequence by $(d(\alpha))_{\alpha\geq 0}$, with the convention that $d(\alpha) = 0$ if $\alpha_k = 0$ for some $k \leq M$.

Telyakovskiĭ considers sequences d that tend to 0 for which there is an auxiliary sequence C that tends to 0 with the properties that $C(\alpha) \geq |\Delta d(\alpha)|$ for all α in Z_+^K and

$$\sum_{\alpha\geq 0} \left[\prod_{k=1}^{M} \alpha_k\right] \left\{\prod_{k=1}^{K} (\alpha_k + 1)\right\} |\Delta C(\alpha)| < \infty. \tag{6.15}$$

He shows that any such sequence C will also have the property that

$$\sum_{\alpha\geq 0} \left[\prod_{k=1}^{M} \alpha_k\right] C(\alpha) < \infty, \tag{6.16}$$

and he points out that this condition is equivalent to the previous one when $\Delta C(\alpha) \geq 0$ for all α. He deduces integrability from the condition that $|\Delta d|$ have a majorant C that tends to 0 and satisfies (6.15).

In this situation, let $D(\alpha) = \sum_{\gamma\geq\alpha} |\Delta C(\gamma)|$ for all α in Z_+^K. Then $D \geq C$ and $\Delta D = |\Delta C|$. It follows that conditions (6.15) and (6.16) both hold with the sequence C replaced by D. Assume without loss of generality that $C = D$. Then C tends to 0 monotonically in the sense that

$$C(\alpha) \geq C(\gamma) \qquad \text{when} \qquad \alpha \geq \gamma. \tag{6.17}$$

The existence of a sequence C that majorizes Δd, that tends to 0 monotonically, and that satisfies (6.16) implies the integrability criteria listed at the end of Section 4.

To verify this, note first that, as in [11, page 208], these conditions imply that

$$\sum_{\beta\geq 0} 2^{|\beta|} \sup\left\{|\Delta d(\gamma)| : \gamma \geq 2^{\beta}\right\} < \infty.$$

This is the version of inequality (6.3) with K variables and with $p = \infty$. It implies the corresponding version of inequality (6.1), which then also holds with $p = 2$. This implies that d is weakly regular. It also follows that the slices d_R with $R \subset \{M + 1, \ldots, K\}$ are weakly regular. Finally, the presence of the factors α_k with $k \leq M$ in (6.16) implies a suitable variant of condition (6.12) in K variables, and inequality (4.7) follows from this.

To summarize, every condition on the sizes of individual mixed differences that was previously known to imply integrability also implies our integrability criteria. Of course, for some of these conditions, like (6.11) and (6.14), the integrability conclusion can be deduced much more simply by the other methods, like summation by parts, that were used in [11] with these conditions. These methods also show that if the coefficients in a multiple trigonometric series tend to 0 and have mixed differences that belong to ℓ^1, then the series converges at all points in the the set I with no components equal to 0. It follows that this pointwise sum is integrable if the coefficients are regular and sufficiently symmetric.

7 Combinations of mixed differences

Our symmetry condition (1.9) can be viewed as a restriction on the sizes of various sums of mixed differences. For single-variable series with regular coefficients, the corresponding symmetry condition is also necessary [2] for integrability. For multiple trigonometric series with regular coefficients, our methods show that integrability occurs if and only if

$$\sum_{\beta \geq 0} |(\tilde{f} - f)|_{E(\beta)} < \infty. \tag{7.1}$$

To force this sum to be finite, we used our symmetry condition to control the L^1-norms of the various series (2.22). We did this by estimating the values of the functionals $\| \cdot \|_\Delta$ and $\| \cdot \|_\Sigma$ applied to the coefficients in these series. Integrability does not imply regularity, however, so that it seems likely that there are cases where these methods yield unduly large overestimates of these L^1-norms. This suggests that sufficient symmetry is not necessary for integrability of multiple series with regular coefficients.

A rather artificial way to return to the position of having a symmetry condition that is also necessary for the integrability of certain series is to strengthen the regularity hypothesis. Call a sequence c on Z^K *very regular* if it is regular and

$$\sum_{0<|S|<K} \sum_{\beta \geq 0} \|\sigma_{[S,2^\beta]} c\|_\Delta < \infty,$$

that is $\sum_{\beta \geq 0} \|\sigma_{[S,2^\beta]} c_S\|_\Delta$ is finite for all proper subsets S of I. The analysis of multiple sine series in the previous section shows that any sequence that tend to 0 and has small enough mixed differences is very regular. The proof of Theorem 1.1 show that a complex trigonometric series with a very regular coefficient sequence c is integrable if and only if

$$\sum_{\beta \geq 0} |\sigma_{[I,2^\beta]} c| < \infty.$$

This applies, for example, to the double series

$$\sum_{(m,n)>0} \frac{1}{[\log(m+n+2)]^2} \sin(mt_1) \sin(nt_2),$$

and shows that it is not integrable.

We conclude by discussing a possible alternative to our method of proof. Given two multiindices α and γ in Z^K and a sequence c defined on that set, let

$$\Delta_\gamma(\alpha) \equiv c(\alpha) - c(\alpha + \gamma).$$

By an argument that goes back to Bernstein's paper [3] on absolute convergence of trigonometric series, it is easy to show, as in [2], that if the sequence c tends to 0 and if

$$\sum_{\beta \geq 0} 2^{-|\beta|/2} \|\Delta_{2^\beta} c\|_2 < \infty, \tag{7.2}$$

then the series (1.1) is integrable. For single-variable sequences c that tend monotonically to 0 at $\pm\infty$, condition (7.2) is equivalent [2] to the combination of regularity and sufficient symmetry.

In any case, this combination implies (7.2). If this could be proved simply, then we would have a less complicated proof of Theorem 1.1. At the moment, however, the simplest way that we have found to deduce (7.2) from our integrability criteria is to examine the proof of Theorem 1.1, and observe that the method really shows that

$$\sum_{\beta \geq 0} \left\{ \left[|f|^2 \right]_{E(\beta)} \right\}^{1/2} < \infty, \tag{7.3}$$

or equivalently that

$$\sum_{\beta \geq 0} 2^{-|\beta|/2} \left\| F \cdot 1_{E(\beta)} \right\|_2 < \infty. \tag{7.4}$$

As in [9], this also follows easily from (7.2); moreover, as in [2] it is also easy to verify that (7.4) implies (7.2).

The situation for Walsh series is similar. The set of Walsh functions of one variable is a group under pointwise multiplication, and this structure transfers [16, §1.2] to a a group operation $\oplus$ on the set of nonnegative integers. Given a sequence a on Z_+ and a nonnegative integer m, let $(W_m)a$ be the sequence mapping n to $a(n \oplus m)$ for all n in Z_+. Extend these notions to multiple Walsh systems in the obvious way. With the appropriate change in the definitions of the sets E_β, our proof of Theorem 1.4 shows that if a is regular, then condition (7.4) holds. As in [13], condition (7.4) is equivalent, for coefficient sequences a that tend to 0, to the counterpart

$$\sum_{\beta \geq 0} 2^{-|\beta|/2} \|a - (W_{2^\beta})a\|_2 < \infty$$

of Bernstein's condition.

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