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An Integrability Theorem for Unbounded Vilenkin Systems

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Abstract. We consider Vilenkin systems $(\chi_n)_0^\infty$ and series $\sum_{n=0}^\infty a_n\chi_n$ with coefficients tending to 0. We suppose that the coefficients satisfy the regularity condition that

$$\sum_{\nu=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\sum_{n=j2^\nu}^{(j+1)2^\nu-1}|a_n-a_{n+1}|\right]^2\right\}^{1/2}<\infty.$$

We show that the series then represents an integrable function on the set $[0,1)$ if and only if the coefficients satisfy a symmetry condition similar to one that arises in the study of integrability of trigonometric series.

The symmetry condition is automatically satisfied by regular sequences if the Vilenkin system is of bounded type. So integrability follows in this special case if the coefficients tend to 0 and their differences satisfy the condition above. This was proved earlier by other methods. Our methods also show that when the generators of the Vilenkin system all have odd order, and the system is enumerated in a natural symmetric way, then the corresponding Fejér kernels form a bounded sequence in L^1, even when it is known that this is *not* true for the Fejér kernels for the conventional enumeration.

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1. Introduction

We state the symmetry condition at the end of this section, after specifying a construction of Vilenkin systems. In Section 2, we provide more information about these systems, and outline the proof of our main result. In Section 3, we complete the proof, except for two lemmas, which we prove in Section 4. We propose and analyse an alternate indexing of some Vilenkin systems in Section 5. We begin this section by mentioning some other work on integrability, and we comment further on related matters in Section 6.

The simplest example of a bounded Vilenkin system is the Walsh system. The condition that the differences $\triangle a_n = a_n - a_{n+1}$ have the property that

$$\sum_{\nu=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\sum_{n=j2^\nu}^{(j+1)2^\nu-1}|\Delta a_n|\right]^2\right\}^{1/2} < \infty. \tag{1.1}$$

was encountered, independently, by N. Tanović-Miller and her fellow workers, and by us in work on Walsh series. Both groups showed [19, 1] that condition (1.1) implies integrability for Walsh series with coefficients (a_n) tending to 0.

Both groups then showed [7, 2] that the same condition implies integrability for cosine series with coefficients (a_n) tending to 0. Finally, both groups also considered sine series with coefficients (b_n) tending to 0 and satisfying the analogue of condition (1.1), and showed [9, 2] that these series are integrable if and only if

$$\sum_{n=1}^{\infty}\frac{|b_n|}{n} < \infty. \tag{1.2}$$

These separate results on cosine and sine series can be combined as a statement about trigonometric series in the complex form $\sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$. If $c_n \to 0$ as $n \to \pm\infty$ and if

$$\sum_{\nu=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\sum_{j2^\nu\le|n|<(j+1)2^\nu}|\triangle c_n|\right]^2\right\}^{1/2} < \infty, \tag{1.3}$$

then the series represents an integrable function if and only if

$$\sum_{n=1}^{\infty}\frac{|c_n - c_{-n}|}{n} < \infty. \tag{1.4}$$

Both groups also showed that conditions (1.1) and (1.3) are strictly weaker than all conditions, on the sizes of *individual* differences, that were previously known, in the presence of symmetry conditions like (1.2) and (1.4), to imply integrability of trigonometric series. In that context, Tanović-Miller's group used summation by parts and estimates for

L^1-norms of sums of Dirichlet kernels to prove their integrability theorems, while we used methods similar to those in the present paper. We will discuss conditions on the sizes of *combinations* of differences in Section 6.

The symmetry condition for unbounded Vilenkin systems resembles conditions (1.2) and (1.4). To state it we need to recall the definition of general Vilenkin systems, and the standard enumeration for them. We will discuss the advantages of a different enumeration of some systems in Section 5. For completeness, we repeat the description given in [6]. See [23 and 24] for much more on this topic. We will use the term *Vilenkin system* for any orthonormal system of functions that can be constructed in the following way.

Begin with a nonatomic probability space like the interval $[0, 1)$ and a sequence $(p_r)_{r=0}^{\infty}$ of prime numbers, and construct a sequence (χ_n) of functions on the space as follows. Let χ_0 be the constant function 1 and let Γ_0 be the singleton set $\{\chi_0\}$. Let ϕ_1 be a function taking each value in the set of p_1-th roots of unity with probability $1/p_1$. For $1 \le n < p_1$ let χ_n be the function ϕ_1^n, and let Γ_1 be the set of all functions χ_n with $0 \le n < p_1$.

Let $m_0 = 1$. For each positive integer r, let $m_r = \prod_{n=1}^{r} p_n$. When $r > 1$, assume that the functions χ_n with $n < m_{r-1}$ have been specified and denote the set of these functions by Γ_{r-1}. Then select three functions α_r, β_r, and γ_r with the following properties:
(i) α_r belongs to the set Γ_{r-1};
(ii) β_r is a p_r-th root of α_r that is constant on each set where α_r is constant;
(iii) γ_r takes each value in the set of p_r-th roots of unity with probability $1/p_r$;
(iv) γ_r is independent of the functions in the set Γ_{r-1}.
Let ϕ_r be the product $\gamma_r\beta_r$. Then $\phi_r^{p_r} = \alpha_r \in \Gamma_{r-1}$. There is a multiple, t_r say, of p_r so that the values taken by ϕ_r run through the set of t_r-th roots of unity, each such value occurring with probability $1/t_r$.

Observe that each integer n in the interval $[m_{r-1}, m_r)$ has a unique representation as $n = jm_{r-1} + k$ with $1 \le j < p_r$ and $0 \le k < m_{r-1}$. Given this representation of n, let $\chi_n = \phi_r^j\chi_k$. Denote the set of functions χ_n with $0 \le n < m_r$ by Γ_r, and continue the construction for all r.

The system of functions constructed in this way is said [24] to be of *bounded type* if the sequence (p_r) is bounded. When $\alpha_r = 1$ for all r the system is said [25] to be of *multiplicative type*. The Walsh system arises when $p_r = 2$ and $\alpha_r \equiv 1$ for all r. Varying the sequence (p_r) and the choices of α_r leads to many other examples. The results in this paper are mainly new when the sequence (p_r) is unbounded, but the methods work for all Vilenkin systems.

Denote the underlying probability space by Ω and the probability measure on it by $d\omega$. Given a function f in $L^1(d\omega)$ and a Vilenkin system (χ_n) let

$$\hat{f}(n) = \int_\Omega f(\omega)\overline{\chi_n(\omega)}\,d\omega.$$

Say that a series $\sum_{n=0}^{\infty} a_n\chi_n$ *represents an integrable function*, or simply that the series is *integrable*, if there is such an f in $L^1(d\omega)$ for which $\hat{f}(n) = a_n$ for all n.

The functions χ_n all have absolute value 1, and they form an orthonormal sequence in $L^2(d\omega)$. By the Riemann-Lebesgue lemma, a necessary condition for integrability is that $a_n \to 0$ as $n \to \infty$. This is not sufficient, however, because [4] there are Walsh series that are not integrable but have coefficients that tend monotonically to 0.

By our main theorem, a Vilenkin series is integrable if its coefficients tend to 0 in a sufficiently regular and balanced way. A particular case of this principle is the fact [26] that a Walsh series with coefficients that form a convex sequence tending to 0 must be integrable. Such convex sequences also satisfy the conditions, on the sizes of first differences, that were shown in [6] and [16] to imply integrability for bounded Vilenkin systems.

We sometimes regard the sequence $(a_n)_{n=0}^{\infty}$ as a function on the set $\Gamma = \{\chi_n\}_{n=0}^{\infty}$, and then use the notation $a(\chi_n)$ rather than a_n. Each element of Γ is a function on the underlying probability space. The complex conjugate of χ_n also belongs to Γ. We follow [23] in using the notation $\tilde{n}$ for the index for which $\overline{\chi_n} = \chi_{\tilde{n}}$. The indices $\tilde{n}$ and n are equal if and only if $(\chi_n)^2$ coincides with the constant function 1.

Call the coefficient sequence (a_n) *regular* if it tends to 0 and satisfies condition (1.1). Call it *sufficiently symmetric* if

$$\sum_{n=1}^{\infty} \frac{|a_n - a_{\tilde{n}}|}{n} < \infty. \tag{1.5}$$

Another notation for the n-th term in the sum above is $\dfrac{|a(\chi_n) - a(\overline{\chi_n})|}{n}$.

Theorem 1. *If a Vilenkin series has a regular coefficient sequence, then the series is integrable if and only if the sequence is sufficiently symmetric.*

In proving this in the next three sections we will sometimes use other forms of the regularity and symmetry conditions. We note here that condition (1.5) follows easily from condition (1.1) if the system is of bounded type. Hence regularity implies integrability in this case, as was shown in a different way in [19] and [1].

On the other hand, given any Vilenkin system that is not of bounded type, one can devise a convex sequence that tends to 0 but is not sufficiently symmetric. Thus regularity does not imply sufficient symmetry in this case. By Theorem 1, the series with these coefficients is *not* integrable; earlier results showing that convexity does not always imply integrability appear in [17].

This also means that the same sequence can satisfy condition (1.5) for some Vilenkin systems, and fail to satisfy the condition for other systems. For a regular coefficient sequence, integrability can depend on the sequence (p_r) of prime numbers, but not on the particular choice of Vilenkin system associated with those prime numbers.

2. Coronas and coronets.

The functions in a Vilenkin system form a denumerable abelian group under pointwise multiplication, with every element in the group having finite order. Vilenkin began [23] with such a group, and also considered its dual group, which is separable, compact and abelian, and becomes a probability space if its Haar measure is suitably normalized.

We will usually suppress the group structure on this space, because that structure should not matter in determining whether a given Vilenkin series is integrable. Instead this depends on the joint distribution of the functions χ_n and the properties of the coefficients in the series. There is a measure-preserving mapping [24, p. 703] that we will call the *standard map* of each Vilenkin group onto the set $[0, 1)$ with Lebesgue measure. We use the properties of the Vilenkin functions transferred to the unit interval by a variant of the standard map. We will specify that variant at the beginning of Section 3.

Let G_r be the set where the functions χ_n with $n < m_r$ all coincide with 1. Then $G_r \subset G_{r-1}$ for all $r > 0$, and G_r has measure $1/m_r$. The standard map and our variant of it both identify G_r with the interval $[0, 1/m_r)$. The sets $G_{r-1}\backslash G_r$ are called *coronas*; they correspond to the intervals $[1/m_r, 1/m_{r-1})$. Each such interval is a union of $p_r - 1$ disjoint translates, by multiples of $1/m_r$, of the deleted interval $[0, 1/m_r)$; we call these translates *lots*.

Our proof of Theorem 1 runs as follows. Consider a formal series

$$S(x) = \sum_{n=0}^{\infty} a_n \chi_n(x) \tag{2.1}$$

with regular coefficients. Define a function h on the interval $(0, 1)$ by letting $h(x)$ be equal to $(1 - \phi_r(x))/m_{r-1}$ when $x \in G_{r-1}\backslash G_r$, and following this pattern in every corona. Consider the formal product $h(x) \cdot S(x)$.

On the corona $G_{r-1}\backslash G_r$ the coefficients in the product series mostly have the form

$$\frac{a_n - a_{n-m_{r-1}}}{m_{r-1}}.$$

It follows from the regularity hypothesis on the sequence (a_n) that the formal product series converges absolutely on each corona. Denote the sum of the product series by $f(x)$.

For our variant of the standard map, there are positive constants c and C so that

$$cx \le |h(x)| \le Cx \quad \text{for all } x \text{ in the interval } (0, 1). \tag{2.2}$$

So, at least formally, the series $S(x)$ is integrable if and only if the function $g : x \mapsto f(x)/x$ is integrable on the set $(0, 1)$. This reduces matters to showing that g is integrable if and only if condition (1.5) holds.

To study this question we split the coronas where $p_r > 3$ into subsets that we call *coronets*, chosen so that the variable x changes by a relatively small amount in each coronet.

We take the smallest coronet in $G_{r-1}\backslash G_r$ to be the union of the first two lots in this corona, the next smallest coronet to be the union of the next four lots, and continue doubling until the next step would use more than half of the vacant lots in the corona. At that point, we combine all the remaining lots to form the last coronet in $G_{r-1}\backslash G_r$. When $p_r \le 3$, we regard the whole corona $G_{r-1}\backslash G_r$ as a coronet. Each coronet then has the property that the ratio of any two numbers in it lies between 1/3 and 3.

The interval $(0,1)$ is a union of disjoint coronets, which we list from right to left as $(I_\nu)_{\nu=1}^{\infty}$. Then

$$\int_0^1 \frac{|f(x)|}{x}\,dx = \sum_{\nu=1}^{\infty} \int_{I_\nu} \frac{|f(x)|}{x}\,dx. \tag{2.3}$$

In each coronet I_ν the variable x is nearly constant, and the measure of I_ν is about equal to any value of x in I_ν. So the sum (2.3) is finite if and only if the sum of the average values of $|f(x)|$ in the various coronets is finite.

To estimate the average value of $|f(x)|$ over I_ν, we expand $f(x)$ as the formal product $S(x)\cdot(1-\phi_r(x))/m_{r-1}$ for the appropriate value of r. We then split this product series as $s_\nu(x)+T_\nu(x)$, where $s_\nu(x)$ is the sum of the terms in the series for which n belongs to a suitable subset of $[0, m_r)$.

In the next section, we show that the regularity hypothesis on (a_n) implies the finiteness of the sum of the averages, over the intervals I_ν, of the sizes of the tails $T_\nu(x)$. We also show that this is true for the sum of the averages of $|s_\nu(x)-s_\nu(0)|$. So a series with regular coefficients is integrable if and only if

$$\sum_{\nu=1}^{\infty} |s_\nu(0)| < \infty. \tag{2.4}$$

We then show for regular coefficient sequences that this sum is finite if and only if the symmetry condition (1.5) holds. Finally, we indicate how to make the formal parts of this outline rigorous.

3. Details of the proof.

We begin by specifying more conventions and notation. By the definition of the set G_{r-1} all the functions in Γ_{r-1} are equal to 1 on it. So β_r is equal to a fixed p_r-th root of 1 on this set. Since γ_r is independent of Γ_{r-1}, the restriction of ϕ_r to G_{r-1} takes each value in the set of p_r-th roots of unity with the same probability. It takes the value 1 on the set G_r, by the definition of that set.

The standard map makes the function ϕ_r constant on each lot in the corona $G_{r-1}\backslash G_r$, and makes its values on successive lots in the corona run counterclockwise through the set of nontrivial p_r-th roots of unity starting with $e^{i2\pi/p_r}$. We modify this mapping so that these constant values run from right to left through the set of nontrivial p_r-th roots of unity. When $p_r = 5$ for instance, we let the successive values taken on successive lots in the corona be $e^{i2\pi/5}$, $e^{-i2\pi/5}$, $e^{i4\pi/5}$, and $e^{-i4\pi/5}$. Then $|(1-\phi_r(x))|$ is nondecreasing on $G_{r-1}\backslash G_r$ and the function h defined in the previous section satisfies condition (2.2).

We also consider coronas and coronets in the index set $Z_+ = \{0, 1, 2, ...\}$. When we are subdividing sets of integers, we use the notation $[a, b)$ to mean the set of integers n for which $a \le n < b$; it should be clear from the context whether we intend $[a, b)$ to be a set of real numbers or a set of integers.

Call the interval $[m_{r-1}, m_r)$ in Z_+ the *r-th corona*. It is a union of disjoint lots that are translates of the interval $[0, m_{r-1})$ by multiples of m_{r-1}. We divide the corona $[m_{r-1}, m_r)$ into the same number of coronets that we used in splitting the interval $[1/m_r, 1/m_{r-1})$, but the procedure looks different because we have not modified the standard enumeration of Vilenkin systems. Again, we only split the coronas in the cases where $p_r > 3$. In these cases, the first coronet is the union of the first and last lots in the corona, the second coronet consists of the next two vacant lots at the beginning and the last two vacant lots at the end, and so on. Again, we keep doubling until this would use more than half of the vacant lots in the corona, and at that point we assign all the remaining vacant lots to one large coronet. We do this for all r, and enumerate the successive coronets as $(J_\nu)_{\nu=1}^{\infty}$. Finally, we let K_ν be the union of the set $\{0\}$ with $\bigcup_{\mu=1}^{\nu} J_\mu$.

Denote the right endpoint of the domain coronet I_ν by $x_{\nu-1}$. If I_ν is the last coronet in $G_{r-1}\backslash G_r$, then $x_{\nu-1} = 1/m_{r-1}$. Thus $(1/m_r)_{r=0}^{\infty}$ is a subsequence of $(x_\nu)_{\nu=0}^{\infty}$. Define a dual sequence (k_ν) of integers by letting k_ν be the smallest positive integer in the complement of the set K_ν. Then $k_\nu = m_r$ when $x_\nu = 1/m_r$, and it turns out that the product $k_\nu x_\nu$ always lies in the interval $(1/2, 1]$.

We will mainly work with the condition that

$$\sum_{\nu=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\sum_{n=jk_\nu}^{(j+1)k_\nu-1}|\Delta a_n|\right]^2\right\}^{1/2} < \infty. \tag{3.1}$$

We say that a sequence (k_ν) of positive integers *grows geometrically* if there are constants c and C in the interval $(1, \infty)$ for which $ck_\nu \le k_{\nu+1} \le Ck_\nu$ for all ν; this is true for the sequence (k_ν) above with $c = 2$ and $C = 6$. For each coefficient sequence (a_n), condition (3.1) holds for one index sequence (k_ν) that grows geometrically if and only if the condition holds for all such index sequences. So, for sequences $(a(n))$ that tend to 0, condition (3.1) is equivalent to regularity.

On the other hand,

$$\sum_{n=k_1}^{\infty} |\Delta a_n| \leq \sum_{\nu=1}^{\infty} \sum_{n=k_\nu}^{6k_\nu - 1} |\Delta a_n|,$$

because the intervals $[k_\nu, 6k_\nu)$ cover the set $[k_1, \infty)$. Since the right side above is finite if condition (3.1) holds, regularity implies that (a_n) has bounded variation.

We now discuss some other forms of the symmetry condition (1.5); they arise more naturally in our proof of Theorem 1. Whenever the prime number p_r is odd, write it in the form $2q_r + 1$. In this case, the r-th corona is a union of $2q_r$ lots. Given a coefficient sequence (a_n) and a value of r for which p_r is odd, form numbers $C_r(s)$ as follows. Let $C_r(1)$ be the average of (a_n) over the first lot in the r-th corona, and $C_r(-1)$ be the average of (a_n) over the last lot in the corona. When $q_r \geq 2$, let $C_r(2)$ be the average of (a_n) over the second lot, and let $C_r(-2)$ be the average of (a_n) over the second-last lot. Continue in this way until $s = \pm q_r$.

For sequences (a_n) with bounded variation, condition (1.5) is equivalent to the requirement that

$$\sum_{p_r > 2} \sum_{s=1}^{q_r} \frac{|C_r(s) - C_r(-s)|}{s} < \infty,$$

and this is equivalent, for such sequences, to the requirement that

$$\sum_{p_r > 2} \sum_{2^i < q_r} |C_r(2^i) - C_r(-2^i)| < \infty. \tag{3.2}$$

Fix an index r, and cover the natural numbers with *blocks* that are disjoint translates of the set $[0, m_r)$; then split each block into lots that are disjoint translates of $[0, m_{r-1})$. For the standard enumeration of the Vilenkin system, $\chi_n \cdot \phi_r = \chi_{n+m_{r-1}}$, unless n lies in the last lot in some block, when $\chi_n \cdot \phi_r = \chi_{n+m_{r-1}-m_r}$ instead. Formally multiplying the series $S(x)$ by $\phi_r(x)$ has the following effect on the coefficients in the series. In each block they are shifted to the right by m_{r-1}, except that each coefficient in the last lot in each block is shifted to the corresponding position in the first lot in the block.

To save on subscripts, we change to function notation for the coefficients $a(n)$. We assume that they form a regular sequence, and hence that $\Delta a \in \ell^1$. We majorize the coefficients $b_r(n)$ in the product series $S(x) \cdot (1 - \phi_r(x))/m_{r-1}$ as follows. In each lot, D say, of a block B, except for the first lot, we have that

$$b_r(n) = \frac{a(n) - a(n - m_{r-1})}{m_{r-1}}. \tag{3.3}$$

In these lots

$$|b_r(n)| \le \frac{1}{m_{r-1}} \left[\sum_{n\in D} |\Delta a(n)| + \sum_{n\in P} |\Delta a(n)| \right], \tag{3.4}$$

where P is the previous lot. In the first lot we have that

$$b_r(n) = \frac{a(n) - a(n - m_{r-1} + m_r)}{m_{r-1}}, \tag{3.5}$$

and that

$$|b_r(n)| \le \frac{1}{m_{r-1}} \sum_{n\in B} |\Delta a(n)|. \tag{3.6}$$

Each lot contains exactly m_{r-1} integers. Hence

$$\sum_{n\in B} |b_r(n)| \le 3 \sum_{n\in B} |\Delta a(n)|. \tag{3.7}$$

It follows from our assumption that $(a(n))$ has bounded variation that the formal series $S(x)\cdot(1-\phi_r(x))/m_{r-1}$ converges absolutely and uniformly.

When $k_\nu \in (m_{r-1}, m_r]$, let s_ν be the sum of the finite number of terms in the series $\sum_{n=0}^\infty b_r(n)\chi_n$ for which $n\in K_\nu$, and let T_ν be the sum of the remaining terms in this series. Since the series for T_ν converges uniformly, its coefficients are the Fourier coefficients of T_ν with respect to the orthonormal system (χ_n). Thus $\widehat{T_\nu}(n) = 0$ if $n \in K_\nu$ and $\widehat{T_\nu}(n) = b_r(n)$ otherwise. Given a sequence $(b(n))$ and a positive integer k, let

$$\|b\|_{1,2,k} = \left\{ \sum_{j=0}^\infty \left[\sum_{n=jk}^{(j+1)k-1} |b(n)| \right]^2 \right\}^{1/2}. \tag{3.8}$$

Note that this differs from the pattern in condition (3.1), because in (3.8) the sum on j starts with $j=0$ rather than $j=1$, and we use the numbers $b(n)$ rather than $\Delta a(n)$. In computing $\|\widehat{T_\nu}\|_{1,2,k_\nu}$, however, we can omit the term with $j=0$, because $\widehat{T_\nu}$ vanishes on the set K_ν, which includes the interval $[0, k_\nu)$. When $k = k_\nu$, the inner sums in (3.8) run over unions of blocks B for which inequality (3.7) holds. So

$$\|\widehat{T_\nu}\|_{1,2,k_\nu} \le 3 \left\{ \sum_{j=1}^\infty \left[\sum_{n=jk_\nu}^{(j+1)k_\nu-1} |\Delta a(n)| \right]^2 \right\}^{1/2}. \tag{3.9}$$

Denote the measure of an interval I in $[0,1)$ by $|I|$.

Lemma 2. Suppose that $T \in L^1[0,1)$ and that $\|\hat{T}\|_{1,2,k_\nu} < \infty$. Then

$$\frac{1}{|I_\nu|} \int_{I_\nu} |T| \le 500 \|\hat{T}\|_{1,2,k_\nu}. \tag{3.10}$$

We will prove this estimate in the next section. It follows easily from (3.9) and (3.10) that

$$\sum_{\nu=1}^{\infty} \int_{I_\nu} \frac{|T_\nu(x)|}{x}\, dx \le 5{,}000 \sum_{\nu=0}^{\infty} \left\{ \sum_{j=1}^{\infty} \left[\sum_{n=jk_\nu}^{(j+1)k_\nu - 1} |\Delta a(n)| \right]^2 \right\}^{1/2} . \tag{3.11}$$

In the rest of the proof of Theorem 1, the only properties of the sequence (a_n) that we need are that it tends to 0 and has bounded variation.

Lemma 3. The sums s_ν have the property that

$$\frac{1}{|I_\nu|} \int_{I_\nu} |s_\nu(x) - s_\nu(0)|\, dx \le 6\pi 2^{-\nu} \sum_{\mu=1}^{\nu} 2^\mu \left[\sum_{n \in J_\mu} |\Delta a(n)| \right] . \tag{3.12}$$

We also delay the proof of this until the next section. Adding these estimates as ν runs from 1 to ∞ and reversing the order of summation on the right yields that

$$\sum_{\nu=1}^{\infty} \frac{1}{|I_\nu|} \int_{I_\nu} |s_\nu(x) - s_\nu(0)|\, dx \le 12\pi \sum_{n=0}^{\infty} |\Delta a(n)|. \tag{3.13}$$

So a series with regular coefficients is integrable if and only if the sum on ν of $|s_\nu(0)|$ is finite, as specified in line (2.4) of the previous section.

Since $\chi_n(0) = 1$ for all n, the sum $s_\nu(0)$ is just $\sum_{n \in K_\nu} b_r(n)$. Fix an index r, and consider the cases where $k_\nu \in (m_{r-1}, m_r]$. If $k_\nu = m_r$, then $K_\nu = [0, m_r)$. Otherwise, K_ν is a union of some but not all of the lots in $[0, m_r)$.

Adopt the convention that the last lot in the block $[0, m_r)$ is the lot "previous" to the first one. By formulas (3.3) and (3.5), for each lot D the sum $\sum_{n \in D} b_r(n)$ is equal to the difference between the average of $a(n)$ over the lot D and the average over the previous lot. The sum of these differences of averages over all the lots in $[0, m_r)$ is equal to 0. This makes $s_\nu(0) = 0$ when $k_\nu = m_r$.

Now consider the other values, if there are any, of k_ν associated with r. For the smallest such k_ν, the set K_ν is the union of $[0, m_{r-1})$ with the first and last lots in $[m_{r-1}, m_r)$, and thus $s_\nu(0) = C_r(1) - C_r(-2)$ in this case. For the next such k_ν, add in the effect of also having the second and third lots in $[m_{r-1}, m_r)$ and the second-last and third-last ones; then $s_\nu(0) = C_r(3) - C_r(-4)$. This pattern continues, and leads to the formula

$$\sum_{k_\nu \in (m_{r-1}, m_r]} |s_\nu(0)| = \sum_{2^i < q_r} |C_r(2^i - 1) - C_r(-2^i)|. \tag{3.14}$$

So a Vilenkin series with regular coefficients is integrable if and only if

$$\sum_{p_r > 2} \sum_{2^i < q_r} |C_r(2^i - 1) - C_r(-2^i)| < \infty. \tag{3.15}$$

Finally, since $(a(n))$ has bounded variation, condition (3.15) is equivalent to condition (3.2) and hence to the condition (1.5) that appears in the definition of sufficient symmetry.

We now justify the formal calculations relating the series $S(x)$ to the function f. Suppose first that the series has regular coefficients, and is integrable, representing F say. Then for each positive integer r, the product $F \cdot (1-\phi_r)/m_{r-1}$ belongs to L^1, and calculating the Fourier coefficients of that product from their definition shows that they are equal to the numbers $b_r(n)$. Since $F \in L^1$,

$$\sum_{\nu=1}^{\infty} \int_{I_\nu} \frac{|f(x)|}{x}\, dx < \infty, \tag{3.16}$$

and then Lemmas 1 and 2 imply that condition (3.2) holds. So sufficient symmetry is necessary for integrability of Vilenkin series with regular coefficients.

To prove the converse, suppose that the coefficient sequence $(a(n))$ is regular and sufficiently symmetric. Then the numbers $b_r(n)$ are all well defined, and each series $\sum_{n=0}^{\infty} b_r(n)\chi_n(x)$ converges uniformly. Define $f(x)$ on the r-the corona to be the sum of this series. Then f satisfies condition (3.16), because of the regularity and symmetry of $(a(n))$. Hence there is a function F in L^1 so that $F = f/h$.

We claim that $\hat{F}(n) = a(n)$ for all n. To verify this, we consider the effect of multiplying the series S and the Fourier series of F by $(1-\phi_r)/m_{r-1}$ for successive values of r. Take the series $S(x)\cdot(1-\phi_1(x))/m_0$ first. It converges absolutely and uniformly to a sum $f_1(x)$ say. Then $\hat{f}_1(n) = b_1(n)$ for all n. Consider a block of terms in the product series, indexed by $[(k-1)m_1, km_1)$ for some positive integer k. The Vilenkin functions in this block all take the same constant value on the set G_1, and the coefficients $b(n)$ in the block add up to 0. So the sum of the terms in the block vanishes on the set G_1, and the same is true for the partial sums of the first Km_1 terms in the series for all positive integers K. Hence f_1 vanishes on G_1, and since f was defined to be equal to to f_1 on $G_0\backslash G_1$, we have that $f_1 = f \cdot 1_{G_0\backslash G_1}$.

On the other hand, we also have that $F\cdot(1-\phi_1)/m_0 = f_1$, because $h = (1-\phi_1)/m_0$ on the set $G_0\backslash G_1$, and $\phi_1 \equiv 1$ on G_1. It then follows by computing the Fourier coefficients of the product $F\cdot(1-\phi_1)/m_0 = f_1$, that the numbers $b_1(n)$ can be written as differences of the Fourier coefficients of F. The pattern is that in each block that is a translate of the set $[0, m_1)$, the coefficient $b_1(n)$ is equal to $\hat{F}(n) - \hat{F}(n-1)$, except at the first number n in the block. Since these coefficients can also be written in the same way as differences of the coefficients $(a(n))$, we conclude that the sequence $(\hat{F}(n) - a(n))$ must be constant on blocks that are translates of $[0, m_1)$.

We can then compare the products $S(x)\cdot(1-\phi_2(x))/m_1$ and $F\cdot(1-\phi_2)/m_1$ in a similar way. From the analysis above, the difference between these product series has coefficients that are constant on lots that are translates of $[0, m_1)$. Now the sum of the Vilenkin functions indexed by a lot $[(k-1)m_1, km_1)$ vanishes off the set G_1. So the sums of the first Km_1 terms of the difference series all vanish off G_1; since the coefficients in the difference series tend to 0, it converges uniformly to 0 off G_1. For both series, the sums of the first Km_2 terms vanish on G_2, so that the difference series also converges uniformly to 0 on G_2.

By the definition of f, the product series $S(x) \cdot (1 - \phi_2(x))/m_1$ converges uniformly to $f(x)$ on the corona G_1/G_2. Meanwhile, the series for $F \cdot (1-\phi_2)/m_1$ must be the series for the function obtained from f by multiplying it by 0 on G_2 and by $(1 - \phi_2)/(hm_1)$ off G_2. That function is bounded and coincides with f on G_1/G_2. Without loss of generality, use our variant of the standard mapping to identify the probability space supporting the Vilenkin functions with the interval $[0, 1)$ with Lebesgue measure. Then the Vilenkin system is complete, and the series for $F \cdot (1-\phi_2)/m_1$ must converge in L^2-norm to a function that coincides almost-everywhere in G_1/G_2 with f. So the difference series converges to 0 in L^2-norm on all of $[0, 1)$.

It follows that the two product series have the same coefficients. Now in each block that is a translate of the set $[0, m_2)$ by a multiple of m_2,

$$b_2(n) = \frac{\hat{F}(n) - \hat{F}(n - m_1)}{m_1},$$

except on the first lot of m_1 numbers n in the block. Since these coefficients can also be written in the same way as differences of the coefficients $(a(n))$, and since the sequence $(\hat{F}(n) - a(n))$ is already known to be constant on lots of length m_1, we conclude that $(\hat{F}(n) - a(n))$ must be constant on blocks that are translates of $[0, m_2)$.

Continuing in this manner shows that the sequence $(\hat{F}(n) - a(n))$ is constant on all the sets $[0, m_r)$. Since this difference sequence tends to 0, it must be identically 0. This completes our proof of Theorem 1, except for the two lemmas.

Remark 1. We delay most of our comments until the later sections of the paper, but we mention here that the methods above also provide estimates for L^1-norms. Given a sequence $(a(n))$, let $\|a\|_\Delta$ be the sum appearing in inequality (1.1) and let $\|a\|_\Sigma$ be the sum appearing in inequality (1.5). Regular, sufficiently symmetric sequences form a Banach space relative to the norm $\| \cdot \|_* \equiv \| \cdot \|_\Delta + \| \cdot \|_\Sigma$. The closed-graph theorem applied to Theorem 1 implies that there must be constants A and B so that the function F represent by a series with such coefficients satisfies

$$\|F\|_1 \leq A\|\hat{F}\|_\Delta + B\|\hat{F}\|_\Sigma. \tag{3.17}$$

In fact, the proof of Theorem 1 can be analysed to yield explicit constants A and B for which this inequality holds for all F in L^1.

This can be used to provide a different justification of the formal calculations used earlier in this section. These calculations are correct for Vilenkin polynomials, and yield the estimates specified above for the L^1-norms of such polynomials. Given a series with coefficients $(a(n))$ that form a regular, sufficiently symmetric sequence, let $a^{(r)}$ coincide with a outside the interval $[0, m_r)$ but be equal to $a(m_r)$ on this interval; note that $a^{(0)} = a$. Then $\|a^{(r)}\|_* \to 0$ as $r \to \infty$, so that there is an index r_1 for which $\|a^{(r_1)}\|_* \leq (1/2)\|a\|_*$.

Let F_1 be the Vilenkin polynomial with coefficients that coincide with $a - a^{(r_1)}$ on the interval $[0, m_{r_1})$ and vanish off this interval. Then

$$\|F_1\|_1 \le (A+B)\left\|\widehat{F_1}\right\|_* = (A+B)\left\|a - a^{(r_1)}\right\|_* \le (A+B)\|a\|_*.$$

Applying this reasoning repeatedly yields an increasing sequence of integers r_i so that $\|a^{(r_i)}\|_* \le (1/2)^i\|a\|_*$. It also yields Vilenkin polynomials F_i so that $\widehat{F_i} = a^{(r_{i-1})} - a^{(r_i)}$. Then

$$\|F_i\|_1 \le (A+B)\left\|\widehat{F_i}\right\|_* \le \left(\frac{1}{2}\right)^{i-1}(A+B)\|a\|_*.$$

So the series $\sum_{i=1}^{\infty}\|F_i\|_1$ converges, and hence $\sum_{i=1}^{\infty} F_i$ converges in L^1-norm to some integrable function F. It follows easily that $\hat{F} = a$.

Remark 2. The proof of Theorem 1 also shows that

$$\|F\|_1 \ge B\|\hat{F}\|_\Sigma - A\|\hat{F}\|_\Delta \tag{3.18}$$

This allows us to correct the erroneous statement in [6] that, for any conventional indexing of a Vilenkin system, the sequence of Fejér kernels is always bounded in L^1. This error resulted from a misreading of [23], where this L^1-boundedness is only asserted and proved for some Vilenkin systems of bounded type. As pointed out in [3], it was shown in [20] that for any conventional indexing of any multiplicative system of unbounded type the sequence of Fejér kernels is *not* bounded in L^1.

The estimates for L^1-norms in the proof of Theorem 1 settle all cases of this question, showing that, for any conventional indexing of a Vilenkin system, the sequence of Fejér kernels is bounded if and only if the system is of bounded type. The fact that this sequence is L^1-bounded in the latter case also follows from [3] or [6].

4. Proofs of the lemmas.

Proof of Lemma 3. Fix indices r and ν for which $k_\nu \in (m_{r-1}, m_r]$. Then

$$s_\nu(x) - s_\nu(0) = \sum_{n \in K_\nu} b_n[\chi_n(x) - \chi_n(0)],$$

so that

$$\frac{1}{|I_\nu|} \int_{I_\nu} |s_\nu(x) - s_\nu(0)|\, dx \leq \sum_{n \in K_\nu} |b_r(n)| \sup_{x \in I_\nu} |\chi_n(x) - \chi_n(0)|. \tag{4.1}$$

We will show below that

$$|\chi_n(x) - \chi_n(0)| \leq 2\pi n x \tag{4.2}$$

for all n in Z_+ and all x in $[0, 1)$. Assuming this, we then have that the n-the term on the right in (4.1) is majorized by $2\pi n x_\nu |b_r(n)|$. We also have the upper bound $2\pi \tilde{n} x_\nu |b_r(n)|$ for this term, because of the fact that $\chi_{\tilde{n}}$ is the complex conjugate of χ_n. If n belongs to J_μ, then so does $\tilde{n}$, and one of them must lie in the first half of J_μ, and therefore be bounded above by k_μ. Then the n-th term on the right in (4.1) is majorized by $2\pi k_\mu x |b_r(n)|$. Summing this bound over all values of n in J_μ and using inequality (3.7) gives at most $6\pi k_\mu x \sum_{n \in J_\mu} |\Delta a(n)|$.

As noted earlier, $k_\nu x \leq 1$ for all x in I_ν, so that $x \leq 1/k_\nu$ for all such x. Then $k_\mu x \leq k_\mu / k_\nu \leq 2^{\mu - \nu}$. Combining these estimates for all values of $\mu \leq \nu$ yields that

$$\frac{1}{|I_\nu|} \int_{I_\nu} |s_\nu(x) - s_\nu(0)|\, dx \leq 6\pi 2^{-\nu} \sum_{\mu=1}^{\nu} 2^\mu \left[\sum_{n \in J_\mu} |\Delta a(n)| \right],$$

as specified in inequality (3.12).

We complete the proof of Lemma 3 by verifying inequality (4.2). It is trivially true if $x = 0$ or $n = 0$, because then $\chi_n(x) = 1$. So we may suppose that n and x belong to coronas, $[m_{r-1}, m_r)$ and $(1/m_s, 1/m_{s-1}]$ say. Our analysis then splits into three cases. If $r < s$, then $\chi_r(x) = 1$, and there is again nothing to prove. If $r > s$, then $2\pi n x \geq 2\pi m_{r-1}/m_s \geq 2\pi$, and inequality (4.2) holds because $|\chi_n(x) - \chi_n(0)| \leq 2$ in any case.

Finally, in the case where $r = s$, use the fact that then $n = j m_{r-1} + k$ with $0 \leq j < p_r$ and $0 \leq k < m_{r-1}$; so χ_n factors as $\phi_r^j \chi_k$. For these values of k, the functions χ_k all coincide with 1 on the interval $[0, 1/m_{r-1})$ and hence at x. This reduces matters to estimating $|\phi_r^j(x) - 1|$ when $0 \leq j < p_r$ and $x \in G_{r-1} \backslash G_r$.

We permuted the lots in the block $G_{r-1} \backslash G_r$ so that the difference $|\phi_r(x) - 1|$ does not decrease as x increases in the interval $[0, 1/m_{r-1})$. This difference vanishes when $x \in G_r$, and it is easy to see that the ratio $|\phi_r(x) - 1|/x$ attains its maximum value in $[0, 1/m_{r-1})$ when $x = 1/m_r$. Therefore

$$\frac{|\phi_r(x) - 1|}{x} \leq (m_r) 2 \sin(\pi/p_r) \leq 2\pi m_{r-1}, \tag{4.3}$$

for all x in G_{r-1}. It follows that, $|\phi_r^j(x) - 1| \le j2\pi x m_{r-1}$ for these values of x. This implies inequality (4.2), because $\chi_n(x) = \phi_r(x)^j$ and $jm_{r-1} \le n$ in this case.

Proof of Lemma 2. We follow the convention that if h is an integrable function and S is a measurable set with positive measure, then h_S denotes the average of the function g over the set S. When $\nu \le 4$, we simply use the Schwarz inequality and the orthonormality of the Vilenkin system χ_n to obtain that

$$|T|_{I_\nu} \le \left[|T^2|_{I_\nu}\right]^{1/2} \le |I_\nu|^{-1/2}\|T\|_2 = |I_\nu|^{-1/2}\|\hat{T}\|_2. \tag{4.5}$$

For each coronet the ratio of the left endpoint to the right endpoint is at least 1/3. It follows that each of the first four coronets has measure at least $(1/3)^4$. Inserting this estimate in inequality (4.5) and using the fact that the norm on the right is bounded above by $\|\hat{T}\|_{1,2,k_\nu}$ yields the first four cases of inequality (3.10), namely that

$$\frac{1}{|I_\nu|}\int_{I_\nu} |T| \le 500\|\hat{T}\|_{1,2,k_\nu}$$

when $\nu \le 4$.

Now fix an index $\nu > 4$, let $i_\nu = k_{\nu-4}$, and let

$$K_\nu(x) = \frac{1}{i_\nu}\sum_{n=0}^{i_\nu - 1} \chi_n(x)$$

for all x in $(0,1]$. Then $nx_\nu \le 2^{-4}$ for all indices n in the sum defining the function K_ν. It follows from inequality (4.2) that the real part of each term in that sum is bounded below by $1/2$ on the set I_ν; so this is also a lower bound for $K_\nu(x)$ on I_ν. Hence $|T|_{I_\nu} \le 2|K_\nu \cdot T|_{I_\nu}$, and applying inequality (4.5) with T replaced by $K_\nu \cdot T$ yields that

$$|T|_{I_\nu} \le 2|I_\nu|^{-1/2}\left\|(K_\nu \cdot T)^\wedge\right\|_2. \tag{4.6}$$

To analyse the coefficients of the product $K_\nu \cdot T$, we transfer the group structure on the set of functions χ_n to the set of nonnegative integers. The subset $[0, m_r) = \Gamma_r$ becomes a subgroup, and the intervals $[jm_r, (j+1)m_r)$ that occur in the definition of $\|\hat{T}\|_{1,2,m_r}$ are cosets of this subgroup; this will simplify the estimation of $\|\hat{T}\|_{k_\nu}$ in the cases where $k_\nu = m_r$ for some value of r. If $m_{r-1} < k_\nu < m_r$, on the other hand, then $k_\nu = 2^s m_{r-1}$ for some integer s, and each interval $[jk_\nu, (j+1)k_\nu)$ is a union of 2^s adjacent cosets of Γ_{r-1}. The coefficients of $K_\nu \cdot T$ are given by the convolution, associated with the group structure on the set $\{\chi_n\}$, of the functions $\widehat{K_\nu}$ and $\hat{T}$. This convolution is well-defined, because $\widehat{K_\nu}$ has finite support.

Let $(\hat{T})_j$ be $\hat{T}$ multiplied by the indicator function of the set $[jk_\nu, (j+1)k_\nu)$. By Young's inequality for convolution

$$\left\|\widehat{K_\nu} * (\hat{T})_j\right\| \le \left\|\widehat{K_\nu}\right\|_2 \left\|(\hat{T})_j\right\|_1 = (i_\nu)^{-1/2}\left\|(\hat{T})_j\right\|_1. \tag{4.7}$$

Fix ν, and suppose temporarily that the various functions $\widehat{K_\nu} * (\hat{T})_j$ have disjoint supports. Then

$$\left(\left\|\widehat{K_\nu} * \hat{T}\right\|_2\right)^2 = \sum_{j=0}^{\infty}\left(\left\|\widehat{K_\nu} * (\hat{T})_j\right\|_2\right)^2 \le \left(\frac{1}{i_\nu}\right)\sum_{j=0}^{\infty}\left(\left\|(\hat{T})_j\right\|_1\right)^2 = \left(\frac{1}{i_\nu}\right)\left(\left\|\hat{T}\right\|_{1,2,k_\nu}\right)^2.$$

Combining this with inequality (4.6) yields that

$$|T|_{I_\nu} \le \frac{2}{\sqrt{i_\nu |I_\nu|}}\left\|\hat{T}\right\|_{1,2,k_\nu}.$$

If $i_\nu = m_r$ for some index r, then the functions $\widehat{K_\nu} * (\hat{T})_j$ have disjoint supports for the following reasons. In this case, the support of $\widehat{K_\nu}$ is included in the subgroup Γ_r, and each interval $[jk_\nu, (j+1)k_\nu)$ is a union of cosets of this subgroup. Then the support of $\widehat{K_\nu} * (\hat{T})_j$ is still included in this union of cosets.

If i_ν differs from all the numbers m_r, on the other hand, then, as we show below, we can split $\hat{T}$ into at most three pieces with disjoint supports so that if $\hat{T}$ is replaced by any of these pieces then the the supports of the corresponding functions $\widehat{K_\nu} * (\hat{T})_j$ are disjoint. We then have that in any case

$$|T|_{I_\nu} \le \frac{6}{\sqrt{i_\nu |I_\nu|}}\left\|\hat{T}\right\|_{1,2,k_\nu}.$$

Inequality (3.10) follows from this, because $|I_\nu| \ge (1/3)x_\nu$ and

$$i_\nu x_\nu \ge \frac{1}{6^4} k_\nu x_\nu \ge \frac{1}{6^4}\frac{1}{2}.$$

Finally, we specify the splitting of $\hat{T}$ when $m_r < i_\nu < m_{r+1}$ for some nonnegative integer r. The support of $\widehat{K_\nu}$ is a subset of the subgroup Γ_{r+1}, so that the convolution of $\widehat{K_\nu}$ with a function vanishing outside some coset of Γ_{r+1} will also vanish outside that coset. We carry out the splitting separately in each coset of Γ_{r+1}.

Such a coset will be covered by the finitely-many successive intervals $[jk_\nu, (j+1)k_\nu)$, with $j = j_0, j_1, \cdots, j_q$ say, that intersect nontrivially with the coset. Split $\hat{T}$ into pieces $\hat{U}$, $\hat{V}$, and $\hat{W}$ so that, on this coset, $\hat{U}$ coincides with the restriction of the function $\hat{T}_{j_0}$ to the coset, while $\hat{V}$ coincides with the restriction of the sum of the functions $\hat{T}_{j_p}$ for odd integers p, and $\hat{W}$ coincides with the restriction of the sum of the functions $\hat{T}_{j_p}$ for *positive* even integers p. This produces large enough gaps in the supports of each of the functions $\hat{U}$, $\hat{V}$, and $\hat{W}$ for their convolutions with $\widehat{K_\nu}$ to have the desired disjointness property.

5. Another Enumeration for Some Vilenkin Systems.

The functions ϕ_r generate the whole Vilenkin system, with every member of the system having a unique representation as a product of finitely-many powers $\phi_r^{q_r}$, where $0 \leq q_r < p_r$ for all r. The key property of these generators is that ϕ_r belongs to the r-th corona $\Gamma_r \backslash \Gamma_{r-1}$ for all r. Different selections of functions with this property generate the same Vilenkin system, but lead to different conventional enumerations of that system. The effect of such changes on a Vilenkin series is to permute the terms in the series. Such permutations have no effect on integrability, but in some cases can change the regularity of the coefficient sequence.

Start, for instance, with any strictly-decreasing convex sequence $(a(n))$ that tends to 0 slowly enough that $\sum_0^\infty a(n) = \infty$. Then choose a sequence (p_r) of prime numbers as follows. Use the fact that $\sum_0^\infty a(n) = \infty$ to choose an integer N_1 so that $\sum_{n=0}^{N_1} a(n) > 2$; then use the fact that $a(n) \to 0$ as $n \to \infty$ to choose p_1 so that $a((p_1 - 1)/2) < (1/2)a(N_1)$. Given the first $j-1$ values of p_r, let m_{j-1} be their product, and choose N_j so that $\sum_{n=m_{j-1}}^{N_j} a(n) > 2m_{j-1}$; then choose p_j so that $a((m_j - 1)/2) < (1/2)a(N_j)$.

Rather than using the functions ϕ_r to generate the Vilenkin system, use their powers $\phi_r^{(p_r+1)/2}$. Part of the effect of this change is to permute the various lots in the r-th corona so that each lot where some coefficients is at least $a(N_r)$ is adjacent to a lot where all the coefficients are at most $a(N_r)/2$. This makes the variation of the permuted coefficient sequence over the r-th corona at least

$$\frac{1}{m_{r-1}} \sum_{n=m_{r-1}}^{N_r} \frac{1}{2} a(n) > 1.$$

Thus the permuted coefficient sequence does not have bounded variation, and hence is not regular.

The standard enumeration of a Vilenkin system uses the fact that each nonnegative integer has a unique representation as a sum of terms $q_r m_{r-1}$ with $0 \leq q_r < p_r$ for all r. Call the Vilenkin system *fully odd* if $p_r > 2$ for all r. In that case, each integer, including the negative ones, has a unique representation as a sum of finitely-many terms $q_r m_{r-1}$, where now $-p_r/2 < q_r < p_r/2$ for all r.

It also turns out that each function in a fully odd Vilenkin system has a unique representation as a product of finitely-many powers $\phi_r^{q_r}$, where again $-p_r/2 < q_r < p_r/2$ for all r. To verify this, argue by induction on the integer s for which the Vilenkin function, χ say, belongs to the set $\Gamma_s \backslash \Gamma_{s-1}$. Then $\chi = \phi_s^{u_s} \chi'$ for a unique integer u_s with $0 < u_s < p_s$ and a unique function χ' in the set Γ_{s-1}. If $u_s < p_s/2$, simply let $q_s = u_s$ and use the corresponding representation of χ' as a product of powers $\phi_r^{q_r}$ with $r < s$ and $-p_r/2 < q_r < p_r/2$ for all such indices r. If $p_s/2 < u_s < p_s$, let $q_s = u_s - p_s$. Then $\phi_s^{u_s} = \phi_s^{q_s} \chi''$, where $\chi'' \in \Gamma_{s-1}$. The product $\chi' \chi''$ also belongs to the subgroup Γ_{s-1} and so also has a unique representation as a product of powers $\phi_r^{q_r}$ with $r < s$ and with $-p/r/2 < q_r < p_r/2$ for all such indices r. Multiplying this by $\phi_s^{q_s}$ then gives the desired representation of χ.

This simple correspondence between representation of integers and representations of functions in a fully-odd Vilenkin system sets up an enumeration of the system using the full set of integers as an index set. Call this enumeration as a *symmetric indexing*, and use the different notation $(\psi_n)_{n=-\infty}^{\infty}$ for the Vilenkin functions when they are enumerated symmetrically.

When the system is fully-odd, symmetric indexing simplifies the description of the notion of *conjugate function* that was introduced in [21]. The idea is to split each corona in half and use multipliers that take different constant values in the two halves of the corona. This was originally done for conventionally-indexed multiplicative systems. For symmetrically indexed fully odd systems, the conjugate series of $\sum_n c_n\psi_n$ is $-i\sum_{n\neq 0}\operatorname{sgn}(n)c_n\psi_n$. The multiplier $-i\cdot\operatorname{sgn}(n)$ is the same as the one used to define the conjugate of the complex form of a trigonometric series.

The following conclusion comes from [10]; as pointed out there, some cases of it were known earlier.

Theorem 4. *For a symmetrically-indexed fully-odd Vilenkin system, if a series and its conjugate are both integrable, then the coefficients (c_n) in the series satisfy the condition that*

$$\sum_{n=1}^{\infty}\frac{|c_n|}{n}<\infty. \tag{5.1}$$

We note that if the coefficient sequence is odd, that is if $c_{-n}=-c_n$ for all n, then this condition is equivalent to the requirement that

$$\sum_{n=1}^{\infty}\frac{|c_n-c_{-n}|}{n}<\infty. \tag{5.2}$$

As in the trigonometric case, we call any coefficient sequence $(c_n)_{n=-\infty}^{\infty}$ *sufficiently symmetric* if condition (5.2) holds. We call the sequence *regular* if it tends to 0 at $\pm\infty$ and its differences Δc_n are small enough that

$$\sum_{\nu=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\sum_{j2^\nu\le|n|<(j+1)2^\nu}|\Delta c_n|\right]\right\}^{1/2}<\infty. \tag{5.3}$$

We do not know if regularity with respect to a conventional indexing implies regularity with respect to a symmetric indexing. A sequence of coefficients can be regular for a symmetric indexing, however, without being regular for any conventional indexing. We specify how this can happen for systems of bounded type, and let the reader modify the example suitably for other systems. Let α be a positive constant, and let

$$c_n=\begin{cases}[1/\log(2+n)]^\alpha & \text{if } n\ge 0;\\ 0 & \text{if } n<0.\end{cases}$$

Then the sequence $(c_n)_{-\infty}^{\infty}$ is convex on the set of nonnegative integers and 0 elsewhere and hence is regular.

Consider the series $\sum_{n=-\infty}^{\infty} c_n\psi_n$ and the corresponding series, $\sum_{k=0}^{\infty} a_n\chi_k$ say, for a conventional indexing using the same coronas. In each corona, half of the coefficients a_k are equal to 0, because they correspond to coefficients $c_{n'}$ where $n' < 0$. It follows from this and the boundedness of the sequence (p_r) that there is a positive constant C so that

$$\|a\|_\Delta \ge C \sum_{r=0}^{\infty} \left\{ \sum_{j=r}^{\infty} \frac{1}{j^{2\alpha}} \right\}^{1/2} .$$

The double sum on the right is finite if and only if $(2\alpha - 1)/2 > 1$, that is $\alpha > 3/2$. So when $0 < \alpha \le 3/2$, the symmetrically indexed series in this class has a regular coefficient sequence but the conventionally indexed series does not. By Theorem 4 above, this series is not integrable when $\alpha \le 1$; by Theorem 5 below, the series is integrable when $\alpha > 1$. When $1 < \alpha \le 3/2$, the series is integrable but does not have regular coefficients for any conventional indexing using the same coronas.

In our proof of Theorem 1, regularity for the conventional indexing is used to partly control differences between coefficients in the two halves of each corona; this accounts for the fact that this form of regularity implies sufficient symmetry when the Vilenkin system is of bounded type. As the example above shows, regularity for a symmetric indexing does not provide as much control on the sizes of differences between coefficients in opposite halves of coronas. Let $n_r = (m_r - 1)/2$, and note that in a symmetric indexing the subgroup Γ_r corresponds to the set of integers in the interval $[-n_r, n_r]$.

Definition. *Given a fully-odd Vilenkin system, call a sequence* $(c_n)_{n=-\infty}^{\infty}$ structurally regular *if it is regular and satisfies the further condition that*

$$\sum_{r=1}^{\infty} |c_{n_r} - c_{-n_r}| \log p_r < \infty. \tag{5.4}$$

For systems of bounded type, structural regularity is equivalent to the combination of regularity and sufficient symmetry. For systems of unbounded type, structural regularity does not imply sufficient symmetry, nor does it follow from sufficient symmetry and regularity.

Theorem 5. *Suppose that the Vilenkin system is fully odd and symmetrically indexed. If a series has structurally regular coefficients and is sufficiently symmetric, then the series is integrable. If a series is integrable and has regular coefficients, then the coefficients are sufficiently symmetric.*

Proof. Suppose first that the coefficients are structurally regular, and consider the methods used to prove Theorem 1. There, the first lot in each corona received special attention, because it had to be regarded as the next lot after the last one in the corona. For symmetric indexing, each corona splits into positive and negative halves, and the leftmost negative lot should be viewed as being the next one after the rightmost positive lot. Condition (5.4) controls the size of the coefficients of the function f in these leftmost negative lots. The factor $\log p_r$ in the condition is appropriate because such estimates are needed for every choice of k_ν between n_{r-1} and n_r. Similarly one needs about $\log p_r$ estimates for the ℓ^1-norm of the restriction of Δc_n to the leftmost and rightmost lots in the corona; condition (5.3) provides estimates for about that many copies of that norm.

So, for symmetric indexing and structurally regular coefficients, the methods used to prove Theorem 1 show that the series is integrable if and only if the coefficients are sufficiently symmetric. All that remains is to show that the "only if" part here persists when structural regularity is weakened to regularity.

If the coefficient sequence is even, that is if $c_{-n} = c_n$ for all n, then conditions 5.3 and 5.4 both hold automatically, and then regularity implies integrability. Every sequence splits uniquely as a sum of an even sequence and an odd sequence. Suppose that the original sequence is regular and that the series is integrable. Then the two series coming from the even and odd parts of the coefficient sequence also have these properties. The conjugate of the odd series is also integrable because that conjugate is even and has regular coefficients. By Theorem 4 applied to the odd part of the series, the coefficient sequence must be sufficiently symmetric. This completes the proof of Theorem 5.

For fully-odd Vilenkin systems of bounded type, structural regularity is equivalent, as noted earlier, to the combination of regularity and sufficient symmetry. So, in this case, integrability of a symmetrically indexed series with regular coefficients is equivalent to sufficient symmetry. This does *not* follow directly from the statement of Theorem 1, because regularity for symmetric indexing does not imply regularity for any conventional indexing, but the method of proof is the same.

As in the case of Theorem 1, the proof of Theorem 5 yields estimates for L^1-norms of Vilenkin polynomials in terms of the quantities (5.2), (5.3), and (5.4). For a symmetric indexing of a fully-odd system, form symmetric partial sums, their $(\mathrm{C}, 1)$ means, and the corresponding Fejér kernels in the usual way; these will *not* be the same as the partial sums, means, and kernels for any conventional indexing. The symmetric Fejér kernels satisfy the hypotheses of Theorem 5 in a uniform way, because their coefficients are even, and the restrictions of these coefficients to the nonnegative integers form convex sequences. It follows that these kernels form a bounded sequence in L^1, even if the Vilenkin system is *not* of bounded type. As noted at the end of Section 3, the corresponding statement is false for any conventional ordering of any system of unbounded type.

6. Related Questions.

In this section, we describe the situations where conclusions about norm convergence in L^1 follow easily from the main results in this paper; we hope to return to the harder cases of these questions in another paper. Then we discuss regularity conditions on sequences that imply integrability of some Vilenkin series but that do not force the sequences to have bounded variation. These conditions do not imply condition (1.5).

Kolmogorov [14] proved that cosine series with (quasi)convex coefficients, $(c_n)_{n=-\infty}^{\infty}$ say, converge in L^1-norm if and only if

$$c_n \log |n| \to 0 \qquad \text{as} \qquad n \to \pm\infty. \tag{6.1}$$

It was shown in [2] that the same criterion for norm convergence holds for integrable trigonometric series with coefficients satisfying weak local forms of the regularity condition (1.5); no symmetry condition was required in that analysis. Earlier papers cited in [2] contained similar results on this question, with forms of regularity that are locally weaker than quasiconvexity but stronger than those used in [2].

We seek counterparts of criterion (6.1) for Vilenkin series. The method of proof in [2] transfers when the system is of bounded type and conventionally indexed, and when the system is fully-odd and symmetrically indexed. To see this, suppose first that the system is of bounded type and that the coefficients, $(a_n)_{n=1}^{\infty}$ say, are regular with respect to some conventional order. Then the series represents some integrable function. It follows that the subsequence of partial sums of order m_r converges in L^1-norm to some function, F say, and the series is the Vilenkin series of F.

As pointed out at the end of Section 3, the sequence of Fejér kernels is norm-bounded in L^1 for systems of bounded type. Then the Fejér means of the series converge in L^1-norm to F, and the same is true for the de la Vallée-Poussin means, $V_N(F)$ say. Follow the standard convention for Vilenkin series by letting $S_N(F)$ denote the sum of all the terms in the series with index *strictly* less than N. If the series converges in L^1-norm, then it must converge to F. So L^1-norm convergence occurs if and only if

$$\|V_N(F) - S_N(F)\|_1 \to 0 \qquad \text{as} \qquad N \to \infty. \tag{6.2}$$

The coefficients of $V_N(F) - S_N(F)$ vanish outside the interval $[N, 2N-2]$; within that interval, they coincide with the coefficients of $V_N(F)$, that is with the coefficients in the series multiplied by the linear sequence that equals 1 at $N-1$ and 0 at $2N-1$. Let

$$G_N(F) = V_N(F) - S_N(F) + a_N D_N, \tag{6.3}$$

where D_N denotes the N-the Dirichlet kernel. As in [2], an analysis of the regularity of the coefficients of $G_N(F)$ show, via Remark 1, that $\|G_N(F)\|_1 \to 0$ as $N \to \infty$. It then follows from equation (6.3) that condition (6.2) holds if and only if

$$\|a_N D_N\|_1 \to 0 \qquad \text{as} \qquad N \to \infty. \tag{6.4}$$

Hence the series converges in L^1-norm if and only if condition (6.4) holds. This corresponds to the norm-convergence criterion for trigonometric series since in that case $\|D_N\|_1$ is of the same order as $\log N$ as $N \to \infty$.

Applying Remark 1 yields a constant C so that $\|D_n\|_1 \leq C \log n$ for all n. So L^1-norm convergence follows here from the condition that $a_n \log n \to 0$ as $n \to \infty$. In some cases, the estimates coming from Remark 1 are too high; for instance, the Dirichlet kernels of order m_r all have L^1-norm 1. For many other indices N, however, the kernels do have order $\log N$. For the Walsh system, this fact and the bounded variation of the coefficient sequence were used [2] to deduce from condition (6.4) that the series converges in L^1-norm only if $a_n \log n \to 0$ as $n \to \infty$.

Now suppose that the Vilenkin system is fully-odd and symmetrically indexed. The discussion above transfers almost verbatim for series with regular *even* coefficients, and yields that such series converge in L^1-norm if and only if

$$\|c_N D_N\|_1 \to 0 \qquad \text{as} \qquad N \to \pm\infty. \tag{6.5}$$

In this context, the symbol D_N denotes the symmetric version of the Dirichlet kernel. When the system is of bounded type, one can transfer a symmetrization argument used in [2, §3] for trigonometric series to reduce the analysis of L^1-norm convergence of any series to the case of even series. It follows that, for fully-odd systems of bounded type, a series with structurally regular coefficients (c_n) converges in L^1-norm if and only if condition (6.5) holds. In this case, and the others discused above, the methods still work when the regularity condition is weakened and made local in the style of [2].

We now discuss the possibility of deducing integrability from versions of condition (1.1) that do not force the sequence (a_n) to have bounded variation. The authors of [7] point out that their method still works for cosine series when the inner sum of absolute values in (1.1) is replaced by the supremum of the quantities

$$\left| \sum_{n=j2^\nu}^{m} \Delta a_n \right| \tag{6.6}$$

over all values of m with $j2^\nu \leq m < (j+1)2^\nu$. It may also be possible to use the methods in [2] and in the present paper with the same weaker version of regularity.

Summing differences, as above, before taking absolute values, controls the sizes of the difference $a_n - a_m$ for certain pairs of integers m and n. As pointed out at the end of [2], the simplest integrability criterion of this kind seems to be a dual version of the condition used by Bernstein [5] and Szasz [22] to prove absolute convergence for certain Fourier series.

To state that dual condition, revert to writing coefficients as functions on the set Γ of all members in the Vilenkin system. Given such a function a on Γ and a member γ of Γ, let $a(\cdot - \gamma)$ denote the function on Γ given by

$$\chi \mapsto a\,(\chi\overline{\gamma}) \qquad \text{for all } \chi \text{ in } \Gamma.$$

This notation confounds multiplication and addition, but it conforms with the use of the additive semigroup of nonnegative integers or the additive group of all integers to index the multiplicative group Γ. Suppose intially that Γ is indexed conventionally, and recall the positive integers k_ν used in Section 3. Say that a function a on Γ belongs to the *homogeneous Besov space* $\dot{B}(1/2,2,1)(\Gamma)$ if

$$\sum_{\nu=1}^{\infty} 2^{-\nu/2} \left\| a - a\left(\cdot - \chi_{k_\nu}\right) \right\|_2 < \infty. \tag{6.7}$$

One of the main properties of these spaces is having a multitude of equivalent definitions [18]. For instance, when Γ is fully odd and symmetrically indexed, one can interlace the sequence (n_r) with a sequence (t_ν), having similar properties to (k_ν), and show that condition (6.7) is then equivalent to requiring that

$$\sum_{\nu=1}^{\infty} 2^{-\nu/2} \left\| a - a\left(\cdot - \psi_{t_\nu}\right) \right\|_2 < \infty. \tag{6.8}$$

This is so even though it rarely happens that $\chi_{k_\nu} = \psi_{t_\nu}$. The most enlightening characterization of $\dot{B}(1/2,2,1)(\Gamma)$ is that the function a should differ by a constant from the coefficients of a function, F say, satisfying the requirement that

$$\sum_{\nu=1}^{\infty} 2^{-\nu/2} \left\| F \cdot 1_{I_\nu} \right\|_2 < \infty. \tag{6.9}$$

This is a key endpoint case of a large family [13] of inclusions and equivalences. The elementary details for the trigonometric version of the equivalence of conditions (6.7) and (6.9) are presented at the end of [2]. That method transfers easily to the present context. It also shows that conditions (6.8) and (6.9) are equivalent for fully-odd systems.

The connection with integrability is that condition (6.9) implies, via Cauchy-Schwarz, that $F \in L^1(0,1)$. On the other hand, in Theorem 1, our deduction of integrability follows from regularity and sufficient symmetry proceeded via Lemmas 2 and 3, which provide estimates for the L^2 and L^∞-norms of restrictions of parts of the function f to the sets I_ν. Estimates for the L^2-norms of the restrictions of F to the same sets follow immediately, and they combine to give condition (6.9). So regularity and sufficient symmetry of the coefficients in a Vilenkin series imply that the coefficients belong to the space $\dot{B}(1/2,2,1)(\Gamma)$.

Arguments using the sizes of the quantities (6.6) also seem to lead to estimates for L^2 and L^∞-norms over the sets I_ν, and then to condition (6.9). This means that the sequences to which these arguments are applied must actually satisfy the weaker and simpler condition (6.7). In principle, it should be possible to deduce (6.7) or (6.8) from regularity and sufficient symmetry by direct estimates on the group Γ, rather than proceeding via (6.9). This program was carried out for the trigonometric case at the end of [2]. It turned out to be nearly as complicated as the route via the counterpart of condition (6.9), and then a separate argument was required for the part of Theorem 1 showing that regularity and integrability imply sufficient symmetry. Presumably that program can be carried out for Vilenkin systems, with the regularity condition weakened as suggested near line (6.6).

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