Week 7 lecture notes
October 25, 26, 27 (2022)

Topics: Optimization

Instructor notes:

- Multiple examples of "flavoured" optimization problems are provided in the third section. We should use the ones (or come up with ones) most appropriate to the class flavour.

Learning Objectives (Global extrema on a closed interval):

- Determine the critical and singular points of a function.
- Find the global extrema of a function on a closed interval.
- Explain how the algorithm can be used in optimization problems.

Problems and takeaways (Global extrema on a closed interval):

Let $f(x) = -x^4 + x^2 + 1$ on the interval $[-5, 0]$.

1. Find all points $x = c$ such that $f'(c) = 0$ or $f'(c)$ does not exist.
2. **Definition**: Let $f(c)$ be defined. If $f'(c) = 0$, $x = c$ is a *critical point* of $f(x)$. If $f'(c)$ does not exist, $x = c$ is a *singular point* of $f(x)$.

   See CLP-1 Definition 3.5.6.
3. Evaluate $f(x)$ at the endpoints of $[-5, 0]$, and at its critical and singular points.
4. Sketch the graph of $f(x)$.
5. Where does $f(x)$ achieve a global maximum on $[-5, 0]$? Where does it achieve a global minimum?
6. **Takeaway**: If $f(x)$ is continuous on a closed interval, then $f(x)$ has a global maximum and a global minimum on that interval.

   See CLP-1 Theorem 3.5.11.

   These global extrema may be found by evaluating the function at critical points, singular points and endpoints.

   See CLP-1 Theorem 3.5.12 and Corollary 3.5.13.
7. What if we considered $f(x)$ on the interval $(-5, 0)$? Does $f(x) = -x^4 + x^2 + 1$ have a global maximum and a global minimum on the interval $(-5, 0)$?
8. **Takeaway**: A continuous function on an open interval does not necessarily have a global maximum and a global minimum on that interval. However, you can still final local and global extrema by determining where the function is increasing and where it is decreasing.

Learning Objectives (Geometric optimization):

- Convert geometric information into a function optimization problem.

Problems and takeaways (Geometric optimization):

1. Suppose you have 100 metres of fencing to enclose a rectangular area against a long, straight wall. What is the largest area you can enclose?

2. **Takeaway**: It is helpful to solve optimization problems in a systematic way.
   (a) Draw and label a picture.
   (b) Write down what you wish to optimize.
   (c) Write down an equation that expresses the thing you wish to optimize in terms of variables.
   (d) If necessary, reduce the equation to one variable.
   (e) Differentiate and solve.
   (f) Do a "reality check" to see if your answer makes sense.

Learning Objectives (Real-world model optimization):

- Interpret model optimization problems based on real-world examples according to their context.

Problems and takeaways (Real-world model optimization):

1. (Applications to Physical Sciences) Two wall lamps are placed in a narrow corridor. The first lamp is placed at point A and provides an illumination of $100/x^2$ lumens at a distance of x metres from the lamp; the second lamp is placed at a point B which is 10 metres away from A and provides an illumination of $50/x^2$ lumens at a distance of x metres from the lamp. At what point in the corridor between A and B is the total illumination minimized?

2. (Applications to Biology and Life Sciences, adapted from L. Keshet's book) For a spherical cell of radius r micrometers, the nutrient absorption and consumption rates, $A(r)$ and $C(r)$ respectively, are:

   $$A(r) = 4k\pi r^2 \quad \text{and} \quad C(r) = \frac{4}{3}\pi k r^3$$

   for some constant $k > 0$. Assuming that $1 \leq r \leq 5$, determine the radius of the cell for which the net rate of increase of nutrients is largest.

3. (Applications to Commerce and Social Sciences) Owners of a car rental company have determined that if they charge customers d dollars per day to rent a car, where $50 \leq d \leq 200$, the number of cars N they rent per day can be modelled by the function $N(d) = 1000 - 5d$. How much should they charge to maximize their revenue?

4. (Applications to Commerce and Social Sciences) A car factory can produce up to 120 units per week. Find the (whole number) quantity q of units which maximizes profit if the total revenue in dollars is $R(q) = 750q - 3q^2$, the total cost in dollars is $C(q) = 10000 + 148q$, and $0 \leq q \leq 120$. (Remember that profit is revenue minus cost.)

Additional problems

- Does $f(x) = \frac{1}{x}$ have a global maximum on the interval $(0, 5)$?
- Find the global maximum and global minimum of $f(x) = \sqrt[3]{x^2 - 64}$ on the interval $[-1, 10]$.

- Suppose we wish to find the dimensions of a cylindrical can that holds a constant volume V of juice and uses the least amount of material. What are the dimensions of the can that minimize the surface area?
- Suppose we wish to find the dimensions of a cylindrical can whose surface area is a constant S and holds the most volume. What are the dimensions of the can that maximize the volume? How is this problem related to the previous problem?
- CLP-1 Problem Book 3.5.1: Q1-Q7.
- CLP-1 Problem Book 3.5.2.: Q2, Q4, Q5.
- CLP-1 Problem Book 3.5.3: Q1-Q15.