Week 7 small class

Topics: Optimization

Instructor notes:

- Extra materials: string, scissors, and rulers.
- The main problem is deliberately broken up into many small steps; this is to encourage the whole class to stay together as the problem is solved.
- *Tip of the week: go for deeper exchanges.* This week's main problem is technically straightforward, and there are many opportunities for reflection afterward – see questions 9-13. Start to encourage extended discussions ... but make sure to paraphrase student contributions, and don't let things go off the rails!

Learning Objectives

- Set up and optimize a function based on geometry.
- Confirm an answer derived with calculus using a method that does not use calculus.

Problems and takeaways:

1. Cut a metre-long piece of string. Then cut that piece of string in two. Form one piece (your choice) into a square, and the other piece into a circle.

   Team work.

2. Calculate the total enclosed area.

   Team work. You can ask students what their areas are and put them on the board. Why are some bigger?

3. The challenge: suppose you can cut one of your two pieces of string one more time, and reallocate string from the circle to the square, or from the square to the circle. Can you make the total enclosed area larger?

   Class discussion. This should be a simple yes/no poll, not an extended discussion.

4. Draw and label a picture of the setup.

   Team work. Ensure that the circumference of the circle is given by x, and that the perimeter of the square is given by $1 - x$. (Teams that choose the other way around should be encouraged to follow through with their version on their own time.) After at least two teams have drawn a well-labelled picture, have a team draw their picture on the board.

5. What do you wish to optimize?

   Class discussion.

   We wish to maximize the enclosed area.

6. Write down an equation that expresses the thing you wish to optimize in terms of x.

   Team work. After at least two teams have a good answer, have a team write down their answer on the board. This may be done in two stages: one where not every length is in terms of x, and one where every length is.

   $A(x) = \frac{x^2}{4\pi} + \frac{(1-x)^2}{16}$.

7. Differentiate.

   Team work. After at least two teams have a good answer, have a team write down their answer on the board.

   $A'(x) = \frac{2x}{4\pi} - \frac{2(1-x)}{16}$.

8. What are the critical and singular points of $A(x)$?

   Team work. Teams should keep their answers to themselves. Teams that finish quickly should be dispersed to help other teams.

   $A(x)$ has one critical point $x = \frac{\pi}{\pi+4}$.

9. Does $x = \frac{\pi}{\pi+4}$ maximize $A(x)$?

   Class discussion. Once the conclusion below is reached, move on to team work.

   Not necessarily: the domain is $0 \leq x \leq 1$, and we need to check the endpoints.

   Team work. Teams should again keep their answers to themselves until the whole class is called together for a discussion.

   $A(x)$ is maximized when $x = 1$; that is, when all the string is used in the circle.

10. What is the shape of the graph of $A(x)$? Why does that tell you that $A(x)$ is *not* maximized at its critical point?

    Class discussion.

    $A(x)$ is a parabola opening upward. Its critical point is at its minimum. $A(x)$ is therefore guaranteed to be maximized at one of the endpoints.

11. **Takeaway**: There are often ways to confirm or even generate answers using non-calculus methods.

12. The circle of circumference 1 m encloses more area than the square of perimeter 1 m. Why?

    Class discussion.

    There are many acceptable answers: $\frac{1}{16} < \frac{1}{4\pi}$, squares have more "wasted space", soap bubbles are spherical and not cubical, André Weil proved the isoperimetric inequality in two dimensions, etc.

13. Do you think this answer – that all the string should be used in the circle — depends on the original length of string? Why or why not?

    Class discussion.

Additional problems:

In all the additional problems, try to predict the answer beforehand.

1. Do the same problem as above, but forming a square and a triangle instead of a square and a circle.

2. Do the same problem as above, but forming two circles instead of a square and a circle.

3. Suppose you wish to form a metre-long piece of string into a rectangle of maximal area. What are the dimensions of the rectangle?

4. Suppose you have formed a metre-long piece of string into a shape that is *not* a circle. Can you guarantee that it does *not* enclose as much area as if the shape were a circle? (This is a challenging question that is posed for your edification; unlike other additional problems, it is not directly relevant to the final exam.)