Math 612: Single Cell Analysis **2019W Term 1**

Lecture 11: October 10

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11.1 Introduction

Duality
Duality is the principle that optimization problems can be seen as two fold: the primal problem and the dual problem. Solving the dual problem yields the lower bound of the solution for the primal problem but the values for both solutions is not always equal. This difference is called the duality gap. For convex optimization problems, this gap is zero.

11.1.1 Primal problem

Consider a general optimization problem, here is the primal problem.

Primal problem:
$$\begin{aligned} \min_{\chi \in \mathbb{R}} \quad & f_o(x) \\ \text{s.t} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

11.1.2 Lagrangian Function

The dual problem is often known as the Lagrangian dual problem. This is formed by taking the Lagrangian of a minimization problem and solving for the primal values that minimize the original objective function. Below we let p^* be the primal optimal value. For any optimization problem there is a Lagrangian function:

$L(\chi, \lambda, \nu) = f_o(x) + \sum_{i=1}^{m} d_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$

L: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \Rightarrow \mathbb{R}$

Define $g(\lambda, \nu) = \quad \underset{x}{\text{minimize}} L(x, \lambda, \nu)$

Facts

- The minimum of multiple linear functions is concave, therefore g is concave
- Lower bound: for any feasible point $\quad \tilde{x} f_i(\tilde{x} \leq 0$ AND $h_i(\tilde{x}) = 0$
  and for any $\lambda \geq 0 i = 1 \ldots m$
- $f_o(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \min_x L(x, \lambda, \nu) = g(\lambda, \nu)$

11.2 Weak and Strong Duality

The duality gap, as previously mentioned, is the difference between the dual optimal value, d^*, and the optimal primal value, p^*. If this gap is zero then strong duality applies. Otherwise, weak duality applies. An example of weak duality is shown in Figure 9.1 below.

The dual optimal value d^* is $\underset{\lambda\nu}{\mathrm{M}ax}\ g(\lambda, \nu)$

$$f_o(\tilde{x}) \geq d^* \tag{11.1}$$
$$p^* \geq d^* \text{Weak duality} \tag{11.2}$$
$$p^* = d^* \text{Strong duality} \tag{11.3}$$

Example Primal problem: $\begin{array}{ll} \min\limits_{X} & X^T X = ||X||_2^2 \\ \text{s.t} & A_x = b \end{array}$

$L(X, \nu) = \nu(AX - b) + X^T X$
$\nabla x L = 2x + A^T \nu = 0$
$X = -\frac{1}{2} A^T \nu$
$g(\nu) = L(-\frac{1}{2} A^T \nu, \nu)$
$g(\nu) = \frac{1}{4} \nu^T A A^T \nu - b^T \nu$
dual: $\underset{\nu}{\mathrm{M}ax}\ g(\nu)$

Weak duality: $p^* \geq d^*$
Strong duality: $p^* = d^*$
True under "constraint qualification" conditions

11.3 Slater's condition

For most convex optimization problems, strong duality often applies only in addition to some conditions. One such condition is Slater's theorem.

Definition: Strong duality holds for convex problems if there is a point $\tilde{x}$ with $f_i(\tilde{x}) < 0$ for all $i = 1, \ldots, m$ (i.e. all constraints are satisfied and the non linear restraints are satisfied with strict inequalities). Figure 9.2 shows cases where there is a non-strictly feasible point (a.), a strictly feasible point (b.), and no feasible point (c.)

Example Optimization problem with 1 constraint: $\begin{array}{ll} \min\limits_{x} & f_o(v) \\ \text{s.t.} & f_i(v) \leq 0 \end{array}$

$G = (f_i(v), f_o(v))$

$g(\lambda) = \min_{(x,y) \in G} Y + \lambda x$

$A = (x, y) : f_i(v) \leq x, f_o(v) \leq y$ for some v

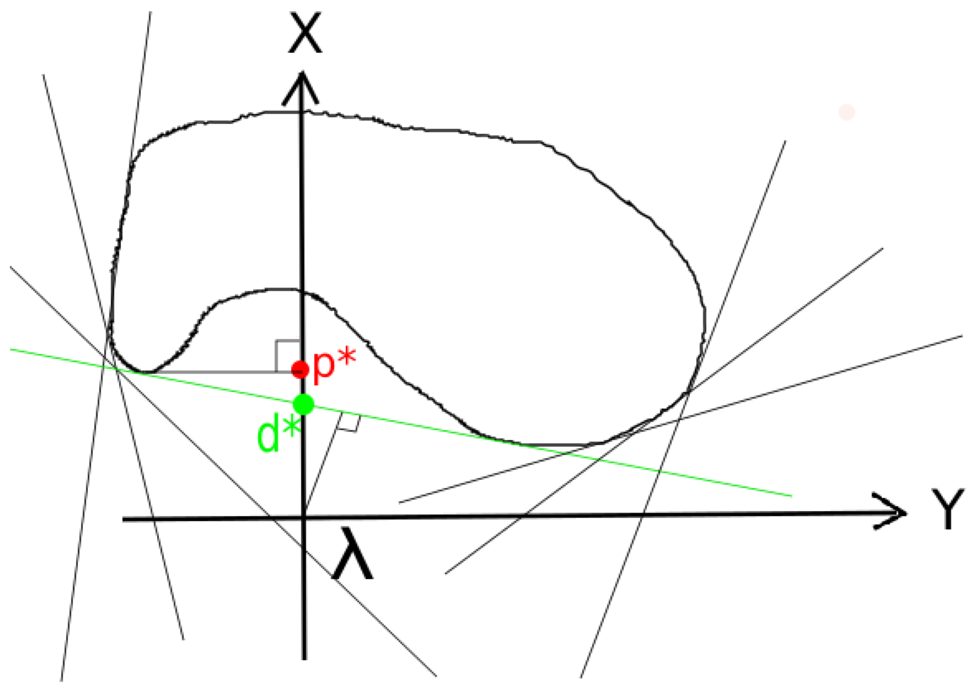

Figure 11.1: Supporting hyperplanes for positive λs

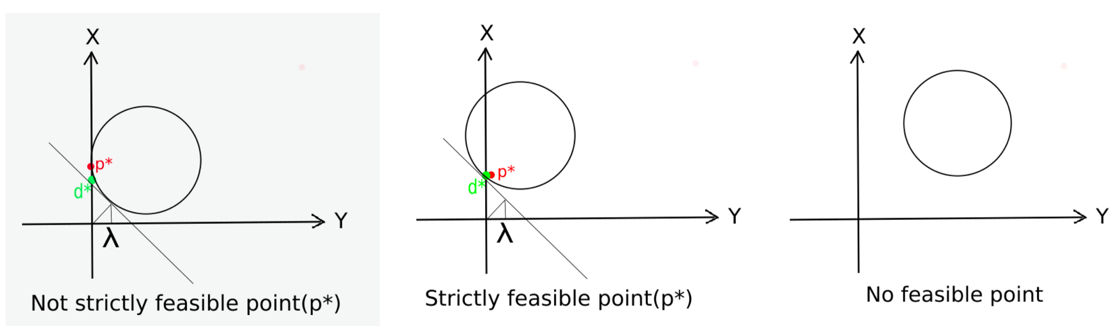

Figure 11.2: Examples of covex problems with and without feasible points

11.3.1 Summary

Slater's condition There is a strictly feasible point at $(0, p^*)$ where the supporting hyperplane is non vertical and $d^* = p^*$, therefore there is strong duality.