Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Number Theory 112 (2005) 298–331
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$

Greg Martin

Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, BC, V6T 1Z2 Canada

Received 18 November 2003; revised 19 May 2004

Communicated by A. Granville

Available online 29 December 2004

Abstract

A formula for the dimension of the space of cuspidal modular forms on $\Gamma_0(N)$ of weight k ($k \geqslant 2$ even) has been known for several decades. More recent but still well-known is the Atkin–Lehner decomposition of this space of cusp forms into subspaces corresponding to newforms on $\Gamma_0(d)$ of weight k, as d runs over the divisors of N. A recursive algorithm for computing the dimensions of these spaces of newforms follows from the combination of these two results, but it would be desirable to have a formula in closed form for these dimensions. In this paper we establish such a closed-form formula, not only for these dimensions, but also for the corresponding dimensions of spaces of newforms on $\Gamma_1(N)$ of weight k ($k \geqslant 2$). This formula is much more amenable to analysis and to computation. For example, we derive asymptotically sharp upper and lower bounds for these dimensions, and we compute their average orders; even for the dimensions of spaces of cusp forms, these results are new. We also establish sharp inequalities for the special case of weight-2 newforms on $\Gamma_0(N)$, and we report on extensive computations of these dimensions: we find the complete list of all N such that the dimension of the space of weight-2 newforms on $\Gamma_0(N)$ is less than or equal to 100 (previous such results had only gone up to 3).

MSC: 11F11; 11F25

Keywords: Modular forms; Cusp forms; Newforms; Congruence groups; Atkin–Lehner decomposition

E-mail address: gerg@math.ubc.ca.

0022-314X/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jnt.2004.10.009

1. Introduction

The study of modular forms on congruence groups was initiated by Hecke and Petersson in the 1930s and, at least when the weight k is an integer exceeding 1, is quite well understood. In particular, formulas for the dimensions of the spaces of modular forms and cusp forms on the congruence groups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \,(\mathrm{mod}\, N) \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \,(\mathrm{mod}\, N),\ c \equiv 0 \,(\mathrm{mod}\, N) \right\}$$

are known [6,7] (see Propositions 12 and 15 below). The structure of these spaces of cusp forms was clarified by the work of Atkin and Lehner [1], who exhibited their orthogonal decomposition with respect to the Petersson inner product into spaces of cuspidal newforms. Until now, however, the dimensions of the spaces of newforms could only be calculated recursively (in terms of the corresponding dimensions for divisors of the level N) and thus were rather poorly understood in general.

In this paper, we present closed formulas for the dimensions of the spaces of weight-k cuspidal newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$, for all integers $k \geqslant 2$. The formulas consist of linear combinations of multiplicative functions of N, with coefficients depending on k; in particular, they have the same level of simplicity as the formulas for the dimensions of the full spaces of cusp forms on these modular groups. As an application of the new formulas, we derive simple upper and lower bounds for the dimensions of these spaces of newforms for all $k \geqslant 2$. We also calculate all positive integers N for which the dimension of the space of newforms of weight 2 on $\Gamma_0(N)$ is at most 100, and we prove the validity of certain inequalities and identities for these dimensions observed empirically by Bennett. Finally, we calculate the average orders both of the dimensions of the spaces of weight-k cusp forms on $\Gamma_0(N)$ and $\Gamma_1(N)$ and of the dimensions of the subspaces of newforms. In addition, we establish analogues of all these results for the numbers of non-isomorphic automorphic representations associated with these spaces of modular forms.

For the sake of the reader using this paper as a reference, we begin by listing together all of the functions for which we establish explicit formulas, upper and lower bounds, and average orders. All of the notation will be explained in more detail subsequently. The following list summarizes of the functions we investigate, after which a table displays the locations of the corresponding results:

- $g_0^\#(k,N)$: the dimension of the space of weight-k newforms on $\Gamma_0(N)$,
- $g_0^*(k,N)$: the number of non-isomorphic automorphic representations associated with $S_k(\Gamma_0(N))$,

- $g_0(k, N)$: the dimension of the full space of weight-k cusp forms on $\Gamma_0(N)$,
- $g_1^{\#}(k, N)$: the dimension of the space of weight-k newforms on $\Gamma_1(N)$,
- $g_1^{*}(k, N)$: the number of non-isomorphic automorphic representations associated with $S_k(\Gamma_1(N))$,
- $g_1(k, N)$: the dimension of the full space of weight-k cusp forms on $\Gamma_1(N)$,
- $\rho_0(k, N)$: the ratio $g_0^{\#}(k, N)/g_0(k, N)$, that is, the proportion of $S_k(\Gamma_0(N))$ occupied by $S_k^{\#}(\Gamma_0(N))$,
- $\rho_1(k, N)$: the ratio $g_1^{\#}(k, N)/g_1(k, N)$, that is, the proportion of $S_k(\Gamma_1(N))$ occupied by $S_k^{\#}(\Gamma_1(N))$,

Function	Formula	Bounds	Average order
$g_0^{\#}(k, N)$	Theorem 1	Theorem 6	Theorem 8
$g_0^{*}(k, N)$	Theorem 4	Theorem 6	Theorem 8
$g_0(k, N)$	Proposition 12	Theorem 6	Theorem 8
$g_1^{\#}(k, N)$	Theorem 13	Theorem 7	Theorem 9
$g_1^{*}(k, N)$	Theorem 14	Theorem 7	Theorem 9
$g_1(k, N)$	Proposition 15	Theorem 7	Theorem 9
$\rho_0(k, N)$		Theorem 10	Theorem 11
$\rho_1(k, N)$		Theorem 10	Theorem 11

We now elaborate on the notation with which our results will be described. Let $S_k(\Gamma_0(N))$ denote the space of cusp forms on $\Gamma_0(N)$ of weight k and $S_k^{\#}(\Gamma_0(N))$ the space of newforms on $\Gamma_0(N)$ of weight k. Let $g_0(k, N)$ and $g_0^{\#}(k, N)$ denote the dimensions of $S_k(\Gamma_0(N))$ and $S_k^{\#}(\Gamma_0(N))$, respectively. Our formula for $g_0^{\#}(k, N)$ involves several multiplicative functions that we shall define shortly. Recall that a function f, not identically zero, is multiplicative if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. It follows that $f(1) = 1$ and that f is completely determined by its values on prime powers. Some common examples of multiplicative functions that will be useful to us are Euler's totient function $\phi(n)$ and the Möbius function $\mu(n)$; also $\omega(n)$, the number of distinct prime factors of n, and $\tau(n)$, the number of positive divisors of n; and finally the delta function at 1,

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Our first theorem shows that $g_0^{\#}(k, N)$ can be expressed as a linear combination of multiplicative functions of N, with the coefficients depending on k.

Theorem 1. *For any even integer $k \geqslant 2$ and any integer $N \geqslant 1$, we have*

$$g_0^{\#}(k, N) = \frac{k-1}{12} N s_0^{\#}(N) - \tfrac{1}{2} v_\infty^{\#}(N) + c_2(k) v_2^{\#}(N) + c_3(k) v_3^{\#}(N) + \delta\left(\frac{k}{2}\right) \mu(N),$$

where the functions $s_0^{\#}$, $v_\infty^{\#}$, $v_2^{\#}$, $v_3^{\#}$, c_2, and c_3 are defined in Definition $1'$ *below.*

We remark that the restriction that k be even is natural, since there are no modular forms on $\Gamma_0(N)$ of odd integer weight, that is, $g_0(k,N)=0$ and hence $g_0^\#(k,N)=0$ when k is odd. We promptly give the definitions of the six functions in the statement of Theorem 1. In the definitions of the multiplicative functions and throughout this paper, p always denotes a prime number.

Definition 1′.

(A) $s_0^\#$ is the multiplicative function satisfying
$s_0^\#(p)=1-\frac{1}{p}$, $s_0^\#(p^2)=1-\frac{1}{p}-\frac{1}{p^2}$, and $s_0^\#(p^\alpha)=\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)$ for $\alpha\geqslant 3$.

(B) $v_\infty^\#$ is the multiplicative function satisfying $v_\infty^\#(p^\alpha)=0$ for α odd, $v_\infty^\#(p^2)=p-2$, and $v_\infty^\#(p^\alpha)=p^{\alpha/2-2}(p-1)^2$ for $\alpha\geqslant 4$ even.

(C) $v_2^\#$ is the multiplicative function satisfying:

- $v_2^\#(2)=-1$, $v_2^\#(4)=-1$, $v_2^\#(8)=1$, and $v_2^\#(2^\alpha)=0$ for $\alpha\geqslant 4$;
- if $p\equiv 1 \pmod 4$ then $v_2^\#(p)=0$, $v_2^\#(p^2)=-1$, and $v_2^\#(p^\alpha)=0$ for $\alpha\geqslant 3$;
- if $p\equiv 3 \pmod 4$ then $v_2^\#(p)=-2$, $v_2^\#(p^2)=1$, and $v_2^\#(p^\alpha)=0$ for $\alpha\geqslant 3$.

(D) $v_3^\#$ is the multiplicative function satisfying:

- $v_3^\#(3)=-1$, $v_3^\#(9)=-1$, $v_3^\#(27)=1$, and $v_3^\#(3^\alpha)=0$ for $\alpha\geqslant 4$;
- if $p\equiv 1 \pmod 3$ then $v_3^\#(p)=0$, $v_3^\#(p^2)=-1$, and $v_3^\#(p^\alpha)=0$ for $\alpha\geqslant 3$;
- if $p\equiv 2 \pmod 3$ then $v_3^\#(p)=-2$, $v_3^\#(p^2)=1$, and $v_3^\#(p^\alpha)=0$ for $\alpha\geqslant 3$.

(E) c_2 is the function defined by $c_2(k)=\frac{1}{4}+\left\lfloor\frac{k}{4}\right\rfloor-\frac{k}{4}$.
(F) c_3 is the function defined by $c_3(k)=\frac{1}{3}+\left\lfloor\frac{k}{3}\right\rfloor-\frac{k}{3}$.

We remark that as this manuscript was being prepared, a paper of Halberstadt and Kraus [5] appeared, in the appendix of which they independently established the special case of Theorem 1 where $k=2$.

The formula given in Theorem 1 provides a method of computing $g_0^\#(k,N)$ that is much faster than the recursive formula (16) below. In Section 5, we show how to use such a computation to determine the complete list of positive integers N such that $g_0^\#(2,N)$ is at most 100. Previously, exhaustive lists of those N for which $g_0^\#(2,N)=j$ had been given [5] only for $j=0,1,2,3$. We also gather evidence supporting the assertion that every non-negative integer is a value of the function $g_0^\#(2,N)$, but we refute this assertion for $g_0(2,N)$ itself—the first omitted value is 150.

Moreover, the formula in Theorem 1 is much more amenable to analysis of the behavior of the function $g_0^\#(k,N)$. For example, the coefficients of the last four multiplicative functions $v_\infty^\#$, $v_2^\#$, $v_3^\#$, and μ in this formula are all bounded functions of k. Therefore we can immediately conclude that when N is fixed, the dimension $g_0^\#(k,N)$ grows roughly linearly with k; more precisely,

$$g_0^\#(k,N)=\frac{Ns_0^\#(N)}{12}k+O_N(1).$$

Two further concrete examples of the usefulness of the explicit formula in Theorem 1 are provided by the following two results. These theorems establish the validity of certain identities and inequalities proposed by Bennett (personal communication) on the basis of numerical observations.

Theorem 2. *For all positive integers N, we have $g_0^\#(2,N) \leqslant (N+1)/12$, with equality holding if and only if either $N = 35$ or N is a prime that is congruent to* $11 \pmod{12}$.

Theorem 3. *Let $N \geqslant 3$ be an odd squarefree integer. Then $g_0^\#(k, 2^\alpha N) = (k-1)2^{\alpha-5}\phi(N)$ for every integer $\alpha \geqslant 4$; in particular, $g_0^\#(k, 32N) = (k-1)\phi(N)$. In addition, we have $g_0^\#(k, 2N) \leqslant (k-1)\phi(N)$.*

The method of proof of Theorem 1 can also be used to establish a similar formula for the number of non-isomorphic automorphic representations associated with $S_k(\Gamma_0(N))$, which we denote by $g_0^*(k,N)$. (See the proof of Theorem 4 in Section 2 for a more precise definition of the number in question.) Our next theorem shows that $g_0^*(k,N)$ can also be expressed as a linear combination of multiplicative functions of N.

Theorem 4. *For any even integer $k \geqslant 2$ and any integer $N \geqslant 1$, we have*

$$g_0^*(k,N) = \frac{k-1}{12} N s_0^*(N) - \tfrac{1}{2} v_\infty^*(N) + c_2(k) v_2^*(N) + c_3(k) v_3^*(N) + \delta\left(\frac{k}{2}\right)\delta(N),$$

where the functions c_2, c_3, s_0^*, v_∞^*, v_2^*, and v_3^* are defined in Definition $1'$(E)–(F) above and Definition $4'$ below.

The definitions of the four new functions in the statement of Theorem 4 are as follows.

Definition 4′.

(A) s_0^* is the multiplicative function satisfying
$$s_0^*(p) = 1 \text{ and } s_0^*(p^\alpha) = 1 - \tfrac{1}{p^2} \text{ for } \alpha \geqslant 2.$$
(B) v_∞^* is the multiplicative function satisfying
$$v_\infty^*(p) = 1 \text{ and } v_\infty^*(p^\alpha) = p^{\lfloor \alpha/2 - 1 \rfloor}(p-1) \text{ for } \alpha \geqslant 2.$$
(C) v_2^* is the multiplicative function satisfying:

- $v_2^*(2) = 0$, $v_2^*(4) = -1$, and $v_2^*(2^\alpha) = 0$ for $\alpha \geqslant 3$;
- if $p \equiv 1 \pmod 4$ then $v_2^*(p) = 1$ and $v_2^*(p^\alpha) = 0$ for $\alpha \geqslant 2$;
- if $p \equiv 3 \pmod 4$ then $v_2^*(p) = -1$ and $v_2^*(p^\alpha) = 0$ for $\alpha \geqslant 2$.

(D) v_3^* is the multiplicative function satisfying:

- $v_3^*(3) = 0$, $v_3^*(9) = -1$, and $v_3^*(3^\alpha) = 0$ for $\alpha \geqslant 3$;
- if $p \equiv 1 \pmod 3$ then $v_3^*(p) = 1$ and $v_3^*(p^\alpha) = 0$ for $\alpha \geqslant 2$;
- if $p \equiv 2 \pmod 3$ then $v_3^*(p) = -1$ and $v_3^*(p^\alpha) = 0$ for $\alpha \geqslant 2$.

Theorem 4 allows a very short proof of a result of Gekeler [4] in the case where the level N is squarefree:

Corollary 5. *Let $k \geqslant 2$ be an even integer, and let $N \geqslant 1$ be a squarefree integer, with $N > 1$ if $k = 2$. Then*

$$g_0^*(k,N) = \frac{k-1}{12}N - \tfrac{1}{2} + c_2(k)\left(\frac{-1}{N}\right) + c_3(k)\left(\frac{-3}{N}\right),$$

where $(\frac{d}{N})$ is Kronecker's extension of the Legendre symbol. In particular, $g_0^*(k,N)$ depends on the residue class of N modulo 12, but not on the prime factorization of N.

We remark that the symbols $(\frac{-1}{N})$ and $(\frac{-3}{N})$ could also be represented by the non-principal characters χ_{-4} and χ_{-3} modulo 4 and 3, respectively. Gekeler used a proof by induction on the number of prime factors of N, which yielded a formula more complicated than, but equivalent to, the formula in Corollary 5. The corollary follows immediately from Theorem 4 by noting that $\delta(\frac{k}{2})\delta(N) = 0$ under the hypothesis $(k,N) \neq (2,1)$ and that $s_0^*(p) = v_\infty^*(p) = 1$, $v_2^*(p) = (\frac{-1}{p})$, and $v_3^*(p) = (\frac{-3}{p})$ for every prime p.

The situation is exactly the same for modular forms on $\Gamma_1(N)$: although the dimensions of spaces of cusp forms on $\Gamma_1(N)$ are well-understood, the dimensions of the corresponding spaces of newforms are more mysterious. Let $S_k(\Gamma_1(N))$ denote the space of cusp forms on $\Gamma_1(N)$ of weight k and $S_k^\#(\Gamma_1(N))$ the space of newforms on $\Gamma_1(N)$ of weight k. Let $g_1(k,N)$ and $g_1^\#(k,N)$ denote the dimensions of $S_k(\Gamma_1(N))$ and $S_k^\#(\Gamma_1(N))$, respectively. Also let $g_1^*(k,N)$ denote the number of non-isomorphic automorphic representations associated with $S_k(\Gamma_1(N))$. The method of proof of Theorems 1 and 4 can also be used to establish formulas for $g_1^\#(k,N)$ and $g_1^*(k,N)$ for any integer $k \geqslant 2$ (not necessarily even). Since the expressions that result are slightly more complicated than the above expressions for $g_0^\#(k,N)$ and $g_0^*(k,N)$, we defer the statements of the formulas to Theorems 13 and 14 in Section 3. The complications arise because the most natural formula for $g_1(k,N)$ holds only for $N \geqslant 5$; the presence of elliptic points and irregular cusps corresponding to $\Gamma_1(N)$ for $1 \leqslant N \leqslant 4$ causes $g_1(k,N)$ to be somewhat different for these small values of N. Unfortunately, the behavior of $g_1^\#(k,N)$ and $g_1^*(k,N)$ depends on the values of $g_1(N',k)$ for all divisors N' of N, and so the exceptional cases $1 \leqslant N' \leqslant 4$ influence every single value of $g_1^\#(k,N)$ and $g_1^*(k,N)$.

The explicit nature of the formulas in these theorems allows us to determine both the precise average orders and sharp asymptotic upper and lower bounds for these counting functions as well. The minimal and maximal orders of these functions are given in the next two theorems. Recall that $\gamma = \lim_{x\to\infty}\left(\sum_{n\leqslant x}\frac{1}{n} - \log x\right) \approx 0.577216$ is Euler's constant.

Theorem 6. *Uniformly for all even integers $k \geqslant 2$ and all integers $N \geqslant 1$, we have*:

(a) $\frac{k-1}{12}N + O(\sqrt{N}\log\log N) < g_0(k,N) < \frac{e^\gamma(k-1)}{2\pi^2}N\log\log N + O(N)$;

(b) $\frac{k-1}{2\pi^2}N + O\left(\frac{\phi(N)}{\sqrt{N}}\right) < g_0^*(k,N) < \frac{k-1}{12}N + O(1)$;

(c) $\frac{A_0^\#(k-1)}{12}\phi(N) + O(\sqrt{N}) < g_0^\#(k,N) < \frac{k-1}{12}\phi(N) + O(2^{\omega(N)})$, *where*

$$A_0^\# = \prod_p \left(1 - \tfrac{1}{p^2-p}\right) \approx 0.373956. \tag{2}$$

Moreover, if N is not a perfect square, then the lower bound can be improved to

$$\frac{A_0^\#(k-1)}{12}\phi(N) + O(2^{\omega(N)}) < g_0^\#(k,N).$$

The product defining $A_0^\#$ in Eq. (2) is an infinite product over all prime numbers p. The upper bounds in Theorem 6 imply in particular that both $g_0^*(k,N)$ and $g_0^\#(k,N)$ are bounded above by a constant multiple of kN, in contrast to the size of $g_0(k,N)$ itself which can be as large as a constant multiple of $kN\log\log N$. We remark that Theorem 6 is stronger and more general than [5, Proposition B.1], which appeared as this manuscript was being prepared. Also, the $k=2$ case of Theorem 6(a) appears (with different error terms) in a manuscript of Csirik, Wetherell, and Zieve [3, Section 3].

Theorem 7. *Uniformly for all integers $k \geqslant 2$ and all integers $N \geqslant 1$, we have*:

(a) $\frac{k-1}{4\pi^2}N^2 + O(N\tau(N)+k) < g_1(k,N) < \frac{k-1}{24}N^2 + O(k)$;

(b) $\frac{A_1^*(k-1)}{24}N^2 + O(N\tau(N)+k) < g_1^*(k,N) \leqslant g_1(k,N)$, *where*

$$A_1^* = \prod_p \left(1 - \frac{2}{p^2}\right) \approx 0.322634; \tag{3}$$

(c) $\frac{A_1^\#(k-1)}{24}N^2 + O(N\tau(N)+k) < g_1^\#(k,N) \leqslant g_1^*(k,N)$, *where*

$$A_1^\# = \prod_p \left(1 - \frac{3}{p^2}\right) \approx 0.125487. \tag{4}$$

To judge the quality of these error terms, recall that both $2^{\omega(N)}$ and $\tau(N)$ are $O(N^\varepsilon)$ for any fixed $\varepsilon > 0$. Although Theorems 6(a) and 7(a) are easy consequences of the well-known formulas for $g_0(k,N)$ and $g_1(k,N)$, the bounds contained therein do not seem to have been recorded in the literature. We remark that all of the bounds given in Theorems 6 and 7 are best possible; the proofs of these theorems in Section 6 are easily modified to produce sequences of values of N asymptotically attaining the indicated upper and lower bounds.

We turn now to the question of the average orders of these various functions. Recall that a function $f(n)$ is said to have average order $g(n)$ if

$$\sum_{n \leqslant x} f(n) \sim \sum_{n \leqslant x} g(n),$$

meaning that the quotient of the two sides approaches 1 as x tends to infinity. It turns out that the average orders of the counting functions associated with $\Gamma_0(N)$ are explicit constant multiples of N.

Theorem 8. *Fix an even integer* $k \geqslant 2$.

(a) *The average order of* $g_0(k, N)$ *is* $5(k-1)N/4\pi^2$.
(b) *The average order of* $g_0^*(k, N)$ *is* $15(k-1)N/2\pi^4$.
(c) *The average order of* $g_0^\#(k, N)$ *is* $45(k-1)N/\pi^6$.

(We remark that the $k = 2$ case of Theorem 8(a) was proved in [3, Section 5].) On the other hand, the average orders of the counting functions associated with $\Gamma_1(N)$ depend on the special value $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$ of the Riemann zeta-function.

Theorem 9. *Fix an integer* $k \geqslant 2$.

(a) *The average order of* $g_1(k, N)$ *is* $(k-1)N^2/24\zeta(3)$.
(b) *The average order of* $g_1^*(k, N)$ *is* $(k-1)N^2/24\zeta(3)^2$.
(c) *The average order of* $g_1^\#(k, N)$ *is* $(k-1)N^2/24\zeta(3)^3$.

Another natural quantity to consider is the relative number of newforms with in the spaces of cusp forms on $S_k(\Gamma_0(N))$ and $S_k(\Gamma_1(N))$. To measure this proportion, define

$$\rho_0(k, N) = \begin{cases} g_0^\#(k, N)/g_0(k, N) & \text{if } g_0(k, N) > 0, \\ 1 & \text{if } g_0(k, N) = 0 \end{cases}$$

and similarly for $\rho_1(k, N)$. We are able to establish asymptotically sharp lower bounds for $\rho_0(k, N)$ and $\rho_1(k, N)$, as well as determine their average orders.

Theorem 10. *Uniformly for all integers* $k \geqslant 2$ *and all integers* $N \geqslant 1$, *we have*:

(a) $\frac{A_0^\# \pi^2}{6e^{2\gamma}(\log\log N)^2} + O\left(\frac{1}{(\log\log N)^3}\right) < \rho_0(k, N) \leqslant 1$, *where* $A_0^\#$ *is defined in Eq.* (2);
(b) $\frac{A_1^\# \pi^2}{6} + O\left(\frac{1}{\log N \log\log N} + \frac{k}{N}\right) < \rho_1(k, N) \leqslant 1$, *where* $A_1^\#$ *is defined in Eq.* (4).

Note that $\frac{A_1^\# \pi^2}{6} \approx 0.206418$; we deduce from the lower bound in Theorem 10(b) that when N is large enough with respect to k, it always the case that at least 20% of the weight-k cusp forms on $\Gamma_1(N)$ are newforms.

Theorem 11. *Fix an integer* $k \geqslant 2$.

(a) *If* k *is even, then the average order of* $\rho_0(k, N)$ *is*

$$B_0 = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)^{-1}\left(1 + \frac{2}{p} - \frac{1}{p^4} - \frac{1}{p^5}\right) \approx 0.444301. \tag{5}$$

(b) *The average order of* $\rho_1(k, N)$ *is*

$$\begin{aligned} B_1 &= \prod_p \left(1 + \frac{1}{p}\right)^{-1}\left(1 + \frac{1}{p} - \frac{2}{p^3} - \frac{2}{p^4} - \frac{2}{p^5} + \frac{1}{p^6} + \frac{1}{p^7} + \frac{1}{p^8}\right) \\ &\approx 0.652036. \end{aligned} \tag{6}$$

In Section 2, we prove the main formulas for $g_0^\#(k, N)$ and $g_0^*(k, N)$ given in Theorems 1 and 4. Subsequently, we investigate the analogous functions for modular forms on $\Gamma_1(N)$ in Section 3, culminating in the statements and proofs of Theorems 13 and 14. Sections 4 and 5 are devoted to the explicit inequalities in Theorems 2 and 3 and to computational results concerning $g_0^\#(2, N)$ and $g_0(2, N)$. We finish by establishing the asymptotic inequalities of Theorems 6, 7, and 10 in Section 6 and the average-order results of Theorems 8, 9, and 11 in Section 7.

2. Notation and proof of Theorems 1 and 4

The dimensions of the spaces of weight-k cusp forms on $\Gamma_0(N)$ are well-known for positive even integers k. The following proposition gives a formula for these dimensions, phrased in the way that is most convenient for our purposes.

Proposition 12. *For any even integer* $k \geqslant 2$ *and any integer* $N \geqslant 1$, *we have*

$$g_0(k, N) = \frac{k-1}{12} N s_0(N) - \tfrac{1}{2} v_\infty(N) + c_2(k) v_2(N) + c_3(k) v_3(N) + \delta\left(\frac{k}{2}\right),$$

where the functions s_0, v_∞, v_2, v_3, c_2, *and* c_3 *are defined in Definition* 12′ *below and Definition* 1′(E)–(F) *above.*

The definitions of the four new functions in the statement of Proposition 12 are as follows.

Definition 12′.

(A) s_0 is the multiplicative function satisfying $s_0(p^\alpha) = 1 + \frac{1}{p}$ for all $\alpha \geqslant 1$.

(B) v_∞ is the multiplicative function satisfying

$$v_\infty(p^\alpha) = \begin{cases} 2p^{(\alpha-1)/2} & \text{if } \alpha \text{ is odd,} \\ p^{\alpha/2} + p^{\alpha/2-1} & \text{if } \alpha \text{ is even.} \end{cases}$$

(C) v_2 is the multiplicative function satisfying:

- $v_2(2) = 1$ and $v_2(2^\alpha) = 0$ for $\alpha \geqslant 2$;
- if $p \equiv 1 \pmod 4$ then $v_2(p^\alpha) = 2$ for $\alpha \geqslant 1$;
- if $p \equiv 3 \pmod 4$ then $v_2(p^\alpha) = 0$ for $\alpha \geqslant 1$.

(D) v_3 is the multiplicative function satisfying:

- $v_3(3) = 1$ and $v_3(3^\alpha) = 0$ for $\alpha \geqslant 2$;
- if $p \equiv 1 \pmod 3$ then $v_3(p^\alpha) = 2$ for $\alpha \geqslant 1$;
- if $p \equiv 2 \pmod 3$ then $v_3(p^\alpha) = 0$ for $\alpha \geqslant 1$.

Proof of Proposition 12. The facts invoked in this proof can be found in many sources; we follow the exposition in Miyake [6]. For now we assume that $N \geqslant 2$. We begin by remarking that the multiplicative function $v_\infty(N)$ denotes the number of (inequivalent) cusps of $\Gamma_0(N)$ and that the multiplicative functions $v_j(N)$ denote the numbers of (inequivalent) elliptic points of $\Gamma_0(N)$ of order j. Formulas for these numbers are given in [6, Theorem 4.2.7] in the form

$$v_\infty(N) = \sum_{d|n} \phi\left(\left(d, \frac{n}{d}\right)\right) = \prod_{p^\alpha \| N} \left\{ \sum_{\beta=0}^{\alpha} \phi\left(p^{\min\{\beta,\alpha-\beta\}}\right) \right\} \tag{7}$$

and

$$v_2(N) = \begin{cases} 0 & \text{if } 4 \mid n, \\ \prod_{p|n} \left(1 + \left(\frac{-1}{p}\right)\right) & \text{otherwise;} \end{cases}$$
$$v_3(N) = \begin{cases} 0 & \text{if } 9 \mid n, \\ \prod_{p|n} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{otherwise.} \end{cases} \tag{8}$$

Here again the symbol $(\frac{a}{p})$ is Kronecker's extension of the Legendre symbol. It is easily verified that the formulas for v_2 and v_3 in Eq. (8) are equivalent to the formulas in Definition 12′(C)–(D). It is also easily verified that since $\alpha \geqslant 1$,

$$\sum_{\beta=0}^{\alpha} \phi\left(p^{\min\{\beta,\alpha-\beta\}}\right) = 2 + (p-1) \sum_{\beta=1}^{\alpha-1} p^{\min\{\beta,\alpha-\beta\}-1} = \begin{cases} 2p^{(\alpha-1)/2} & \text{if } \alpha \text{ is odd,} \\ p^{\alpha/2} + p^{\alpha/2-1} & \text{if } \alpha \text{ is even,} \end{cases}$$

and so the formula in Eq. (7) is the same as the formula in Definition 12′(B).

Next, if we let g_N denote the genus of the (compactified) quotient of the upper half-plane by $\Gamma_0(N)$, then we have the formula [6, Theorem 4.2.11]

$$g_N = \frac{\mu_N}{12} - \frac{v_\infty(N)}{2} - \frac{v_2(N)}{4} - \frac{v_3(N)}{3} + 1, \tag{9}$$

where μ_N is the index of $\overline{\Gamma}_0(N)$ in $\overline{SL}_2(\mathbb{Z})$, and $\overline{G}$ denotes the quotient of the group G by its center. According to [6, Theorem 4.2.5],

$$\mu_N = N \prod_{p|N} \left(1 + \frac{1}{p}\right) = Ns_0(N)$$

as defined in Definition 12′(A).

Now the dimension $g_0(k, N)$ of the space of weight-k cusp forms on $\Gamma_0(N)$ can be calculated from this information by the Riemann–Roch theorem. From [6, Theorem 2.5.2] we see that $g_0(2, N) = g_N$ and

$$g_0(k, N) = (k-1)(g_N - 1) + \left(\frac{k}{2} - 1\right) v_\infty(N) + \sum_{j \geqslant 2} \left\lfloor \frac{k}{2}\left(1 - \frac{1}{j}\right) \right\rfloor v_j(N)$$

for every even integer $k \geqslant 4$. Only the terms $j = 2, 3$ are present in the sum due to [6, Lemma 4.2.6], and so the equation for $g_0(k, N)$ becomes

$$g_0(k, N) = (k-1)(g_N - 1) + \left(\frac{k}{2} - 1\right) v_\infty(N) + \left\lfloor \frac{k}{4} \right\rfloor v_2(N) + \left\lfloor \frac{k}{3} \right\rfloor v_3(N).$$

Combining this with the formula (9) and collecting the multiples of $v_\infty(N)$, $v_2(N)$, and $v_3(N)$ yields

$$\begin{aligned} g_0(k, N) \;=\; & \frac{k-1}{12} Ns_0(N) - \tfrac{1}{2} v_\infty(N) + \left(\tfrac{1}{4} - \frac{k}{4} + \left\lfloor \frac{k}{4} \right\rfloor\right) v_2(N) \\ & + \left(\tfrac{1}{3} - \frac{k}{3} + \left\lfloor \frac{k}{3} \right\rfloor\right) v_3(N), \end{aligned} \tag{10}$$

which is the same as the assertion of the proposition (when $k \geqslant 4$) in light of the Definitions 1′(E)–(F) of c_2 and c_3. It is easily checked that the formula holds for $k = 2$ as well. Finally, all of this discussion assumed that $N \geqslant 2$, but the special case $N = 1$ is worked through in detail in [6, Section 4.1], and the formula [6, Corollary 4.1.4] can be seen to agree with the assertion of the proposition as well. □

We may now prove Theorems 1 and 4.

Proof of Theorem 1. If $f(z)$ is a cusp form on $\Gamma_0(d)$, then $f(mz)$ is a cusp form on $\Gamma_0(N)$ for any multiple N of dm. Therefore for every triple (m, d, N) of positive integers such that $dm \mid N$, we have an injection $i_{m,d,N} : S_k(\Gamma_0(d)) \to S_k(\Gamma_0(N))$ defined by $i_{m,d,N}(f)(z) = f(mz)$. As shown by Atkin and Lehner [1], we may write

$$S_k(\Gamma_0(N)) = \bigoplus_{d|N} \bigoplus_{m|N/d} i_{m,d,N}\left(S_k^{\#}(\Gamma_0(d))\right) \tag{11}$$

(in fact, summands corresponding to distinct divisors d are orthogonal with respect to the Petersson inner product). In particular, the dimensions of these spaces satisfy

$$g_0(k, N) = \sum_{d|N} \sum_{m|N/d} g_0^{\#}(k, d) = \sum_{d|N} g_0^{\#}(k, d)\tau(N/d). \tag{12}$$

This equation can be written more simply using the Dirichlet convolution

$$f * g(n) = \sum_{d|n} f(d)g(n/d). \tag{13}$$

Recall that the set of arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ forms a ring under the usual addition of functions and the Dirichlet convolution as the multiplication operation, with the function δ defined in Eq. (1) as the multiplicative identity. In fact, the set of multiplicative functions forms a multiplicative subgroup—the Dirichlet convolution of two multiplicative functions f, g is again multiplicative. Indeed, the values of $f * g$ on prime powers can be computed easily from the values of f and g on prime powers using the identity

$$f * g(p^\alpha) = \sum_{\beta=0}^{\alpha} f(p^\beta)g(p^{\alpha-\beta}), \tag{14}$$

which is a special case of Eq. (13). We also remark that the characteristic property of the Möbius μ function, often phrased as the Möbius inversion formula, is that it is the inverse (under Dirichlet convolution) of the function $1(n)$ that takes the value 1 at all positive integers:

$$(\mu * 1)(n) = \sum_{d|n} \mu(d) = \delta(n).$$

Now in this notation, Eq. (12) says simply that $g_0 = g_0^{\#} * \tau$ for every fixed k. Define λ to be the inverse (under Dirichlet convolution) of τ. Since $\tau = 1 * 1$, we see that

$\lambda = \mu * \mu$. Equivalently, λ is the multiplicative function satisfying

$$\lambda(p) = -2, \quad \lambda(p^2) = 1, \quad \lambda(p^\alpha) = 0 \text{ for } \alpha \geqslant 3 \tag{15}$$

as can be seen by applying the formula (14) with $f = g = \mu$. It follows that $g_0^\# = g_0 * \lambda$ for every fixed k, that is,

$$g_0^\#(k, N) = \sum_{d|N} g_0(k, d)\lambda(N/d). \tag{16}$$

However, since $g_0^\#(k, N)$ is a linear combination of multiplicative functions of N (with coefficients depending on k), it is more natural to take the convolution of λ with the right-hand side of the formula given in Proposition 12. We obtain

$$\begin{aligned} g_0^\#(k, N) = {} & \frac{k-1}{12} Ns_0(N) * \lambda(N) - \tfrac{1}{2}(v_\infty * \lambda)(N) \\ & + c_2(k)(v_2 * \lambda)(N) + c_3(k)(v_3 * \lambda)(N) + \delta\left(\frac{k}{2}\right)(1 * \lambda)(N). \end{aligned}$$

We immediately note that $1 * \lambda = 1 * \mu * \mu = \mu$. Furthermore, the functions $v_\infty * \lambda$, $v_2 * \lambda$, and $v_3 * \lambda$ are all multiplicative; by using the formula (14) we see that they are equal to the functions $v_\infty^\#$, $v_2^\#$, and $v_3^\#$ defined in Definition 1′(B)–(D). Finally, it can be verified using (14) that

$$p^\alpha s_0(p^\alpha) * \lambda(p^\alpha) = \sum_{\beta=0}^{\alpha} p^\beta s_0(p^\beta)\lambda(p^{\alpha-\beta}) = p^\alpha s_0^\#(p^\alpha),$$

where $s_0^\#$ is defined in Definition 1′(A); therefore the multiplicative function $Ns_0(N) * \lambda(N)$ is equal to $Ns_0^\#(N)$. This establishes the theorem. □

Proof of Theorem 4. The spaces of cusp forms $S_k(\Gamma_0(N))$ have bases consisting of modular forms that are eigenforms for all but finitely many Hecke operators. An isomorphism class of automorphic representations corresponds to an equivalence class of eigenforms, where two eigenforms are equivalent if all but finitely many Hecke operators act upon them with the same eigenvalues, or equivalently if both eigenforms are the image of the same newform under two injections $i_{m_1,d,N}$ and $i_{m_2,d,N}$. Therefore, if we define the subspace $S_k^*(\Gamma_0(N))$ of $S_k(\Gamma_0(N))$ to be

$$S_k^*(\Gamma_0(N)) = \bigoplus_{d|N} i_{1,d,N}\left(S_k^\#(\Gamma_0(d))\right), \tag{17}$$

then the dimension of $S_k^*(\Gamma_0(N))$ can be interpreted as the number of non-isomorphic automorphic representations associated with $S_k(\Gamma_0(N))$, which we have denoted by

$g_0^*(k,N)$. From here, the proof is very similar to the proof of Theorem 1. The dimensions of these spaces satisfy

$$g_0^*(k,N) = \sum_{d|N} g_0^\#(k,d);$$

in other words, g_0^* is simply the convolution $g_0^\# * 1$ for every fixed k. We saw in the proof of Theorem 1 that $g_0^\# = g_0 * \lambda$ for every fixed k, and hence $g_0^* = g_0 * \lambda * 1 = g_0 * \mu$, that is,

$$g_0^*(k,N) = \sum_{d|N} g_0(k,d)\mu(N/d).$$

Again, since $g_0(k,N)$ is a linear combination of multiplicative functions of N (with coefficients depending on k), it is natural to use Proposition 12 to write

$$\begin{aligned} g_0^*(k,N) = {} & \frac{k-1}{12} Ns_0(N) * \mu(N) - \tfrac{1}{2}(v_\infty * \mu)(N) \\ & + c_2(k)(v_2 * \mu)(N) + c_3(k)(v_3 * \mu)(N) + \delta\left(\frac{k}{2}\right)(1 * \mu)(N). \end{aligned}$$

We immediately note that $1 * \mu = \delta$. Furthermore, the functions $v_\infty * \mu$, $v_2 * \mu$, and $v_3 * \mu$ are all multiplicative; by using the formula (14) we see that they are equal to the functions v_∞^*, v_2^*, and v_3^* defined in Definition 4′(B)–(D). Finally, using (14) we verify that

$$p^\alpha s_0(p^\alpha) * \mu(p^\alpha) = \sum_{\beta=0}^{\alpha} p^\beta s_0(p^\beta)\mu(p^{\alpha-\beta}) = p^\alpha s_0(p^\alpha) - p^{\alpha-1}s_0(p^{\alpha-1}) = p^\alpha s_0^*(p^\alpha),$$

where s_0^* is defined in Definition 4′(A); therefore the multiplicative function $Ns_0(N) * \mu(N)$ is equal to $Ns_0^*(N)$. This establishes the theorem. □

3. Formulas for $g_1^\#$ and g_1^*

In this section, we state and prove formulas for modular forms on $\Gamma_1(N)$ that are analogous to Theorems 1 and 4.

Theorem 13. *For any integer* $k \geqslant 2$ *and any integer* $N \geqslant 1$, *we have*

$$g_1^\#(k,N) = \frac{k-1}{24} N^2 s_1^\#(N) - \tfrac{1}{4}u^\#(N) + \delta\left(\frac{k}{2}\right)\mu(N) + \sum_{\substack{1 \leqslant i \leqslant 4 \\ i|N}} b_i(k)\lambda(N/i),$$

where the functions $s_1^\#$, $u^\#$, b_1, b_2, b_3, *and* b_4 *are defined in Definition* 13′ *below.*

Recall that the multiplicative function $\lambda = \mu * \mu$ was defined in Eq. (15) above. The definitions of the six functions in the statement of Theorem 13 are as follows.

Definition 13′.

(A) $s_1^\#$ is the multiplicative function satisfying

$$s_1^\#(p) = 1 - \tfrac{3}{p^2},\ s_1^\#(p^2) = 1 - \tfrac{3}{p^2} + \tfrac{3}{p^4},\ \text{and}\ s_1^\#(p^\alpha) = \left(1 - \tfrac{1}{p^2}\right)^3 \ \text{for}\ \alpha \geqslant 3.$$

(B) $u^\#$ is the multiplicative function satisfying $u^\#(p) = 2p - 4$, $u^\#(p^2) = 3p^2 - 8p + 6$, and

$$u^\#(p^\alpha) = p^{\alpha-4}(p-1)^3((\alpha+1)p - \alpha + 3) \ \text{for}\ \alpha \geqslant 3.$$

(C) The functions $b_i(k)$ are defined as follows:

- $b_1(k) = \frac{(-1)^k(k-7)}{24} + \begin{cases} c_2(k) + c_3(k) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd;} \end{cases}$
- $b_2(k) = \frac{1}{2}\left((-1)^k \left\lfloor \frac{k}{4} - 1 \right\rfloor + c_2(k)\right);$
- $b_3(k) = c_3(k);$
- $b_4(k) = -c_2(2k).$

There are many equivalent ways to write the formulas defining the functions $b_i(k)$. Our choices were motivated by the desire to make the sizes of the functions $b_i(k)$ as k grows immediately apparent, knowing that the functions $c_2(k)$ and $c_3(k)$ are bounded in absolute value by $\frac{1}{2}$.

Theorem 14. *For any integer $k \geqslant 2$ and any integer $N \geqslant 1$, we have*

$$g_1^*(k, N) = \frac{k-1}{24} N^2 s_1^*(N) - \tfrac{1}{4} u^*(N) + \delta\left(\frac{k}{2}\right) \delta(N) + \sum_{\substack{1 \leqslant i \leqslant 4 \\ i \mid N}} b_i(k) \mu(N/i),$$

where the functions s_1^*, u^*, b_1, b_2, b_3, and b_4 are defined in Definition 14′ *below and Definition* 13′(C) *above.*

The definitions of the two new functions in the statement of Theorem 14 are as follows.

Definition 14′.

(A) s_1^* is the multiplicative function satisfying

$$s_1^*(p) = 1 - \tfrac{2}{p^2} \ \text{and}\ s_1^*(p^\alpha) = \left(1 - \tfrac{1}{p^2}\right)^2 \ \text{for}\ \alpha \geqslant 2.$$

(B) u^* is the multiplicative function satisfying $u^*(p) = 2p - 4$ and

$$u^*(p^\alpha) = p^{\alpha-3}(p-1)^2((\alpha+1)p - \alpha + 2) \ \text{for}\ \alpha \geqslant 2.$$

As in the previous section, our starting point is a formula for $g_1(k, N)$, the dimension of the space $S_k(\Gamma_1(N))$ of weight-k modular forms on $\Gamma_1(N)$.

Proposition 15. *For any integer* $k \geqslant 2$ *and any integer* $N \geqslant 1$, *we have*

$$g_1(k,N) = \frac{k-1}{24} N^2 s_1(N) - \tfrac{1}{4} u(N) + \delta\left(\frac{k}{2}\right) + \sum_{\substack{1\leqslant i\leqslant 4 \\ i\mid N}} b_i(k)\delta(N/i), \tag{18}$$

where the functions s_1, u, b_1, b_2, b_3, *and* b_4 *are defined in Definition* 15′ *below and Definition* 13′(C) *above.*

The definitions of the two new functions in the statement of Proposition 15 are as follows.

Definition 15′.

(A) s_1 is the multiplicative function satisfying $s_1(p^\alpha) = 1 - \frac{1}{p^2}$ for all $\alpha \geqslant 1$.
(B) u is the multiplicative function satisfying

$$u(p^\alpha) = p^{\alpha-2}(p-1)((\alpha+1)p - \alpha + 1) \quad \text{for all } \alpha \geqslant 1.$$

Proof. As in the proof of Proposition 12, our main task is simply to gather together the known facts about $\Gamma_1(N)$. For now we assume that $N \geqslant 5$. In this case, by [6, Theorem 4.2.9], we know both that $\Gamma_1(N)$ has no elliptic elements and that the number of (inequivalent) cusps of $\Gamma_1(N)$ is given by the formula $\frac{1}{2}\sum_{d\mid n}\phi(d)\phi(n/d)$. We calculate that

$$\begin{aligned}
\sum_{d\mid n}\phi(d)\phi(n/d) &= \prod_{p^\alpha \| n} \sum_{d\mid p^\alpha} \phi(d)\phi(p^\alpha/d) \\
&= \prod_{p^\alpha \| n} \sum_{\beta=0}^{\alpha} \phi(p^\beta)\phi(p^{\alpha-\beta}) \\
&= \prod_{p^\alpha \| n} \left(2p^{\alpha-1}(p-1) + (\alpha-1)p^{\alpha-2}(p-1)^2\right) \\
&= \prod_{p^\alpha \| n} p^{\alpha-2}(p-1)((\alpha+1)p - \alpha + 1).
\end{aligned}$$

Thus this expression for the number of cusps is nothing other than $\frac{1}{2}u(n)$ as defined in Definition 15′(B).

We now let g_N denote the genus of the quotient of the upper half-plane by $\Gamma_1(N)$ and μ_N the index of $\overline{\Gamma}_1(N)$ in $\overline{SL}_2(\mathbb{Z})$, superceding the notation in the proof of Proposition 12. From [6, Theorem 4.2.5], we have that

$$\mu_N = \frac{\phi(N)}{2} N \prod_{p\mid N}\left(1 + \frac{1}{p}\right) = \tfrac{1}{2} N^2 \prod_{p\mid N}\left(1 - \frac{1}{p^2}\right) = \tfrac{1}{2} N^2 s_1(N)$$

according to Definition 15′(A). The formula (9) then becomes

$$g_N = \frac{N^2 s_1(N)}{24} - \frac{u(N)}{4} + 1.$$

Using [6, Theorem 2.5.2] again, we discover that $g_1(2, N) = g_N$ and that for even $k \geqslant 4$,

$$g_1(k, N) = \frac{k-1}{24} N^2 s_1(N) - \tfrac{1}{4} u(N)$$

in analogy with Eq. (10). We may combine these two facts into the single equation

$$g_1(k, N) = \frac{k-1}{24} N^2 s_1(N) - \tfrac{1}{4} u(N) + \delta\left(\frac{k}{2}\right), \tag{19}$$

in agreement with the assertion of the proposition (note that the sum in Eq. (18) is zero when $N \geqslant 5$). An appeal to [6, Theorem 2.5.3] shows that this equation holds when $k \geqslant 3$ is odd as well. This establishes the proposition when $N \geqslant 5$.

Unfortunately, the groups $\Gamma_1(N)$ for $1 \leqslant N \leqslant 4$ are exceptional, and the general formula just derived does not give the correct answer. When $1 \leqslant N \leqslant 4$ we have $\overline{\Gamma}_1(N) \cong \overline{\Gamma}_0(N)$, and so the true values of $g_1(k, N)$ for these small N are equal to the values $g_1(k, N)$ when $k \geqslant 2$ is even. Calculating these values explicitly from Proposition 12, we have

$$\begin{aligned} g_1(k, 1) &= \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor - \frac{k}{2} + \delta\left(\frac{k}{2}\right), \\ g_1(k, 2) &= \left\lfloor \frac{k}{4} \right\rfloor - 1 + \delta\left(\frac{k}{2}\right), \\ g_1(k, 3) &= \left\lfloor \frac{k}{3} \right\rfloor - 1 + \delta\left(\frac{k}{2}\right), \\ g_1(k, 4) &= \left\lfloor \frac{k-3}{2} \right\rfloor - 1 + \delta\left(\frac{k}{2}\right) \end{aligned}$$

for even integers $k \geqslant 2$. When $k \geqslant 3$ is odd, we know that $g_1(k, 1) = g_1(k, 2) = 0$ since $\Gamma_1(1) = SL_2(\mathbb{Z})$ and $\Gamma_1(2)$ both contain the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. By carefully working through the details in [6, Section 4.2], we see that the above formulas for $g_1(k, 3)$ and $g_1(k, 4)$ are also correct when $k \geqslant 3$ is odd. In other words, the formulas

$$g_1(k, 1) = \left(\frac{1 + (-1)^k}{2}\right) \left(\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor - \frac{k}{2} \right) + \delta\left(\frac{k}{2}\right),$$

$$g_1(k,2) = \left(\frac{1+(-1)^k}{2}\right)\left(\left\lfloor\frac{k}{4}\right\rfloor - 1\right) + \delta\left(\frac{k}{2}\right),$$
$$g_1(k,3) = \left\lfloor\frac{k}{3}\right\rfloor - 1 + \delta\left(\frac{k}{2}\right),$$
$$g_1(k,4) = \left\lfloor\frac{k-3}{2}\right\rfloor + \delta\left(\frac{k}{2}\right)$$

are valid for all $k \geqslant 2$.

The formula (19) gives the false values $\frac{k-7}{24} + \delta(\frac{k}{2})$, $\frac{k-5}{8} + \delta(\frac{k}{2})$, $\frac{k-4}{3} + \delta(\frac{k}{2})$, and $\frac{2k-7}{4} + \delta(\frac{k}{2})$ for $g_1(k,1)$, $g_1(k,2)$, $g_1(k,3)$, and $g_1(k,4)$, respectively. One can check that

$$\left(\frac{1+(-1)^k}{2}\right)\left(\left\lfloor\frac{k}{4}\right\rfloor + \left\lfloor\frac{k}{3}\right\rfloor - \frac{k}{2}\right) - \frac{k-7}{24} = b_1(k),$$
$$\left(\frac{1+(-1)^k}{2}\right)\left(\left\lfloor\frac{k}{4}\right\rfloor - 1\right) - \frac{k-5}{8} = b_2(k),$$
$$\left\lfloor\frac{k}{3}\right\rfloor - 1 - \frac{k-4}{3} = b_3(k),$$
$$\left\lfloor\frac{k-3}{2}\right\rfloor - \frac{2k-7}{4} = b_4(k)$$

using the Definition 13′(C) of the functions $b_i(k)$. Therefore we can write

$$g_1(k,N) = \frac{k-1}{24} N^2 s_1(N) - \tfrac{1}{4}u(N) + \delta\left(\frac{k}{2}\right) + \begin{cases} b_1(k) & \text{if } N = 1, \\ b_2(k) & \text{if } N = 2, \\ b_3(k) & \text{if } N = 3, \\ b_4(k) & \text{if } N = 4, \\ 0 & \text{if } N \geqslant 5, \end{cases}$$

which is equivalent to the assertion of the proposition for all $N \geqslant 1$ and $k \geqslant 2$. □

We may now prove Theorems 13 and 14.

Proof of Theorems 13 and 14. We proceed as in the proofs of Theorems 1 and 4. Again we have the Atkin–Lehner decomposition

$$S_k(\Gamma_1(N)) = \bigoplus_{d|N} \bigoplus_{m|N/d} i_{m,d,N}\left(S_k^{\#}(\Gamma_1(n))\right).$$

Calculating the dimensions of both sides yields

$$g_1(k,N) = \sum_{d|N} \sum_{m|N/d} g_1^{\#}(k,d) = \sum_{d|N} g_1^{\#}(k,d)\tau(N/d).$$

This implies that $g_1^{\#} = g_1 * \lambda$ for every fixed k (recall the definition (15) of the multiplicative function λ), that is,

$$g_1^{\#}(k,N) = \sum_{d|N} g_1(k,d)\lambda(N/d).$$

Using the formula for $g_1(k,N)$ given in Proposition 15, this becomes

$$\begin{aligned} g_1(k,N) \;=\; & \frac{k-1}{24} N^2 s_1(N) * \lambda(N) - \tfrac{1}{4}(u*\lambda)(N) + \delta\left(\frac{k}{2}\right) \\ & + \sum_{\substack{1 \leqslant i \leqslant 4 \\ i|N}} b_i(k)\left(\delta(N/i) * \lambda(N)\right). \end{aligned}$$

We immediately note that the expression $\delta(N/i) * \lambda(N)$ equals simply $\lambda(N/i)$ in the case where i divides N. Furthermore, the function $u * \lambda$ is multiplicative; by using the formula (14) we see that it is equal to the function $u^{\#}$ defined in Definition 13′(B). Finally, it can be verified using (14) that

$$(p^\alpha)^2 s_1(p^\alpha) * \lambda(p^\alpha) = \sum_{\beta=0}^{\alpha} p^{2\beta} s_2(p^\beta)\lambda(p^{\alpha-\beta}) = (p^\alpha)^2 s_1^{\#}(p^\alpha),$$

where $s_1^{\#}$ is defined in Definition 13′(A); therefore the multiplicative function $N^2 s_1(N) * \lambda(N)$ is equal to $N^2 s_1^{\#}(N)$. This establishes Theorem 13.

The proof of Theorem 14 combines the techniques of the above proof with the proof of Theorem 4, using as a starting point the subspace $S_k^*(\Gamma_1(N))$ of $S_k(\Gamma_1(N))$ defined by

$$S_k^*(\Gamma_1(N)) = \bigoplus_{d|N} i_{1,d,N}\left(S_k^{\#}(\Gamma_1(n))\right),$$

whose dimension can be interpreted as the number of non-isomorphic automorphic representations associated with $S_k(\Gamma_1(N))$. We omit the details, as by now the method has been amply illustrated. □

4. Explicit bounds

We begin this section by using the formula in Theorem 1 to extract some explicit bounds on the function $g_0^\#(2,N)$, culminating in a proof of Theorem 2. In the following lemmas, we prove that Theorem 2 holds for certain conveniently chosen classes of integers N, after which we combine the results of these lemmas with a modest finite calculation to prove the theorem.

Lemma 16. *For every prime p, we have $g_0^\#(2,p) \leqslant \frac{p+1}{12}$, with equality if and only if $p \equiv 11 \pmod{12}$.*

Proof. We directly verify the claim for $p=2$ and $p=3$, so that we may assume $p \geqslant 5$. From Theorem 1 applied with $k=2$, we have

$$g_0^\#(2,p) = \tfrac{1}{12}(p-1) + \left\{ \begin{matrix} \frac12 & \text{if } p \equiv 3 \pmod 4 \\ 0 & \text{if } p \equiv 1 \pmod 4 \end{matrix} \right\} + \left\{ \begin{matrix} \frac23 & \text{if } p \equiv 2 \pmod 3 \\ 0 & \text{if } p \equiv 1 \pmod 3 \end{matrix} \right\} - 1.$$

This establishes the corollary and in fact more, namely that $g_0^\#(p) - p/12$ is a constant depending only on the residue class of $p \pmod{12}$. □

Lemma 17. *We have $Ns_0^\#(N) \leqslant \phi(N)$, $|v_2^\#(N)| \leqslant 2^{\omega(N)}$, $|v_3^\#(N)| \leqslant 2^{\omega(N)}$, and $0 \leqslant v_\infty^\#(N) \leqslant \sqrt{N}$ for all positive integers N.*

Proof. Since all terms in these four inequalities are multiplicative functions, the asserted inequalities can be checked on prime powers directly from the Definitions 1′(A)–(D) of the functions $s_0^\#$ and $v_i^\#$. We omit the straightforward verifications. □

Corollary 18. *We have $g_0^\#(2,N) \leqslant \frac{1}{12}\phi(N) + \frac{7}{12}2^{\omega(N)} + 1$.*

Proof. This follows directly from Theorem 1 and the bounds given in Lemma 17, together with the fact that $|\mu(N)| \leqslant 1$. □

Lemma 19. *Suppose that N is a composite number with at most two distinct prime factors. Then $g_0^\#(2,N) \leqslant \frac{N+1}{12}$, with equality if and only if $N = 35$.*

Proof. Since N is composite, it has a divisor $1 < d \leqslant \sqrt{N}$. There are $N/d \geqslant \sqrt{N}$ multiples of d less than N, none of which is relatively prime to N, and hence we have the inequality $\phi(N) \leqslant N - \sqrt{N}$. From Corollary 18 and the assumption that $\omega(N) \leqslant 2$, we then have

$$g_0^\#(2,N) \leqslant \tfrac{1}{12}(N - \sqrt{N}) + \tfrac{7}{12}2^2 + 1 = \frac{N+1}{12} - \left(\tfrac{1}{12}\sqrt{N} - \tfrac{13}{4} \right).$$

The quantity $(\frac{1}{12}\sqrt{N} - \frac{13}{4})$ is positive as soon as $N > 1521$, and so the lemma holds for these large N. A direct calculation of $g_0^\#(2,N)$ for $N \leqslant 1521$ (which discovers the case of equality $N = 35$) then shows that the lemma holds for these small N as well. $\Box$

Lemma 20. *Suppose that N is divisible by the sixth power of a prime. Then* $g_0^{\#}(2, N) \leqslant \frac{N-6}{12}$.

Proof. Suppose that $p_0^{\alpha_0}$ is a prime power divisor of N with $\alpha_0 \geqslant 6$. Then

$$\frac{N}{2^{\omega(N)}} = \prod_{p^r \| N} \frac{p^r}{2} \geqslant \frac{p_0^{\alpha_0}}{2} \geqslant \frac{2^{\alpha_0 - 1} p_0}{2} \geqslant 16 p_0,$$

which is the same as $N/p_0 \geqslant 16 \cdot 2^{\omega(N)}$. Noting that $\phi(N) = N \prod_{p|N} (1 - \frac{1}{p}) \leqslant N(1 - \frac{1}{p_0})$, this implies that

$$\begin{aligned} N - 6 &= N\left(1 - \frac{1}{p_0}\right) + \frac{N}{p_0} - 6 \geqslant \phi(N) + 16 \cdot 2^{\omega(N)} - 6 \\ &\geqslant \phi(N) + 7 \cdot 2^{\omega(N)} + 9 \cdot 2^1 - 6 \geqslant \phi(N) + 7 \cdot 2^{\omega(N)} + 12. \end{aligned}$$

Dividing both sides by 12 and invoking Corollary 18 establishes the lemma. $\Box$

Lemma 21. *Suppose that N has at least three distinct prime factors, two of which exceed* 5. *Then* $g_0^{\#}(2, N) \leqslant \frac{N-9}{12}$.

Proof. Suppose that $p_0 < p_1 < p_2$ are three distinct prime factors of N with $p_1 > 5$, so that $p_1 \geqslant 7$ and $p_2 \geqslant 11$. Then

$$\frac{N}{2^{\omega(N)}} \geqslant \prod_{p|N} \frac{p}{2} \geqslant \frac{p_2}{2} \frac{p_1}{2} \frac{p_0}{2} \geqslant \frac{77 p_0}{8},$$

which is the same as $\frac{N}{p_0} \geqslant \frac{77}{8} \cdot 2^{\omega(N)}$. This implies that

$$\begin{aligned} N - 9 &= N\left(1 - \frac{1}{p_0}\right) + \frac{N}{p_0} - 9 \geqslant \phi(N) + \tfrac{77}{8} \cdot 2^{\omega(N)} - 9 \\ &\geqslant \phi(N) + 7 \cdot 2^{\omega(N)} + \tfrac{21}{8} 2^3 - 9 \geqslant \phi(N) + 7 \cdot 2^{\omega(N)} + 12 \end{aligned}$$

since $\omega(N) \geqslant 3$. Dividing both sides by 12 and invoking Corollary 18 establishes the lemma. $\Box$

Lemma 22. *If* $(N, 6) > 1$ *and N has a prime factor exceeding* 41, *then* $g_0^{\#}(2, N) \leqslant \frac{N}{12}$.

Proof. Since either $2 \mid N$ or $3 \mid N$, we have $\phi(N) \leqslant \frac{2N}{3}$. Let $p > 41$ be a prime factor of N. Then

$$\frac{N}{2^{\omega(N)}} \geqslant \prod_{p|N} \frac{p}{2} \geqslant \tfrac{43}{2},$$

which is the same as $\frac{7}{12} \cdot 2^{\omega(N)} \leqslant \frac{7N}{258}$. Then by Corollary 18,

$$g_0^{\#}(2,N) \leqslant \tfrac{1}{12}\phi(N) + \tfrac{7}{12}2^{\omega(N)} + 1 \leqslant \tfrac{1}{12}\frac{2N}{3} + \frac{7N}{258} + 1 = \frac{N}{12} - \left(\frac{N}{1548} - 1\right).$$

This establishes the lemma for $N \geqslant 1548$, and we check by direct calculation that the lemma holds for $N < 1548$. □

Proof of Theorem 2. Lemmas 16 and 19 show that if $\omega(N) \leqslant 2$, then $g_0^{\#}(2,N) \leqslant \frac{N+1}{12}$ with equality if and only if either $N = 35$ or N is a prime that is congruent to $11 \pmod{12}$. It remains to show that $g_0^{\#}(2,N) < \frac{N+1}{12}$ when $\omega(N) \geqslant 3$. This inequality follows from Lemma 21 if two of the prime factors of N exceed 5; therefore we need only consider numbers of the form $N = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}p^{\alpha_4}$ with $p > 5$ and at least three of the α_i positive. No such integer can be relatively prime to 6, however; thus if $p > 41$, we have $g_0^{\#}(2,N) < \frac{N+1}{12}$ by Lemma 22. Furthermore, if any $\alpha_i \geqslant 6$, then $g_0^{\#}(2,N) < \frac{N+1}{12}$ by Lemma 20.

Consequently, the only integers N for which we have not verified the theorem are those of the form $N = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}p^{\alpha_4}$ with $7 \leqslant p \leqslant 41$, where each $0 \leqslant \alpha_i \leqslant 5$ and at least three of the α_i are positive. There are 10,125 integers of this form, and a direct calculation verifies that $g_0^{\#}(2,N) \leqslant \frac{N}{12} - \frac{3}{2}$ for these integers. This establishes the theorem. □

We now turn to the evaluation of $g_0^{\#}(k, 2^{\alpha}N)$.

Proof of Theorem 3. Let $N \geqslant 3$ be an odd squarefree integer, and let $\alpha \geqslant 4$ be an integer. Then from Theorem 1,

$$\begin{aligned} g_0^{\#}(k,2^{\alpha}N) = \frac{k-1}{12}\, 2^{\alpha}N s_0^{\#}(2^{\alpha})s_0^{\#}(N) - \tfrac{1}{2}v_{\infty}^{\#}(2^{\alpha})v_{\infty}^{\#}(N) \\ + c_2(k)v_2^{\#}(2^{\alpha})v_2^{\#}(N) + c_3(k)v_3^{\#}(2^{\alpha})v_3^{\#}(N) + \delta\left(\frac{k}{2}\right)\mu(2^{\alpha})\mu(N). \quad (20) \end{aligned}$$

From Definitions 1′(B)–(D), we see that $v_{\infty}^{\#}(2^{\alpha}) = v_2^{\#}(2^{\alpha}) = v_3^{\#}(2^{\alpha}) = \mu(2^{\alpha}) = 0$ since $\alpha \geqslant 4$ and $N \geqslant 3$ is squarefree. Also, from Definition 1′(A),

$$s_0^{\#}(2^{\alpha})s_0^{\#}(N) = \left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{2^2}\right)\prod_{p|N}\left(1 - \frac{1}{p}\right) = \tfrac{3}{8}\,\frac{\phi(N)}{N}.$$

We conclude that $g_0^{\#}(k,2^{\alpha}N) = \frac{k-1}{12}2^{\alpha}N \cdot \frac{3}{8}\frac{\phi(N)}{N} = (k-1)2^{\alpha-4}\phi(N)$ as asserted.

Now considering Eq. (20) in the case $\alpha = 1$, we have

$$\begin{aligned} g_0^\#(k, 2N) &= \frac{k-1}{12} 2N s_0^\#(2) s_0^\#(N) - \tfrac{1}{2} v_\infty^\#(2) v_\infty^\#(N) \\ &\quad + c_2(k) v_2^\#(2) v_2^\#(N) + c_3(k) v_3^\#(2) v_3^\#(N) + \delta\left(\frac{k}{2}\right) \mu(2)\mu(N) \\ &= \frac{k-1}{12} \phi(N) - c_2(k) v_2^\#(N) - 2c_3(k) v_3^\#(N) - \delta\left(\frac{k}{2}\right)\mu(N). \end{aligned}$$

By the bounds given in Lemma 17 and the Definitions 1′(E)–(F) of $c_2(k)$ and $c_3(k)$,

$$g_0^\#(k, 2N) \leqslant \frac{k-1}{12}\phi(N) + \tfrac{1}{4} 2^{\omega(N)} + \tfrac{2}{3} 2^{\omega(N)} + 1 = \frac{k-1}{12}\phi(N) + \tfrac{11}{12} 2^{\omega(N)} + 1.$$

Since $\phi(N) = \prod_{p|N}(p-1)$ and $2^{\omega(N)} = \prod_{p|N} 2$, we have $2^{\omega(N)} \leqslant \phi(N)$ with equality if and only if $N = 3$. We verify by hand that $g_0^\#(k, 6) \leqslant 2(k-1)$, which takes care of the case $N = 3$. When $N > 3$, we have

$$g_0^\#(k, 2N) < \frac{k-1}{12}\phi(N) + \tfrac{11}{12}\phi(N) + 1 \leqslant (k-1)\phi(N) + 1,$$

which establishes the last claim of the theorem. □

5. Calculations of values of $g_0^\#(2, N)$ and $g_0(2, N)$

Using the formula given in Theorem 1, we can derive explicit inequalities for the function $g_0^\#(2, N)$. We can thus determine the precise preimage of any fixed value of $g_0^\#(2, N)$ by combining these inequalities with finite computations. We remark that Halberstadt and Kraus [5] independently employed similar methods in their calculations of the set of integers for which $g_0^\#(2, N) \leqslant 3$.

We begin by stating a few lemmas giving simple but explicit inequalities for the multiplicative functions that concern us. We remind the reader of the Definition (2) of the constant $A_0^\#$:

$$A_0^\# = \prod_p \left(1 - \frac{1}{p^2 - p}\right) \approx 0.373956.$$

Lemma 23. *We have* $N s_0^\#(N) > A_0^\# \phi(N)$ *for all integers* $N \geqslant 1$.

Proof. From the Definition 1′(A) of $s_0^\#$, we see that on prime powers

$$p^r s_0^\#(p^r) \geqslant p^r \left(1 - \frac{1}{p} - \frac{1}{p^2}\right) = \frac{p^{r-1}(p^2-p-1)(p-1)}{p(p-1)} = \phi(p^r)\left(1 - \frac{1}{p^2-p}\right).$$

Therefore

$$Ns_0^\#(N) = \prod_{p^r \| N} p^r s_0^\#(p^r) \geqslant \prod_{p^r \| N} \phi(p^r) \prod_{p|N} \left(1 - \frac{1}{p^2-p}\right) > \phi(N) \cdot A_0^\#$$

as claimed. □

Lemma 24. *We have* $2^{\omega(N)} \leqslant 2^{4-\frac{\log 16}{\log 11}} N^{\frac{\log 2}{\log 11}}$ *for all* $N \geqslant 1$.

Proof. We have

$$2^{\omega(N)} = \left(\prod_{\substack{p|N \\ p\leqslant 7}} 2\right)\left(\prod_{\substack{p|N \\ p\geqslant 11}} 2\right) \leqslant \left(\prod_{\substack{p|N \\ p\leqslant 7}} 2\left(\tfrac{p}{2}\right)^{\frac{\log 2}{\log 11}}\right)\left(\prod_{\substack{p|N \\ p\geqslant 11}} p^{\frac{\log 2}{\log 11}}\right)$$

$$\leqslant \left(\prod_{\substack{p|N \\ p\leqslant 7}} 2^{1-\frac{\log 2}{\log 11}}\right)\left(\prod_{p|N} p^{\frac{\log 2}{\log 11}}\right) \leqslant 2^{4(1-\frac{\log 2}{\log 11})} N^{\frac{\log 2}{\log 11}}$$

as claimed. □

Lemma 25. *We have* $\phi(N) \geqslant \frac{N \log 2}{\log 2N}$ *for* $N \geqslant 2$.

Proof. This is Theorem 3.1(g) of Bressoud and Wagon [2]. □

Proposition 26. *We have* $g_0^\#(2,N) > 100$ *for all* $N > 132{,}000$.

Proof. Suppose first that N is not a perfect square. Then $v_\infty^\#(N) = 0$ by Definition 1′(B), while $c_2(2) = -\frac{1}{4}$ and $c_3(2) = -\frac{1}{3}$ by Definition 1′(E)–(F). Therefore the formula in Theorem 1, applied with $k = 2$, implies the inequality

$$g_0^\#(2,N) \geqslant \tfrac{1}{12} N s_0^\#(N) - \tfrac{1}{4}|v_2^\#(N)| - \tfrac{1}{3}|v_3^\#(N)| - \left|\delta\left(\frac{k}{2}\right)\mu(N)\right|.$$

Applying Lemmas 17 and 23, and noting that $|\delta(\frac{k}{2})\mu(N)| \leqslant 1$, gives

$$g_0^\#(2,N) > \frac{A_0^\#}{12}\phi(N) - \tfrac{7}{12}2^{\omega(N)} - 1. \tag{21}$$

From Lemmas 24 and 25 we conclude that

$$g_0^\#(2,N) > \frac{A_0^\# N\log 2}{12\log 2N} - \tfrac{7}{12}2^{4-\frac{\log 16}{\log 11}}N^{\frac{\log 2}{\log 11}} - 1.$$

It can be verified that the right-hand side is an increasing function of N for $N > 9000$ and takes a value exceeding 100 when $N = 132{,}000$. This establishes the theorem in the case where N is not a perfect square.

Suppose now that $N = M^2$ is a perfect square, where $M \geqslant 1$. Then the formula in Theorem 1, applied with $k = 2$, implies

$$g_0^\#(2,M^2) \geqslant \tfrac{1}{12}M^2 s_0^\#(M^2) - \tfrac12 v_\infty^\#(M^2) - \tfrac14|v_2^\#(M^2)| - \tfrac13|v_3^\#(M^2)|$$

since $\mu(M^2) = 0$. Applying Lemmas 17 and 23 gives

$$g_0^\#(2,M^2) > \frac{A_0^\#}{12}\phi(M^2) - \tfrac12\sqrt{M^2} - \tfrac{7}{12}2^{\omega(M^2)} = \frac{A_0^\#}{12}M\phi(M) - \tfrac12 M - \tfrac{7}{12}2^{\omega(M)}, \quad (22)$$

using the elementary facts that $\phi(M^2) = M\phi(M)$ and $\omega(M^2) = \omega(M)$. From Lemmas 24 and 25 we conclude that

$$g_0^\#(2,M^2) > \frac{A_0^\# M^2\log 2}{12\log 2M} - \tfrac12 M - \tfrac{7}{12}2^{4-\frac{\log 16}{\log 11}}M^{\frac{\log 2}{\log 11}} - 1.$$

It can be verified that the right-hand side is an increasing function of M for $M > 170$ and takes a value exceeding 100 when $M = 280$. Since $280^2 = 78{,}400 < 132{,}000$, this establishes the theorem in this case as well. □

Using the formula in Theorem 1, it takes only a couple of minutes to compute $g_0^\#(2,N)$ for all $N \leqslant 132{,}000$. We discover that there are exactly 2965 integers N for which $g_0^\#(2,N) \leqslant 100$. For example, there are exactly 40 solutions to the equation $g_0^\#(2,N) = 100$, namely

N = 1213, 1331, 2169, 2583, 2662, 2745, 3208, 3232, 3465, 3608, 4040, 4302, 4338, 4772, 4804, 4848, 5084, 5092, 5166, 5252, 5324, 5490, 5572, 5904, 6336, 6820, 6930, 7056, 7188, 7212, 7920, 8052, 8484, 8652, 8676, 8940, 9060, 10332, 10980, 13860.

We found that for every integer $0 \leqslant k \leqslant 100$ there are at least 13 solutions to the equation $g_0^\#(2,N) = k$, and there are only 13 solutions for $k = 86$. The largest number of solutions for k in this range is 68, attained by $k = 96$.

As N ranges from 1 to 132,000, the values taken by $g_0^\#(2, N)$ include every non-negative integer up to and including 4,361. In total, we found 9,566 of the integers less than 10,000 among the values of $g_0^\#(2, N)$ during this calculation, and of course extending the range of computation further would likely increase this number. The following assertion therefore seems reasonable:

Conjecture 27. *For every non-negative integer k, there is at least one positive integer N such that* $g_0^\#(2, N) = k$.

However, the analogous conjecture turns out to be false for $g_0(2, N)$ itself. Using the methods of this section, we can derive explicit upper bounds for $g_0(2, N)$ which, when combined with direct computations, show that several integers are never realized as values of $g_0(2, N)$. The omitted values up to 1000 are

$$150, 180, 210, 286, 304, 312, 336, 338, 348, 350, 480, 536, 570, 598,$$
$$606, 620, 666, 678, 706, 730, 756, 780, 798, 850, 876, 896, 906, 916, 970.$$

Csirik et al. [3] have extended these calculations and proven some related results. They discover, for example, that the first several thousand omitted values are all even, but not every odd integer is realized as a value (49,267 is the first omitted odd value). Furthermore, they prove that, contrary to the impression given by the small values, the set of values of the function $g_0(2, N)$ actually has density zero in the positive integers. They also discuss the distribution of these values among residue classes to various moduli.

6. Minimal and maximal orders

In this section, we provide the arguments necessary to convert the exact formulas for g_0, g_0^*, $g_0^\#$, ρ_0, and g_1 and its variants into asymptotic upper and lower bounds. We remark again that each of these bounds is sharp, and the avid reader will be able to convert the proofs below into constructions of sequences of integers that attain the bounds in question. We begin with three simple lemmas concerning the order of growth of some of the multiplicative functions we have encountered.

Lemma 28. *We have*

$$\prod_{p \leqslant y} \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2} + O\left(\frac{1}{y}\right)$$

for all $y \geqslant 2$.

Proof. The product in question converges to $\prod_p (1 - \frac{1}{p^2}) = \frac{1}{\zeta(2)}$ as y tends to infinity. To assess the error term for the partial product, note that

$$\sum_{p>y} \log \left(1 - \frac{1}{p^2}\right)^{-1} \ll \sum_{p>y} \frac{1}{p^2} < \sum_{n>y} \frac{1}{n^2} \ll \frac{1}{y}.$$

Therefore

$$\prod_{p>y} \left(1 - \frac{1}{p^2}\right)^{-1} = \exp\left(O\left(\frac{1}{y}\right)\right) = 1 + O\left(\frac{1}{y}\right),$$

which implies that

$$\prod_{p \leqslant y} \left(1 - \frac{1}{p^2}\right) = \prod_{p} \left(1 - \frac{1}{p^2}\right) \prod_{p>y} \left(1 - \frac{1}{p^2}\right)^{-1}$$
$$= \frac{1}{\zeta(2)} \left(1 + O\left(\frac{1}{y}\right)\right) = \frac{6}{\pi^2} + O\left(\frac{1}{y}\right),$$

since $\zeta(2) = \frac{\pi^2}{6}$. □

Lemma 29. *We have*

$$1 \leqslant s_0(N) \leqslant \frac{6e^\gamma}{\pi^2} \log\log N + O(1)$$

uniformly for all integers $N \geqslant 2$.

Proof. The lower bound $s_0(N) \geqslant 1$ is trivial. For the upper bound, first we consider the special case where N has the form $N_y = \prod_{p \leqslant y} p$. In this case,

$$s_0(N_y) = \prod_{p \leqslant y} \left(1 + \frac{1}{p}\right) = \prod_{p \leqslant y} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \leqslant y} \left(1 - \frac{1}{p^2}\right).$$

An asymptotic formula for the first product on the right-hand side is well known: Mertens' formula is

$$\prod_{p \leqslant y} \left(1 - \frac{1}{p}\right)^{-1} = e^\gamma \log y + O(1).$$

Therefore

$$s_0(N_y) = \left(e^\gamma \log y + O(1)\right)\left(\frac{6}{\pi^2} + O\left(\frac{1}{y}\right)\right) = \frac{6e^\gamma}{\pi^2}\log y + O(1)$$

by Lemma 28. On the other hand, the prime number theorem tells us that

$$\log N_y = \sum_{p \leqslant y} \log p = y\left(1 + O\left(\frac{1}{\log y}\right)\right)$$

(in fact we could be much more generous with the error term if need be). Therefore

$$s_0(N_y) = \frac{6e^\gamma}{\pi^2}\log\log N_y + O(1),$$

which establishes the lemma for integers of the form N_y.

Now consider an arbitrary integer $N \geqslant 2$. Choose y to be the $\omega(N)$th prime number, and set $N_y = \prod_{p \leqslant y} p$ as before. Then $N \geqslant N_y$, and the various prime factors of N are at least as large as the corresponding prime factors of N_y. Therefore

$$\begin{aligned} s_0(N) = \prod_{p \mid N}\left(1 + \frac{1}{p}\right) \leqslant \prod_{p \leqslant y}\left(1 + \frac{1}{p}\right) &= \frac{6e^\gamma}{\pi^2}\log\log N_y + O(1) \\ &\leqslant \frac{6e^\gamma}{\pi^2}\log\log N + O(1) \end{aligned}$$

as desired. □

Lemma 30. *We have $t(N) \leqslant u(N) \leqslant N\tau(N)$ for all $N \geqslant 1$.*

Proof. Since all three functions are multiplicative and non-negative, it suffices to show that $t(p^\alpha) \leqslant u(p^\alpha) \leqslant p^\alpha \tau(p^\alpha)$ for all prime powers p^α. This is easily verified by hand when $\alpha = 1$ and $\alpha = 2$. When $\alpha \geqslant 3$, we need to show that

$$p^{\alpha-4}(p-1)^3((\alpha+1)p + \alpha - 3) \leqslant p^{\alpha-2}(p-1)((\alpha+1)p + \alpha - 1) \leqslant p^\alpha(\alpha+1)$$

for all primes $p \geqslant 2$. The first inequality follows from the obvious inequality

$$(p-1)^2((\alpha+1)p + \alpha - 3) \leqslant p^2((\alpha+1)p + \alpha - 1)$$

upon multiplying through by $p^{\alpha-4}(p-1)$, and the second inequality similarly follows from

$$(p-1)((\alpha+1)p + \alpha - 1) = p^2(\alpha+1) - (\alpha+1)p + \alpha - 1 \leqslant p^2(\alpha+1)$$

upon multiplying through by $p^{\alpha-2}$. □

Proof of Theorem 6. Starting with the formula

$$g_0(k,N) = \frac{k-1}{12}\,Ns_0(N) - \tfrac12 v_\infty(N) + c_2(k)v_2(N) + c_3(k)v_3(N) + \delta\left(\frac{k}{2}\right)$$

given by Proposition 12, we use the inequalities in Lemma 17 to deduce that

$$\begin{aligned} &\frac{k-1}{12}\,Ns_0(N) - \tfrac12\sqrt{N}\,s_0(N) - |c_2(k)|2^{\omega(N)} - |c_3(k)|2^{\omega(N)} \\ &\quad \leqslant g_0(k,N) \\ &\quad \leqslant \frac{k-1}{12}\,Ns_0(N) + |c_2(k)|2^{\omega(N)} + |c_3(k)|2^{\omega(N)} + 1. \end{aligned}$$

The coefficients $c_2(k)$ and $c_3(k)$ are uniformly bounded, and $2^{\omega(N)} \ll \sqrt{N}$. Therefore we may write these inequalities as

$$\frac{k-1}{12}\,Ns_0(N) + O(\sqrt{N}\,s_0(N)) \leqslant g_0(k,N) \leqslant \frac{k-1}{12}\,Ns_0(N) + O(2^{\omega(N)}).$$

By Lemma 29, we conclude that

$$\begin{aligned} &\frac{k-1}{12}\,N + O(\sqrt{N}\log\log N) \\ &\qquad \leqslant g_0(k,N) \leqslant \frac{k-1}{12}\,N\left(\frac{6e^\gamma}{\pi^2}\log\log N + O(1)\right) + O(2^{\omega(N)}), \end{aligned}$$

which establishes Theorem 6(a).

In a similar way, combining the formula

$$g_0^*(k,N) = \frac{k-1}{12}\,Ns_0^*(N) - \tfrac12 v_\infty^*(N) + c_2(k)v_2^*(N) + c_3(k)v_3^*(N) + \delta\left(\frac{k}{2}\right)\delta(N),$$

from Theorem 4 with the easily verifiable inequalities

$$\frac{6}{\pi^2} = \frac{1}{\zeta(2)} < s_0^*(N) \leqslant 1,\ |v_2^*(N)| \leqslant 1,\ |v_3^*(N)| \leqslant 1, \quad \text{and } 0 \leqslant v_\infty^*(N) \leqslant \frac{\phi(N)}{\sqrt{N}}$$

establishes Theorem 6(b). Moreover, combining the formula

$$g_0^\#(k,N) = \frac{k-1}{12}\,Ns_0^\#(N) - \tfrac12 v_\infty^\#(N) + c_2(k)v_2^\#(N) + c_3(k)v_3^\#(N) + \delta\left(\frac{k}{2}\right)\delta(N),$$

from Theorem 4 with the inequalities from Lemma 17 and the additional inequality

$$Ns_0^\#(N) \geqslant \phi(N) \prod_{p|N} \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{1}{p} - \frac{1}{p^2}\right) = \prod_{p|N} \left(1 - \frac{1}{p^2 - p}\right) > A_0^\#,$$

which follows from the definition (2) of $A_0^\#$, establishes Theorem 6(c). □

The proof of Theorem 7 is very similar, and we omit the details except to mention that Lemma 30 plays a role in simplifying the error terms. As for Theorem 10, we can investigate the size of $\rho_0(k, N)$ (for example) using the information discovered in the proof of Theorem 6. We saw that

$$g_0^\#(k, N) = \frac{k-1}{12} Ns_0^\#(N) + O(\sqrt{N}) = \frac{k-1}{12} Ns_0^\#(N) \left(1 + O\left(\frac{\log\log N}{\sqrt{N}}\right)\right)$$

and similarly $g_0(k, N) = \frac{k-1}{12} Ns_0(N)(1 + O(\frac{\log\log N}{\sqrt{N}}))$. Therefore when $g_0(k, N) \neq 0$, we have

$$\rho_0(k, N) = \frac{g_0^\#(k, N)}{g_0(k, N)} = \frac{s_0^\#(N)}{s_0(N)} \left(1 + O\left(\frac{\log\log N}{\sqrt{N}}\right)\right).$$

The size of the multiplicative function $\frac{s_0^\#(N)}{s_0(N)}$ can be investigated as in the proof of Lemma 29. We find that

$$\frac{A_0^\# \pi^2}{6e^{2\gamma}(\log\log N)^2} \left(1 + O\left(\frac{1}{\log\log N}\right)\right) < \frac{s_0^\#(N)}{s_0(N)} \leqslant 1,$$

which is enough to establish Theorem 10(a). The proof of Theorem 10(b) is quite similar.

7. Average orders

In this final section, we prove Theorems 8, 9, and 11. As it happens, the multiplicative functions under consideration are all in a class of multiplicative functions whose average orders can be calculated rather easily. The following proposition is representative of the average-order theorems for multiplicative functions in the literature; we include a proof for the sake of completeness.

Proposition 31. *Suppose that $h(n)$ is a multiplicative function with the property that for some positive constant η, we have $(h * \mu)(n) \ll n^{-\eta}$. Then for any $\beta > -1$,* *we have*

$$\sum_{n \leqslant x} n^\beta h(n) \sim \frac{c(h)x^{\beta+1}}{\beta+1},$$

where

$$c(h) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right).$$

In particular, the average order of the function $n^\beta h(n)$ is $c(h)n^\beta$.

Proof. Let g denote the convolution $h * \mu$, so that $h(n) = \sum_{d|n} g(d)$ by the Möbius inversion formula; we note that g is multiplicative as well. For $x \geqslant 1$ we have

$$\begin{aligned}\sum_{n \leqslant x} n^\beta h(n) &= \sum_{n \leqslant x} n^\beta \sum_{d|n} g(d) = \sum_{d \leqslant x} g(d) \sum_{\substack{n \leqslant x \\ d|n}} n^\beta \\ &= \sum_{d \leqslant x} g(d) \sum_{md \leqslant x} (dm)^\beta = \sum_{d \leqslant x} d^\beta g(d) \sum_{m \leqslant x/d} m^\beta.\end{aligned}$$

Using the fact that

$$\sum_{m \leqslant y} m^\beta = \frac{y^{\beta+1}}{\beta+1} + O(y^\beta)$$

for any fixed $\beta > -1$, we see that

$$\begin{aligned}\sum_{n \leqslant x} n^\beta h(n) &= \sum_{d \leqslant x} d^\beta g(d) \left(\frac{(x/d)^{\beta+1}}{\beta+1} + O\left((x/d)^\beta\right)\right) \\ &= \frac{x^{\beta+1}}{\beta+1} \sum_{d \leqslant x} \frac{g(d)}{d} + O\left(x^\beta \sum_{d \leqslant x} |g(d)|\right).\end{aligned} \tag{23}$$

Since $g(d) \ll d^{-\eta}$, the sum in the main term is a truncation of a convergent sum, as the tail can be estimated by

$$\sum_{d > x} \frac{g(d)}{d} \ll \sum_{d > x} d^{-\eta-1} \ll x^{-\eta}.$$

Moreover, since g is multiplicative we can write

$$\sum_{n=1}^{\infty} \frac{g(d)}{d} = \prod_p \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right). \tag{24}$$

Since $h(p^\alpha) - h(p^{\alpha-1}) = g(p^\alpha)$, it is easily seen that

$$\left(1 - \frac{1}{p}\right)\left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right) = 1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots,$$

where convergence is ensured by the hypothesis $g(p^\alpha) \ll p^{-\eta\alpha}$. Therefore Eq. (24) becomes

$$\sum_{d=1}^{\infty} \frac{g(d)}{d} = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right) = c(h).$$

Finally, we have the estimate

$$\sum_{d \leqslant x} |g(d)| \ll \sum_{d \leqslant x} d^{-\eta} \ll E_\eta(x),$$

where

$$E_\eta(x) = \begin{cases} x^{1-\eta} & \text{if } 0 < \eta < 1, \\ \log x & \text{if } \eta = 1, \\ 1 & \text{if } \eta > 1. \end{cases}$$

Assembling this information and applying it to Eq. (23) yields

$$\begin{aligned} \sum_{n \leqslant x} n^\beta h(n) &= \frac{x^{\beta+1}}{\beta+1}\left(\sum_{d=1}^{\infty} \frac{g(d)}{d} + O\left(\sum_{d>x} \frac{g(d)}{d}\right)\right) + O\left(x^\beta \sum_{d \leqslant x} |g(d)|\right) \\ &= \frac{x^{\beta+1}}{\beta+1} c(h) + O(x^{\beta+1-\eta} + x^\beta E_\eta(x)) \\ &= \frac{c(h)x^{\beta+1}}{\beta+1} + O(x^\beta E_\eta(x)), \end{aligned}$$

which establishes the proposition. □

To apply this proposition to prove Theorem 8(a), for example, we start with the equation $g_0(k,N) = \frac{k-1}{12} N s_0(N) + O(\sqrt{N}\log\log N)$. It follows that

$$\sum_{N\leqslant x} g_0(k,N) = \frac{k-1}{12} \sum_{N\leqslant x} N s_0(N) + O(x^{3/2}\log\log x). \tag{25}$$

We note that the function $s_0 * \mu$ is multiplicative and satisfies $s_0(p) = \frac{1}{p}$ and $s_0(p^\alpha) = 0$ for $\alpha \geqslant 2$. Therefore the hypothesis of Proposition 31 is satisfied with $\eta = 1$, and so we conclude that

$$\sum_{N\leqslant x} N s_0(N) \sim \tfrac{1}{2} c(s_0) x^2,$$

where

$$\begin{aligned} c(s_0) &= \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{s_0(p)}{p} + \frac{s_0(p^2)}{p^2} + \cdots\right) \\ &= \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \left(1 + \frac{1}{p}\right)\left(\frac{1}{p} + \frac{1}{p^2} + \cdots\right)\right) \\ &= \prod_p \left(1 + \frac{1}{p^2}\right) \\ &= \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} \prod_p \left(1 - \frac{1}{p^4}\right) \\ &= \frac{\zeta(2)}{\zeta(4)} = \frac{\pi^2/6}{\pi^4/90} = \frac{15}{\pi^2}. \end{aligned}$$

Combining this with Eq. (25), we conclude that

$$\sum_{N\leqslant x} g_0(k,N) \sim \frac{k-1}{12}\frac{15}{\pi^2}\frac{x^2}{2} = \frac{5(k-1)x^2}{8\pi^2},$$

which implies that the average order of $g_0(k,N)$ is indeed $\frac{5(k-1)N}{4\pi^2}$. The proofs of the other seven average-order assertions in Theorems 8, 9, and 11 all follow this outline, and we omit the details of the calculations.

Acknowledgments

The author is grateful to Mike Bennett and Nike Vatsal for sharing their expertise and bringing these problems to his attention, and to the anonymous referee for comments that improved the presentation in the paper. The author acknowledges the support of the Natural Sciences and Engineering Research Council.

References

[1] A.O.L. Atkin, J. Lehner, Hecke operators on $\Gamma_0(m)$, Math. Ann. 185 (1970) 134–160.
[2] D. Bressoud, S. Wagon, A Course in Computational Number Theory, Key College Publishing, Emeryville, CA, 2000.
[3] J.A. Csirik, J.L. Wetherell, M.E. Zieve, On the genera of $X_0(N)$, preprint, arXiv:math.NT/0006096.
[4] E.-U. Gekeler, A remark on dimensions of spaces of modular forms, Arch. Math. (Basel) 65 (6) (1995) 530–533.
[5] E. Halberstadt, A. Kraus, Courbes de Fermat: résultats et problèmes, J. Reine Angew. Math. 548 (2002) 167–234.
[6] T. Miyake, Modular forms, Springer, Berlin, 1989 (Translated from Japanese by Yoshitaka Maeda).
[7] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Kanô Memorial Lectures, No. 1, Publications of the Mathematical Society of Japan, No. 11, Iwanami Shoten Publishers, Tokyo, 1971.