Putnam Practice Problems #1

September 19, 2017

(in no particular order)

(1) Eighteen dominoes are placed on a 6×6 checkerboard so that each domino covers two adjacent squares and every square is covered by a domino. Prove that regardless of how the dominoes are placed, the checkerboard can be cut along one of its horizontal or vertical lines in such a way that no domino is cut.

(2) Suppose that g is a function on the positive integers that takes positive integral values, that satisfies $g(g(n)) + 2g(n) = 3n + 4$ for all positive integers n. Find, with proof, $g(2017)$.

(3) Let $h(t)$ be a continuous function, defined for all real numbers t, that is periodic with period 2017 (that is, $h(t + 2017) = h(t)$ for all real numbers t). Show that there exists a real number x such that $h(x + 1) = h(x)$.

(4) Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of four distinct positive integers, and let $s_A = a_1 + a_2 + a_3 + a_4$. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ such that $a_i + a_j$ divides s_A. Find the largest possible value of n_A, and find all sets A for which n_A equals this largest possible value.

(5) Given real numbers $v_0, v_1, \ldots, v_n$, define $E(v_0, v_1, \ldots, v_n)$ to be the value $p(n + 1)$ where $p(x)$ is the unique polynomial of degree at most n such that $p(0) = v_0$, $p(1) = v_1$, $\ldots$, and $p(n) = v_n$. Define a sequence $\{y_n\}$ of real numbers recursively by $y_0 = 1$ and $y_{n+1} = E(y_0, y_1, \ldots, y_n) + 1$ for $n \geq 0$. Find and prove a formula for y_n.

(6) Let $f(x) = x - \lfloor x \rfloor$ denote the fractional part function. For example, $f(12.345) = 0.345$ and $f(\pi) = \pi - 3$ and $f(9876) = 0$. Find all pairs of positive integers m and n for which $f(\sqrt{m}) = f(\sqrt{n})$.

(7) Let T be a triangle with the following property: if T' is any triangle with the same perimeter and area as T, then T' is actually congruent to T. Show that T is equilateral.

(8) For each integer $n \geq 2$, define

$$x_n = \frac{(n^{2017} + 1^{2016})(n^{2017} + 2^{2016}) \cdots (n^{2017} + (n-1)^{2016})(n^{2017} + n^{2016})}{(n^{2017} - 1^{2016})(n^{2017} - 2^{2016}) \cdots (n^{2017} - (n-1)^{2016})(n^{2017} - n^{2016})}$$

Calculate, with proof, $\lim_{n \to \infty} x_n$.