Putnam Practice Problems #2
October 3, 2017
(in no particular order)

(1) The game Even Steven begins with a pile of 2017 checkers. The two players alternate turns, each player taking either 1 or 2 checkers from the pile. They continue until all the checkers are taken, at which point one of the players has an even number of checkers and one has an odd number (since the total number of checkers is odd). The winner is the player with the even number of checkers at the end of the game. Which player has a winning strategy?

(2) Let the Fibonacci numbers be defined as usual by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that there is a Fibonacci number whose last 2017 digits are all 9s.

(3) Given a positive integer n, call a subset S of $\{1, 2, \ldots, n\}$ "ample" if every integer from 2 to 2n, inclusive, can be written as the sum of two (not necessarily distinct) elements of S. Define $f(n)$ to be the number of elements in the smallest ample subset of $\{1, 2, \ldots, n\}$. Calculate, with proof,

$$\lim_{n\to\infty} \frac{\log f(n)}{\log n}.$$

(4) Let $\overline{XY}$ be a diameter of a circle $\mathcal{C}$, and let P be a point inside $\mathcal{C}$. Let $XAPB$ and $YDPE$ be squares with the given diagonals $\overline{XP}$ and $\overline{YP}$. If A lies on the circle $\mathcal{C}$, prove that D or E also lies on $\mathcal{C}$.

(5) Prove that for any positive real number t,

$$\int_0^t \big(\ln(x+1)\big)^{1/\pi}\,dx + \int_0^{2017/t} \big(e^{x^\pi} - 1\big)\,dx \geq 2017.$$

(6) Let n, A, and B be positive integers satisfying $n^2 < A < B < (n+1)^2$. Prove that AB is not a perfect square.

(7) Let X be the set of all subsets of $\{1, 2, \ldots, 63\}$. Call a function $f : X \to \mathbb{R}$ "coordinated" if it has the following property: for any three subsets A, B, C of $\{1, 2, \ldots, 63\}$ that are pairwise disjoint,

$$f(A\cup B\cup C) + f(A) + f(B) + f(C) = f(A\cup B) + f(B\cup C) + f(C\cup A).$$

Suppose that $f_1, f_2, \ldots, f_{2017}$ are all coordinated functions from X to $\mathbb{R}$. Prove that there exist real numbers $a_1, a_2, \ldots, a_{2017}$, not all zero, such that

$$a_1 f_1(D) + a_2 f_2(D) + \cdots + a_{2017} f_{2017}(D) = 0 \quad \text{for all } D \in X.$$

(8) Evaluate

$$\lim_{t\to\infty}\left(\frac{1}{e^t \ln t}\int_0^t\int_0^t \frac{e^x - e^y}{x - y}\,dx\,dy\right).$$