Putnam Practice Problems #3
October 17, 2017
(in no particular order)

(1) Show that there are distinct positive integers A, B_1, B_2, $\ldots$, B_{2017} such that each of the integers
$$(B_1)^2 + A, \quad (B_2)^2 + A, \quad \ldots, \quad (B_{2017})^2 + A$$
is a perfect square.

(2) Suppose that $1 < x < 2$. Show that there are infinitely many integers k such that $\lfloor kx \rfloor$ is a power of 2. (The expression $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.)

(3) Is there a permutation $a_1, a_2, a_3, \ldots$ of the positive integers such that the average of each initial segment
$$(a_1 + a_2 + \cdots + a_n)/n$$
is an integer for every $n \geq 1$?

(4) If s is a positive real number, a "cube of side s" is a set in three-dimensional space that is congruent to the set $\{(x, y, z) : 0 \leq x \leq s,\ 0 \leq y \leq s,\ 0 \leq z \leq s\}$ (in other words, it is any translated and rotated copy of that set). Find, with proof, the smallest number N such that a cube of side 7 can be contained in the union of N cubes of side 4.

(5) Given a set J of 2017 consecutive positive integers, define $g(J)$ to be the number of distinct integers in the set $\{\gcd(m, n) \colon m \in J,\ n \in J\}$. Determine, with proof, the maximum possible value of $g(J)$ over all sets J of 2017 consecutive positive integers.

(6) Determine, with proof, whether the series $\displaystyle\sum_{n=1}^{\infty} \left(e - \left(1 + \frac{1}{n}\right)^n \right)$ converges or diverges.

(7) Let A be the 2017×2017 matrix whose (i, j)th entry equals j^{i-1} if $i \geq j$ and 0 otherwise. Calculate the sum of the entries of A^{-1}.

(8) Consider the squares on a 2017×2017 checkerboard. A subset of the board's squares is called "neighborly" if every square *not* in the subset has at most one adjacent square in the subset. ("Adjacent" means directly adjacent to the north, south, east, or west.) For example, any rectangle of squares forms a neighborly set, but the set consisting of two squares touching at a corner is not a neighborly set. For any 2016 individual squares on the checkerboard, prove that there is a neighborly set containing all of them that is smaller than the entire 2017×2017 checkerboard.