Putnam Practice Problems #4
October 31, 2017
(in no particular order)

(1) Suppose that $f(x)$ is a polynomial of degree 2015 that satisfies
$$f(1) = 1\,, \quad f(2) = \frac{1}{2}\,, \quad f(3) = \frac{1}{3}\,, \quad \dots\,, \quad f(2016) = \frac{1}{2016}.$$
Find, with proof, the value $f(2017)$.

(2) Evaluate
$$\int_1^\infty \frac{dx}{x^2 + x^{2017}} + \int_1^\infty \frac{dx}{x^4 + x^{2017}} + \int_1^\infty \frac{dx}{x^6 + x^{2017}} + \cdots + \int_1^\infty \frac{dx}{x^{4032} + x^{2017}}.$$

(3) A *tromino* is a rectangle whose dimensions are 1 cm by 3 cm. Let n be of the form $3k+1$ where k is a positive integer. Consider an $n \times n$ square grid of little squares each 1 cm on a side. How many of the little squares have the following property: if deleted, the rest of the board can be perfectly covered by non-overlapping trominoes?

(4) Sofia and Rico are playing a two-person game that begins with a pile of 1000 stones. Sofia goes first and can remove any number of stones between 1 and 999. Thereafter they alternate turns removing stones; if the other player removed k stones on the previous turn, then the current player can remove anywhere from 1 to $2k-1$ stones in this turn. (For example, Sofia could start by removing 100 stones; then Rico can remove anywhere from 1 to 199 stones on their turn. If Rico chooses to remove 2 stones, Sofia can then remove anywhere from 1 to 3 stones.) The winner is the player who takes the last stone. Which player can be assured of winning, and what is the winning strategy?

(5) Let P be a convex nondegenerate pentagon, all of whose vertices have integer coordinates. Prove that the area of P is at least $5/2$. (Nondegenerate means that no three consecutive vertices lie on a common line.)

(6) Two lottery machines each contain ping-pong balls numbered $0, 1, 2, \dots, 2017$. The winning lottery number, which is an integer between 0 and 4034, will be determined by picking a ping-pong ball from each of the two machines and adding their numbers together. You are a mechanical wizard and can rig each of the machines to give out its ping-pong balls with any probabilities you want (as long as the total probability is 1 of course), and you can do this separately for each machine if you like. Can you possibly rig the machines so that each of the possible lottery numbers $0, 1, 2, \dots, 4034$ happens with equal probability $1/4035$?

(7) Prove that there exists a triangle that can be cut into exactly 2017 congruent triangles.

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(8) Let M be the 2016×2016 matrix whose (i, j)th entry m_{ij} is defined to be

$$m_{ij} = \begin{cases} i - j, & \text{if } i + j \text{ is prime,} \\ 0, & \text{if } i + j \text{ is composite} \end{cases}$$

for all $1 \leq i, j \leq 2016$. Prove that $|\det M|$ is a perfect square.

(9) Suppose that x is a real number such that 1^x, 2^x, 3^x, $\ldots$ are all integers. Prove that x is a nonnegative integer.