Primitive sets

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joint work with a Celebrated Person and William D. Banks

Elementary, analytic, and algorithmic number theory:
Research inspired by the mathematics of Carl Pomerance
Athens, GA
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slides can be found on my web page
`www.math.ubc.ca/∼gerg/index.shtml?slides`


ATHENS
CARL

Outline

1. What are primitive sets, and how thick can they be?
2. Construction of thick primitive sets (with C.P.)
3. Primitive sets with restricted primes (with B.B.)

Primitive sets

Definition

A primitive set is a set $\mathcal{S} \subset \{2, 3, 4, \dots\}$ with no element dividing another: if m, n are distinct elements of $\mathcal{S}$, then $m \nmid n$.

Examples:

- $\{m, m+1, m+2, \dots, 2m-1\}$ for any $m \ge 2$
- the primes $\mathcal{P} = \{2, 3, 5, 7, 11, \dots\}$
- $\mathcal{P}_k = \{n \in \mathbb{N} \colon \Omega(n) = k\}$ for any $k \ge 2$, where $\Omega(n)$ is the number of prime factors of n counted with multiplicity. For example, $\mathcal{P}_2 = \{4, 6, 9, 10, 14, 15, 21, 22, 25, 26, \dots\}$.

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- $\mathcal{S} = \{2\} \cup \{3p : p \geq 3 \text{ prime}\} \cup \{5p_1p_2 : p_1 \geq p_2 \geq 5 \text{ prime}\} \cup \{7p_1p_2p_3 : p_1 \geq p_2 \geq p_3 \geq 7 \text{ prime}\} \cup \cdots$
- "Primitive abundant numbers": abundant numbers ($\sigma(n) > 2n$) without any abundant divisors

Nonexample:

- the Fibonacci numbers

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Density of primitive sets

Theorem (Erdős, 1935)

If $\mathcal{S}$ is a primitive set, then $\sum_{n\in\mathcal{S}} \frac{1}{n\log n}$ *converges.*

It seems like this would imply that every primitive set has density 0, but not quite. It certainly implies that every primitive set has lower density 0.

A counterintuitive set

On the other hand, Besicovitch gave a construction of primitive sets with upper density greater than $\frac{1}{2} - \delta$ for any $\delta > 0$.

In other words, if $S(x) = \#\{s \in \mathcal{S} : s \le x\}$, then $S(x) > (\frac{1}{2} - \delta)x$ for arbitrarily large x.

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A set that only works when it has to

Besicovitch's primitive sets

- Contained in $[x_1, 2x_1) \cup [x_2, 2x_2) \cup [x_3, 2x_3) \cup \cdots$ for a rapidly increasing sequence $\{x_1, x_2, x_3, \dots\}$
- Obtained from this union of integrals greedily
- $S(2x_j) > \frac{1}{2} - \delta$ for j sufficiently large
- Most of the time, the counting function $S(x)$ is very small (since $\{x_j\}$ grows so fast)

Question

How large can a primitive set's counting function be consistently?

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Theorem (Ahlswede/Khachatrian/Sárközy, 1999)

A primitive set $\mathcal{S}$ exists with $S(x) \gg \dfrac{x}{(\log\log x)(\log\log\log x)^{1+\varepsilon}}$.

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Partial summation

$$\sum_{n\in\mathcal{S}} \frac{1}{n\log n} \text{ converges if and only if } \int_2^\infty \frac{S(x)}{x^2\log x}\,dx \text{ converges.}$$

Consequently:

- By Erdős: if $\mathcal{S}$ is primitive, then $\displaystyle\int_2^\infty \frac{S(x)}{x^2\log x}\,dx$ converges.
- Impossible to have $S(x) \gg \dfrac{x}{(\log\log x)(\log\log\log x)}$, say.

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A sort of converse

Erdős plus partial summation

If $\mathcal{S}$ is primitive, then $\displaystyle\int_2^\infty \frac{S(x)}{x^2 \log x}\,dx$ converges.

Theorem (M.–Pomerance, 2011)

If $F(x)$ is a "nice" function such that $\displaystyle\int_2^\infty \frac{F(x)}{x^2 \log x}\,dx$ converges, then there exists a primitive set $\mathcal{S}$ with $S(x) \asymp F(x)$.

Corollary

For any $\varepsilon > 0$, there exists a primitive set $\mathcal{S}$ with

$$S(x) \gg \frac{x}{(\log\log x)(\log\log\log x)\cdots(\log_{69} x)(\log_{70} x)^{1+\varepsilon}}.$$

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A particular construction of primitive sets

Definition

Fix a sequence $p_1 < p_2 < \cdots$ of primes, and define

$$\mathcal{S}_k = \{n \in \mathbb{N} \colon \Omega(n) = k;\ p_k \mid n;\ p_1 \nmid n, \dots, p_{k-1} \nmid n\}.$$

Example

If $\{p_j\}$ is all the primes, then $\mathcal{S}_1 = \{2\}$, $\mathcal{S}_2 = \{3p \colon p \ge 3 \text{ prime}\}$, $\mathcal{S}_3 = \{5p_1p_2 \colon p_1 \ge p_2 \ge 5 \text{ prime}\}$, etc.

Then $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k$ is primitive.

Proof.

If $m, n \in \mathcal{S}$ are distinct and $m \mid n$, then $\Omega(m) < \Omega(n)$; but then $p_{\Omega(m)}$ divides m but not n. □

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Thick primitive sets

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Fix a sequence $p_1 < p_2 < \cdots$ of primes, with $\sum_{i=1}^{\infty} \frac{1}{p_i} < \frac{1}{2}$, whose growth rate is tied to $F(x)$. Define

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$$S_k(x) \geq \frac{1}{p_k} \frac{x}{\log x} \frac{(\log\log x)^{k-2}}{(k-2)!} \left(1 - \frac{k-3}{\log\log x} \prod_{j=1}^{k-1} \frac{1}{p_j}\right)$$

$$\gg \frac{1}{p_k} \frac{x}{\log x} \frac{(\log\log x)^{k-2}}{(k-2)!} \text{ for } k < \tfrac{3}{2} \log\log x.$$

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One constant to rule them all

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$E(\mathcal{S}) = \sum_{n \in \mathcal{S}} \frac{1}{n \log n}$ for any $\mathcal{S} \subset \{2, 3, \dots\}$.

For $\mathcal{S}$ primitive, Erdős proved more than that $E(\mathcal{S})$ is finite; he proved that $E(\mathcal{S})$ is bounded by an absolute constant. (Erdős/Zhang, 1993: the constant 1.84 suffices.)

Conjecture (Erdős, 1988)

If $\mathcal{S}$ is primitive, then $E(\mathcal{S}) \le E(\mathcal{P}) = 1.63\dots$.

Known for certain classes of primitive $\mathcal{S}$ [Zhang 1991, 1993]:

- all elements $n \in \mathcal{S}$ satisfy $\Omega(n) \le 4$; or
- $\mathcal{S}$ is homogeneous: for $n \in \mathcal{S}$, the quantity $\Omega(n)$ depends only on smallest prime factor of n (e.g., earlier construction)

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For $\mathcal{S}$ primitive, Erdős proved more than that $E(\mathcal{S})$ is finite; he proved that $E(\mathcal{S})$ is bounded by an absolute constant. (Erdős/Zhang, 1993: the constant 1.84 suffices.)

Conjecture (Erdős, 1988)

If $\mathcal{S}$ is primitive, then $E(\mathcal{S}) \le E(\mathcal{P}) = 1.63\dots$.

Known for certain classes of primitive $\mathcal{S}$ [Zhang 1991, 1993]:

- all elements $n \in \mathcal{S}$ satisfy $\Omega(n) \le 4$; or
- $\mathcal{S}$ is homogeneous: for $n \in \mathcal{S}$, the quantity $\Omega(n)$ depends only on smallest prime factor of n (e.g., earlier construction)

One constant to rule them all

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Primitive sets with restricted primes

Definition

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Notation: integers with restricted prime factors

For any $\mathcal{Q} \subset \mathcal{P}$, define $\mathbb{N}(\mathcal{Q}) = \{n \ge 2\colon \text{if } p \mid n, \text{ then } p \in \mathcal{Q}\}$.

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$$E(\mathcal{S}) = \sum_{n\in\mathcal{S}} \frac{1}{n\log n} \quad \text{and} \quad E_t(\mathcal{S}) = \sum_{n\in\mathcal{S}} \frac{1}{n^t}$$

Observation / first-year calculus

$$\frac{1}{n\log n} = \int_1^\infty \frac{dt}{n^t}; \text{ therefore, } E(\mathcal{S}) = \int_1^\infty E_t(\mathcal{S})\,dt$$

If we want to show that $E(\mathcal{S}) \le E(\mathcal{Q})$ for every primitive $\mathcal{S} \subset \mathbb{N}(\mathcal{Q})$, it suffices to show that $E_t(\mathcal{S}) \le E_t(\mathcal{Q})$ for all $t > 1$.

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If $E_1(\mathcal{Q}) \le 1 + \sqrt{1 - E_2(\mathcal{Q})}$, then the conjecture holds.

Corollary

If $E_1(\mathcal{Q}) = \sum_{p \in \mathcal{Q}} \frac{1}{p} \le 1.74$, then the conjecture holds.

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$$\mathcal{T} = \{\text{all twin primes}\}, \quad \mathcal{T}_3 = \mathcal{T} \setminus \{3\}$$

Corollary

If $\mathcal{S}$ is a primitive subset of $\mathbb{N}(\mathcal{T}_3)$, then $E(\mathcal{S}) \le E(\mathcal{T}_3)$.

Brun's constant

Define B to be the sum of the reciprocals of the twin primes:
$B = (\frac{1}{3} + \frac{1}{5}) + (\frac{1}{5} + \frac{1}{7}) + (\frac{1}{11} + \frac{1}{13}) + (\frac{1}{17} + \frac{1}{19}) + (\frac{1}{29} + \frac{1}{31}) + \cdots$.

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- Best bound known: $B < 2.347$ (Crandall/Pomerance)

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Out of all the primitive subsets of $\mathbb{N}(\mathcal{T}_3)$, we can identify the one that maximizes $\sum \frac{1}{n \log n}$ . . . even though we can't say whether that optimal set is finite or infinite!

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Suppose that $B < 2.09596$. *If* $\mathcal{S}$ *is a primitive subset of* $\mathbb{N}(\mathcal{T})$, *then* $E(\mathcal{S}) \le E(\mathcal{T})$.

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The end

The two papers described in this talk, as well as these slides, are available for downloading.

Primitive sets with large counting functions (with C.P.)

www.math.ubc.ca/~gerg/index.shtml?abstract=PSLCF

Optimal primitive sets with restricted primes (with B.B.)

www.math.ubc.ca/~gerg/index.shtml?abstract=OPSRP

These slides

www.math.ubc.ca/~gerg/index.shtml?slides