Absolutely abnormal numbers

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Outline

Simply normal numbers

A real number is simply normal to the base b if each digit occurs in its b-ary expansion with the expected asymptotic frequency.

- $N(\alpha; b, a, x) = \#\{1 \le n \le x\colon$ the nth digit in the base-b expansion of α is $a\}$

Definition

α is simply normal to the base b if for each $0 \le a < b$,

$$\lim_{x\to\infty} \frac{N(\alpha; b, a, x)}{x} = \frac{1}{b}.$$

- b-adic rational numbers α (those for which $b^j\alpha$ is an integer for some j) have two b-ary expansions; but α is not simply normal for either one

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A number is normal to the base b if it is simply normal to each of the bases $b, b^2, b^3, \ldots$.

Equivalently:

For any finite string $a_1a_2\ldots a_\ell$ of base-b digits, the limiting frequency of occurrences of this string in the b-ary expansion of α exists and equals $1/b^\ell$.

- The set of numbers normal to any base b has full Lebesgue measure, and thus the same is true of the set of absolutely normal numbers.
- Proving specific numbers normal is notoriously hard.

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How hard?

Champernowne's number

$$0.12345678910111213141516171819202122232425 26\ldots$$

is normal to the base 10 (and hence to bases 100, 1000, etc.).

Theorem (Stoneham, 1973; Bailey–Crandall 2002)

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$ *is normal to the base* b *if* $\gcd(b,c)=1$.

The bad news

No real number has ever been proved normal to two multiplicatively independent bases.

But they all are.... (pretty pictures)

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Absolute abnormality

Definition

A number is absolutely abnormal if it is not normal to any base $b \geq 2$.

- Every rational number is absolutely abnormal.
- The set of absolutely abnormal numbers (while Lebesgue measure 0) is uncountable and dense.

Well, then:

Can we write down an irrational, absolutely abnormal number?

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A sequence of integers

Definition (recursive)

- $d_2 = 2^2, \quad d_3 = 3^2, \quad d_4 = 4^3, \quad d_5 = 5^{16},$
- $d_6 = 6^{30,517,578,125}, \quad \dots$
- $d_j = j^{d_{j-1}/(j-1)} \quad (j \ge 3).$

Explicitly:

$$d_4 = 4^{3^{2-1}}, \quad d_5 = 5^{4^{(3^{2-1}-1)}}, \quad d_6 = 6^{5^{\left(4^{(3^{2-1}-1)}-1\right)}}, \dots$$

and in general,

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A sequence of rational numbers

Definition

$$\alpha_k = \prod_{j=2}^{k} \left(1 - \frac{1}{d_j}\right)$$

- $\alpha_2 = \frac{3}{4}$, $\alpha_3 = \frac{2}{3}$, $\alpha_4 = \frac{21}{32}$, $\alpha_5 = \frac{100{,}135{,}803{,}222}{152{,}587{,}890{,}625}$, ...

Some nice cancellation

It seems that the denominator of α_k should contain powers of $2, 3, \dots, k$. But in the listed terms, the denominator contains only powers of k; in other words, α_k is a k-adic fraction.

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A little elementary number theory

Fun fact (for you to prove if you get bored)

$(k+1)^{k^m} - 1$ is divisible by k^{m+1} for any integers $k, m \geq 1$.

Lemma

$(k+1)^{d_k/k} - 1$ *is divisible by* d_k *for any integer* $k \geq 2$.

Proof.

Since $d_k = k^{d_{k-1}/(k-1)}$,

$$(k+1)^{d_k/k} - 1 = (k+1)^{k^{d_{k-1}/(k-1)-1}} - 1;$$

apply the fun fact with $m = d_{k-1}/(k-1) - 1$. □

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Lemma

$d_k\alpha_k$ is an integer for each $k \geq 2$. In particular, α_k is a k-adic fraction (since d_k is a power of k).

Proof by induction on k.

$$d_{k+1}\alpha_{k+1} = d_{k+1}\prod_{j=2}^{k+1}\left(1-\frac{1}{d_j}\right) = (d_{k+1}-1)\alpha_k$$
$$= \left((k+1)^{d_k/k}-1\right)\alpha_k = \left(\frac{(k+1)^{d_k/k}-1}{d_k}\right)d_k\alpha_k$$

The fraction is an integer by the lemma on the last slide; so if $d_k\alpha_k$ is an integer, then $d_{k+1}\alpha_{k+1}$ is also an integer. □

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Our candidate

Definition

$$\alpha = \lim_{k\to\infty} \alpha_k = \prod_{j=2}^{\infty} \left(1 - \frac{1}{d_j}\right)$$

- In Peter's honour, I've memorized the first twenty-three billion decimal places of α:

Compare α to its partial products

$\alpha_4 = \frac{21}{32} = 0.65625 \qquad \alpha_5 = \frac{100{,}135{,}803{,}222}{152{,}587{,}890{,}625} = 0.65624999999956992$

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easy exercise

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Definition (reminder)

- $d_j = j^{d_{j-1}/(j-1)}$
- $\alpha = \prod_{j=2}^{\infty} \left(1 - 1/d_j\right)$

$\{d_j\}$ grows ridiculously quickly

One can show that $d_{j+1} > d_j^{d_{j-1}}$ for all $j \ge 5$.

$$\alpha_k - \alpha = \alpha_k \left(1 - \prod_{j=k+1}^{\infty} \left(1 - \frac{1}{d_j}\right)\right) < \alpha_k \sum_{j=k+1}^{\infty} \frac{1}{d_j} < \frac{2}{d_{k+1}} < \frac{2}{d_k^{d_{k-1}}}$$

- These are rational approximations of α by fractions with denominators d_k.
- Since $d_{k-1} \to \infty$, the number α is a Liouville number, hence irrational (transcendental even).

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$\{d_j\}$ grows ridiculously quickly

One can show that $d_{j+1} > d_j^{d_{j-1}}$ for all $j \geq 5$.

$$\alpha_k - \alpha = \alpha_k \left(1 - \prod_{j=k+1}^{\infty} \left(1 - \frac{1}{d_j}\right)\right) < \alpha_k \sum_{j=k+1}^{\infty} \frac{1}{d_j} < \frac{2}{d_{k+1}} < \frac{2}{d_k^{d_{k-1}}}$$

- These are rational approximations of α by fractions with denominators d_k.
- Since $d_{k-1} \to \infty$, the number α is a Liouville number, hence irrational (transcendental even).

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Abnormal for one moment at least

Illustrating example: $b = 4$

- $\alpha_4 = (0.222)_{\text{base } 4}$
- $\alpha_4 - \alpha = (0.000000000000000000102322210110\dots)_{\text{base } 4}$
- $\alpha = (0.221333333333333333231011123223\dots)_{\text{base } 4}$

- $\alpha_b - \alpha < 2/d_b^{d_{b-1}}$ for all $b \geq 5$
- α_b terminates in base b, after D digits say
- α is a tiny bit less than α_b: the difference starts with about $d_{b-1}D$ base-b digits equaling 0
- So α has a long string of base-b digits equaling $b-1$.

How long?

Among the first $d_{b-1}D$ base-b digits of α, at least a proportion $1 - C/d_{b-1}$ of them equal $b-1$ (for some absolute constant C).

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Now change b to b^2:

- Among the first $?_{b^2}$ base-b^2 digits of α, at least a proportion $1 - C/d_{b^2-1}$ of them equal $b^2 - 1$.
- Thus among the first $2?_{b^2}$ base-b digits of α, at least a proportion $1 - C/d_{b^2-1}$ of them equal $b-1$.

We find that α has lots of (disjoint) long strings of base-b digits equaling $b-1$, coming from changing b to $b^2, b^3, b^4, \dots$.

(and sometimes more)

Such strings also come from k if all prime factors of k divide b: k-adic fractions are also b-adic fractions. For example, in base 10 we already saw long strings of 9s coming from α_4 and α_5.

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α is absolutely abnormal

Definition (counting base-b digits equaling a)

$N(\alpha; b, a, x) = \#\{1 \le n \le x\colon$ the nth digit in the base-b expansion of α is $a\}$

- We've found a sequence $\{x_1, x_2, \dots\} = \{?_b, 2?_{b^2}, \dots\}$ such that $N(\alpha; b, b-1, x_j) > (1 - C/d_{b^j-1})x_j$.
- So $\limsup_{x\to\infty} \frac{N(\alpha; b, b-1, x)}{x} \ge \limsup_{j\to\infty}(1 - C/d_{b^j-1}) = 1$.
- This conflicts with $\lim_{x\to\infty} \frac{N(\alpha; b, b-1, x)}{x} = \frac{1}{b}$.

Theorem (M., 2001)

α is an irrational number that fails to be (simply) normal to any base $b \ge 2$.

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Different parameters

The original construction

- $d_2 = 2^2$
- $d_j = j^{d_{j-1}/(j-1)}$
- $\alpha_2 = 3/d_2$
- $\alpha_k = \alpha_2 \prod_{j=3}^{k} \left(1 - 1/d_j\right)$

We can generalize the construction in two ways:

- Start with any dyadic fraction $\alpha_2 = n_1/2^{n_2}$ in place of $3/4$.
- Insert positive integer multiples n_j in the recursion for d_j.

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- The numbers $n_3, n_4, \ldots$ just accelerate the convergence of α, and all the inequalities are still satisfied and more.
- In particular, $\alpha_2 > \alpha > \alpha_2 - 2/d_2$. So by choosing n_1 and n_2 suitably, we can ensure that α ends up in any prescribed interval.
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One neat example: $n_1 = n_2 = 1$ and $n_j = \phi(j-1)$ for $j \ge 3$

- The result is that $d_j = j^{\phi(d_{j-1})}$ for $j \ge 3$.
- The key divisibility $d_k \mid \left((k+1)^{\phi(d_k)} - 1\right)$ just follows from Euler's theorem.

The only choice that doesn't work: $n_j = 1$ for all $j \ge 1$

- The result is that $d_j = j$ for $j \ge 2$, and we get the telescoping product $\alpha_k = \frac{1}{2} \prod_{j=3}^{k} \left(1 - \frac{1}{j}\right) = \frac{1}{k}$.
- In this case $\alpha = 0$. (Well, it is absolutely abnormal!)

Particular parameters

The generalized construction

- $d_2 = 2^{n_2}$
- $\alpha_2 = n_1/d_2$
- $d_j = j^{n_j d_{j-1}/(j-1)}$
- $\alpha_k = \alpha_2 \prod_{j=3}^{k} \left(1 - 1/d_j\right)$

One neat example: $n_1 = n_2 = 1$ and $n_j = \phi(j-1)$ for $j \ge 3$

- The result is that $d_j = j^{\phi(d_{j-1})}$ for $j \ge 3$.
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The end

The paper *Absolutely abnormal numbers*, as well as these slides, are available for downloading:

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