Prime number races

Greg Martin
University of British Columbia

Pure Math Seminar
UNSW Canberra
November 29, 2018

Outline

A historical document

Lettre de M. le professeur Tchébychev à M. Fuss, sur un nouveau théorème relatif aux nombres premiers contenus dans les formes $4n+1$ et $4n+3$.

11 (23) mars 1853.

(Bull. phys.-mathém., T. XI, p. 208).

La bienveillance, avec laquelle vous avez toujours agréé mes recherches, m'engage à vous présenter un nouveau résultat relatif aux nombres premiers et que je viens de trouver. En cherchant l'expression limitative des fonctions qui déterminent la totalité des nombres premiers de la forme $4n+1$ et de ceux de la forme $4n+3$, pris au-dessous d'une limite très grande, je suis parvenu à reconnaître que ces deux fonctions diffèrent notablement entre elles par leurs seconds termes, dont la valeur, pour les nombres $4n+3$, est plus grande que celle pour les nombres $4n+1$; ainsi, si de la totalité des nombres premiers de la forme $4n+3$, on retranche celle des nombres premiers de la forme $4n+1$, et que l'on divise ensuite cette différence par la quantité $\frac{\sqrt{x}}{\log x}$, on trouvera plusieurs valeurs de x telles, que ce quotient s'approchera de l'unité aussi près qu'on le voudra. Cette différence dans la répartition des nombres premiers de la forme $4n+1$ et $4n+3$, se manifeste clairement dans plusieurs cas. Par exemple, 1) à mesure que c s'approche de zéro, la valeur de la série

$$e^{-3c} - e^{-5c} + e^{-7c} + e^{-11c} - e^{-13c} - e^{-17c} + e^{-19c} + e^{-23c} + \dots$$

s'approche de $+\infty$; 2) la série

$$f(3) - f(5) + f(7) + f(11) - f(13) - f(17) + f(19) + f(23) + \dots$$

où $f(x)$ est une fonction constamment décroissante, ne peut être convergente, à moins que la limite du produit $x^{\frac{1}{2}} f(x)$, pour $x = \infty$, ne soit zéro.

Je suis parvenu à ces résultats en traitant une certaine équation, relative aux nombres premiers, et qui comprend, comme cas particulier celle que M. A. de Polignac et moi, indépendamment l'un de l'autre, nous avons trouvée dans nos recherches sur les nombres premiers.

Agréez etc.

Signé: P. Tchébychev.

Le 10 mars 1853.

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement (translated from French):

> *In seeking the limiting expression of the functions that determine the totality of the prime numbers of the form $4n+1$ and those of the form $4n+3$, taken below a very large limit, I have come to recognize that these two functions differ notably between them in their second terms, the value of which, for the numbers $4n+3$, is greater than that for the numbers $4n+1$. . . .*

Let's see two graphs of this "mod 4 race" . . .

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement (translated from French):

In seeking the limiting expression of the functions that determine the totality of the prime numbers of the form $4n+1$ and those of the form $4n+3$, taken below a very large limit, I have come to recognize that these two functions differ notably between them in their second terms, the value of which, for the numbers $4n+3$, is greater than that for the numbers $4n+1$. . . .

Let's see two graphs of this "mod 4 race" . . .

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement (translated from French):

In seeking the limiting expression of the functions that determine the totality of the prime numbers of the form $4n+1$ and those of the form $4n+3$, taken below a very large limit, I have come to recognize that these two functions differ notably between them in their second terms, the value of which, for the numbers $4n+3$, is greater than that for the numbers $4n+1$. . . .

Let's see two graphs of this "mod 4 race" . . .

More data

The race between #{primes of the form $4n+1$ up to x} and #{primes of the form $4n+3$ up to x}

The $4n+1$ primes take the lead at	The $4n+1$ primes lose the lead at
$x = 26{,}861$	$x = 26{,}863$
$x = 616{,}481$	$x = 633{,}798$
$x = 12{,}306{,}137$	$x = 12{,}382{,}326$
$x = 951{,}784{,}481$	$x = 952{,}223{,}506$
$x = 6{,}309{,}280{,}697$	$x = 6{,}403{,}150{,}362$
$x = 18{,}465{,}126{,}217$	$x = 19{,}033{,}524{,}538$

(Leech, Bays & Hudson) Since then, "notable differences" have been observed between primes of various other forms $qn + a$, where q and a are constants.

Let's see graphs of races modulo 3, 8, 10, and 12 . . .

More data

The race between #{primes of the form $4n+1$ up to x} and #{primes of the form $4n+3$ up to x}

The $4n+1$ primes take the lead at	The $4n+1$ primes lose the lead at
$x = 26{,}861$	$x = 26{,}863$
$x = 616{,}481$	$x = 633{,}798$
$x = 12{,}306{,}137$	$x = 12{,}382{,}326$
$x = 951{,}784{,}481$	$x = 952{,}223{,}506$
$x = 6{,}309{,}280{,}697$	$x = 6{,}403{,}150{,}362$
$x = 18{,}465{,}126{,}217$	$x = 19{,}033{,}524{,}538$

(Leech, Bays & Hudson) Since then, "notable differences" have been observed between primes of various other forms $qn+a$, where q and a are constants.

Let's see graphs of races modulo 3, 8, 10, and 12 . . .

More data

The race between #{primes of the form $4n+1$ up to x} and #{primes of the form $4n+3$ up to x}

The $4n+1$ primes take the lead at	The $4n+1$ primes lose the lead at
$x = 26{,}861$	$x = 26{,}863$
$x = 616{,}481$	$x = 633{,}798$
$x = 12{,}306{,}137$	$x = 12{,}382{,}326$
$x = 951{,}784{,}481$	$x = 952{,}223{,}506$
$x = 6{,}309{,}280{,}697$	$x = 6{,}403{,}150{,}362$
$x = 18{,}465{,}126{,}217$	$x = 19{,}033{,}524{,}538$

(Leech, Bays & Hudson) Since then, “notable differences” have been observed between primes of various other forms $qn+a$, where q and a are constants.

Let’s see graphs of races modulo 3, 8, 10, and 12 …

More data

The race between $\#\{$primes of the form $4n+1$ up to $x\}$ and $\#\{$primes of the form $4n+3$ up to $x\}$

The $4n+1$ primes take the lead at	The $4n+1$ primes lose the lead at
$x = 26{,}861$	$x = 26{,}863$
$x = 616{,}481$	$x = 633{,}798$
$x = 12{,}306{,}137$	$x = 12{,}382{,}326$
$x = 951{,}784{,}481$	$x = 952{,}223{,}506$
$x = 6{,}309{,}280{,}697$	$x = 6{,}403{,}150{,}362$
$x = 18{,}465{,}126{,}217$	$x = 19{,}033{,}524{,}538$

(Leech, Bays & Hudson) Since then, "notable differences" have been observed between primes of various other forms $qn+a$, where q and a are constants.

Let's see graphs of races modulo 3, 8, 10, and 12 . . .

More data

The race between #{primes of the form $4n+1$ up to x} and #{primes of the form $4n+3$ up to x}

The $4n+1$ primes take the lead at	The $4n+1$ primes lose the lead at
$x = 26{,}861$	$x = 26{,}863$
$x = 616{,}481$	$x = 633{,}798$
$x = 12{,}306{,}137$	$x = 12{,}382{,}326$
$x = 951{,}784{,}481$	$x = 952{,}223{,}506$
$x = 6{,}309{,}280{,}697$	$x = 6{,}403{,}150{,}362$
$x = 18{,}465{,}126{,}217$	$x = 19{,}033{,}524{,}538$

(Leech, Bays & Hudson) Since then, "notable differences" have been observed between primes of various other forms $qn+a$, where q and a are constants.

Let's see graphs of races modulo 3, 8, 10, and 12 . . .

More data

The race between #{primes of the form $4n+1$ up to x} and #{primes of the form $4n+3$ up to x}

The $4n+1$ primes take the lead at	The $4n+1$ primes lose the lead at
$x = 26{,}861$	$x = 26{,}863$
$x = 616{,}481$	$x = 633{,}798$
$x = 12{,}306{,}137$	$x = 12{,}382{,}326$
$x = 951{,}784{,}481$	$x = 952{,}223{,}506$
$x = 6{,}309{,}280{,}697$	$x = 6{,}403{,}150{,}362$
$x = 18{,}465{,}126{,}217$	$x = 19{,}033{,}524{,}538$

(Leech, Bays & Hudson) Since then, “notable differences” have been observed between primes of various other forms $qn+a$, where q and a are constants.

Let’s see graphs of races modulo 3, 8, 10, and 12 …

More data

The race between #{primes of the form $4n+1$ up to x} and #{primes of the form $4n+3$ up to x}

The $4n+1$ primes take the lead at	The $4n+1$ primes lose the lead at
$x = 26{,}861$	$x = 26{,}863$
$x = 616{,}481$	$x = 633{,}798$
$x = 12{,}306{,}137$	$x = 12{,}382{,}326$
$x = 951{,}784{,}481$	$x = 952{,}223{,}506$
$x = 6{,}309{,}280{,}697$	$x = 6{,}403{,}150{,}362$
$x = 18{,}465{,}126{,}217$	$x = 19{,}033{,}524{,}538$

(Leech, Bays & Hudson) Since then, “notable differences” have been observed between primes of various other forms $qn+a$, where q and a are constants.

Let’s see graphs of races modulo 3, 8, 10, and 12 . . .

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

...

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5$, or $6 \pmod 7$ over primes that are $1, 2$, or $4 \pmod 7$
- Primes that are $3, 5$, or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7$, or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

. . .

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

...

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

. . .

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are 5, 7, or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

. . .

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are 5, 7, or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

…

Notation

- $\pi(x; q, a)$ denotes the number of primes $p \le x$ such that $p \equiv a \pmod{q}$

Example

$$\pi(x; 4, 1) = \#\{\text{primes of the form } 4n + 1 \text{ up to } x\}$$
$$\pi(x; 4, 3) = \#\{\text{primes of the form } 4n + 3 \text{ up to } x\}$$

- $\pi(x) = \pi(x; 1, 1)$ denotes the total number of primes $p \le x$

Example: $\pi(x) = \pi(x; 4, 1) + \pi(x; 4, 3) + 1$ for $x \ge 2$

- $\phi(q) = \#\{1 \le a \le q\colon\ \gcd(a, q) = 1\}$

Example

$\phi(4) = 2$; and there are 2 "reasonable contestants", $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$, in the race (mod 4) … the contestants $\pi(x; 4, 0)$ and $\pi(x; 4, 2)$ give up quickly

Notation

- $\pi(x; q, a)$ denotes the number of primes $p \le x$ such that $p \equiv a \pmod q$

Example

$$\pi(x; 4, 1) = \#\{\text{primes of the form } 4n+1 \text{ up to } x\}$$
$$\pi(x; 4, 3) = \#\{\text{primes of the form } 4n+3 \text{ up to } x\}$$

- $\pi(x) = \pi(x; 1, 1)$ denotes the total number of primes $p \le x$

Example: $\pi(x) = \pi(x; 4, 1) + \pi(x; 4, 3) + 1$ for $x \ge 2$

- $\phi(q) = \#\{1 \le a \le q \colon \gcd(a, q) = 1\}$

Example

$\phi(4) = 2$; and there are 2 "reasonable contestants", $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$, in the race (mod 4) … the contestants $\pi(x; 4, 0)$ and $\pi(x; 4, 2)$ give up quickly

Notation

- $\pi(x; q, a)$ denotes the number of primes $p \le x$ such that $p \equiv a \pmod{q}$

Example

$$\pi(x; 4, 1) = \#\{\text{primes of the form } 4n+1 \text{ up to } x\}$$
$$\pi(x; 4, 3) = \#\{\text{primes of the form } 4n+3 \text{ up to } x\}$$

- $\pi(x) = \pi(x; 1, 1)$ denotes the total number of primes $p \le x$

Example: $\pi(x) = \pi(x; 4, 1) + \pi(x; 4, 3) + 1$ for $x \ge 2$

- $\phi(q) = \#\{1 \le a \le q \colon \gcd(a, q) = 1\}$

Example

$\phi(4) = 2$; and there are 2 “reasonable contestants”, $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$, in the race (mod 4) … the contestants $\pi(x; 4, 0)$ and $\pi(x; 4, 2)$ give up quickly

Notation

- $\pi(x; q, a)$ denotes the number of primes $p \le x$ such that $p \equiv a \pmod{q}$

Example

$$\pi(x; 4, 1) = \#\{\text{primes of the form } 4n+1 \text{ up to } x\}$$
$$\pi(x; 4, 3) = \#\{\text{primes of the form } 4n+3 \text{ up to } x\}$$

- $\pi(x) = \pi(x; 1, 1)$ denotes the total number of primes $p \le x$

Example: $\pi(x) = \pi(x; 4, 1) + \pi(x; 4, 3) + 1$ for $x \ge 2$

- $\phi(q) = \#\{1 \le a \le q\colon\ \gcd(a, q) = 1\}$

Example

$\phi(4) = 2$; and there are 2 “reasonable contestants”, $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$, in the race (mod 4) … the contestants $\pi(x; 4, 0)$ and $\pi(x; 4, 2)$ give up quickly

Notation

- $\pi(x; q, a)$ denotes the number of primes $p \le x$ such that $p \equiv a \pmod{q}$

Example

$$\pi(x; 4, 1) = \#\{\text{primes of the form } 4n + 1 \text{ up to } x\}$$
$$\pi(x; 4, 3) = \#\{\text{primes of the form } 4n + 3 \text{ up to } x\}$$

- $\pi(x) = \pi(x; 1, 1)$ denotes the total number of primes $p \le x$

Example: $\pi(x) = \pi(x; 4, 1) + \pi(x; 4, 3) + 1$ for $x \ge 2$

- $\phi(q) = \#\{1 \le a \le q\colon \gcd(a, q) = 1\}$

Example

$\phi(4) = 2$; and there are 2 "reasonable contestants", $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$, in the race (mod 4) … the contestants $\pi(x; 4, 0)$ and $\pi(x; 4, 2)$ give up quickly

Notation

- $\pi(x; q, a)$ denotes the number of primes $p \leq x$ such that $p \equiv a \pmod{q}$

Example

$$\pi(x; 4, 1) = \#\{\text{primes of the form } 4n+1 \text{ up to } x\}$$
$$\pi(x; 4, 3) = \#\{\text{primes of the form } 4n+3 \text{ up to } x\}$$

- $\pi(x) = \pi(x; 1, 1)$ denotes the total number of primes $p \leq x$

Example: $\pi(x) = \pi(x; 4, 1) + \pi(x; 4, 3) + 1$ for $x \geq 2$

- $\phi(q) = \#\{1 \leq a \leq q \colon \gcd(a, q) = 1\}$

Example

$\phi(4) = 2$; and there are 2 “reasonable contestants”, $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$, in the race (mod 4) . . . the contestants $\pi(x; 4, 0)$ and $\pi(x; 4, 2)$ give up quickly

Past results: computational

Notation (always $\gcd(a,q)=1$)

$\pi(x;q,a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod{q}\}$

Further computation (mostly in the 1970s by Bays and Hudson) reveals that there are occasional periods of triumph for the lagging residue classes over leading residue classes:

- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
- $\pi(x;3,1) > \pi(x;3,2)$ for $316{,}889{,}212$ integers between $x = 608{,}981{,}813{,}029$ and $x = 610{,}968{,}213{,}796$ (its first lead)
- $\pi(x;24,1) > \pi(x;24,13)$ sometime just before $x = 979{,}400{,}000{,}000$ —but still only in 7th place out of the $\phi(24) = 8$ contestants
- no specific value of x is known for which $\pi(x;12,1)$ is ahead of any of $\pi(x;12,5)$, $\pi(x;12,7)$, or $\pi(x;12,11)$! (although at least one lead change happens before 10^{84} in each race)

Past results: computational

Notation (always $\gcd(a,q) = 1$)

$\pi(x;q,a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod{q}\}$

Further computation (mostly in the 1970s by Bays and Hudson) reveals that there are occasional periods of triumph for the lagging residue classes over leading residue classes:

- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
- $\pi(x;3,1) > \pi(x;3,2)$ for $316{,}889{,}212$ integers between $x = 608{,}981{,}813{,}029$ and $x = 610{,}968{,}213{,}796$ (its first lead)
- $\pi(x;24,1) > \pi(x;24,13)$ sometime just before $x = 979{,}400{,}000{,}000$ —but still only in 7th place out of the $\phi(24) = 8$ contestants
- no specific value of x is known for which $\pi(x;12,1)$ is ahead of any of $\pi(x;12,5)$, $\pi(x;12,7)$, or $\pi(x;12,11)$! (although at least one lead change happens before 10^{84} in each race)

Past results: computational

Notation (always $\gcd(a,q)=1$)

$\pi(x;q,a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod q\}$

Further computation (mostly in the 1970s by Bays and Hudson) reveals that there are occasional periods of triumph for the lagging residue classes over leading residue classes:

- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
- $\pi(x;3,1) > \pi(x;3,2)$ for $316{,}889{,}212$ integers between $x = 608{,}981{,}813{,}029$ and $x = 610{,}968{,}213{,}796$ (its first lead)
- $\pi(x;24,1) > \pi(x;24,13)$ sometime just before $x = 979{,}400{,}000{,}000$ —but still only in 7th place out of the $\phi(24) = 8$ contestants
- no specific value of x is known for which $\pi(x;12,1)$ is ahead of any of $\pi(x;12,5)$, $\pi(x;12,7)$, or $\pi(x;12,11)$! (although at least one lead change happens before 10^{84} in each race)

Past results: computational

Notation (always $\gcd(a, q) = 1$)

$\pi(x; q, a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod{q}\}$

Further computation (mostly in the 1970s by Bays and Hudson) reveals that there are occasional periods of triumph for the lagging residue classes over leading residue classes:

- $\pi(x; 8, 1) > \pi(x; 8, 5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x; 8, 1)$ still lags behind $\pi(x; 8, 3)$ and $\pi(x; 8, 7)$
- $\pi(x; 3, 1) > \pi(x; 3, 2)$ for $316{,}889{,}212$ integers between $x = 608{,}981{,}813{,}029$ and $x = 610{,}968{,}213{,}796$ (its first lead)
- $\pi(x; 24, 1) > \pi(x; 24, 13)$ sometime just before $x = 979{,}400{,}000{,}000$ —but still only in 7th place out of the $\phi(24) = 8$ contestants
- no specific value of x is known for which $\pi(x; 12, 1)$ is ahead of any of $\pi(x; 12, 5)$, $\pi(x; 12, 7)$, or $\pi(x; 12, 11)$! (although at least one lead change happens before 10^{84} in each race)

Past results: computational

Notation (always $\gcd(a,q)=1$)

$\pi(x;q,a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod{q}\}$

Further computation (mostly in the 1970s by Bays and Hudson) reveals that there are occasional periods of triumph for the lagging residue classes over leading residue classes:

- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
- $\pi(x;3,1) > \pi(x;3,2)$ for $316{,}889{,}212$ integers between $x = 608{,}981{,}813{,}029$ and $x = 610{,}968{,}213{,}796$ (its first lead)
- $\pi(x;24,1) > \pi(x;24,13)$ sometime just before $x = 979{,}400{,}000{,}000$ —but still only in 7th place out of the $\phi(24) = 8$ contestants
- no specific value of x is known for which $\pi(x;12,1)$ is ahead of any of $\pi(x;12,5)$, $\pi(x;12,7)$, or $\pi(x;12,11)$! (although at least one lead change happens before 10^{84} in each race)

Dirichlet's theorem

It was already known in Chebyshev's time that each contestant in these prime number races could run forever:

Theorem (Dirichlet, 1837)

If $\gcd(a,q)=1$*, then there are infinitely many primes* $p \equiv a \pmod q$*.*

But it wasn't until the turn of the 20th century that it was shown that all of these reasonable residue classes had, asymptotically, the same number of primes:

Theorem (combining work of Dirichlet, Riemann, Hadamard, de la Vallée Poussin)

If $\gcd(a,q)=1$*, then* $\displaystyle\lim_{x\to\infty} \frac{\pi(x;q,a)}{\pi(x)} = \frac{1}{\phi(q)}$.

Dirichlet's theorem

It was already known in Chebyshev's time that each contestant in these prime number races could run forever:

Theorem (Dirichlet, 1837)

If $\gcd(a, q) = 1$*, then there are infinitely many primes* $p \equiv a \pmod{q}$.

But it wasn't until the turn of the 20th century that it was shown that all of these reasonable residue classes had, asymptotically, the same number of primes:

Theorem (combining work of Dirichlet, Riemann, Hadamard, de la Vallée Poussin)

If $\gcd(a, q) = 1$*, then* $\lim_{x\to\infty} \frac{\pi(x; q, a)}{\pi(x)} = \frac{1}{\phi(q)}$.

Dirichlet's theorem

It was already known in Chebyshev's time that each contestant in these prime number races could run forever:

Theorem (Dirichlet, 1837)

If $\gcd(a,q) = 1$*, then there are infinitely many primes* $p \equiv a \pmod{q}$*.*

But it wasn't until the turn of the 20th century that it was shown that all of these reasonable residue classes had, asymptotically, the same number of primes:

Theorem (combining work of Dirichlet, Riemann, Hadamard, de la Vallée Poussin)

If $\gcd(a,q) = 1$*, then* $\displaystyle\lim_{x\to\infty} \frac{\pi(x;q,a)}{\pi(x)} = \frac{1}{\phi(q)}$.

Dirichlet's theorem

It was already known in Chebyshev's time that each contestant in these prime number races could run forever:

Theorem (Dirichlet, 1837)

If $\gcd(a, q) = 1$*, then there are infinitely many primes* $p \equiv a \pmod{q}$*.*

But it wasn't until the turn of the 20th century that it was shown that all of these reasonable residue classes had, asymptotically, the same number of primes:

Theorem (combining work of Dirichlet, Riemann, Hadamard, de la Vallée Poussin)

If $\gcd(a, q) = 1$*, then* $\displaystyle\lim_{x\to\infty} \frac{\pi(x; q, a)}{\pi(x)} = \frac{1}{\phi(q)}$.

A few goals

Learning to "handicap" prime number races means understanding the following questions:

Question

When is $\pi(x;q,a)$ bigger than $\pi(x;q,b)$?

More fundamental question

Given q and a, how fast does $\pi(x;q,a)$ grow as a function of x?

Since $\lim_{x\to\infty} \frac{\pi(x;q,a)}{\pi(x)} = \frac{1}{\phi(q)}$, this question reduces to:

Even more fundamental question

How fast does $\pi(x)$ grow as a function of x?

So let's talk about how many primes there are up to x.

A few goals

Learning to "handicap" prime number races means understanding the following questions:

Question

When is $\pi(x;q,a)$ bigger than $\pi(x;q,b)$?

More fundamental question

Given q and a, how fast does $\pi(x;q,a)$ grow as a function of x?

Since $\lim_{x\to\infty} \frac{\pi(x;q,a)}{\pi(x)} = \frac{1}{\phi(q)}$, this question reduces to:

Even more fundamental question

How fast does $\pi(x)$ grow as a function of x?

So let's talk about how many primes there are up to x.

A few goals

Learning to "handicap" prime number races means understanding the following questions:

Question

When is $\pi(x;q,a)$ bigger than $\pi(x;q,b)$?

More fundamental question

Given q and a, how fast does $\pi(x;q,a)$ grow as a function of x?

Since $\lim_{x\to\infty} \frac{\pi(x;q,a)}{\pi(x)} = \frac{1}{\phi(q)}$, this question reduces to:

Even more fundamental question

How fast does $\pi(x)$ grow as a function of x?

So let's talk about how many primes there are up to x.

How many primes?

Question

Approximately how many primes are there less than some given number x?

Notation

We write $f(x) \sim g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

A “good” answer to the question will mean finding a simple, smooth function $g(x)$ such that $\pi(x) \sim g(x)$.

How many primes?

Question

Approximately how many primes are there less than some given number x?

Notation

We write $f(x) \sim g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

A "good" answer to the question will mean finding a simple, smooth function $g(x)$ such that $\pi(x) \sim g(x)$.

How many primes?

Question

Approximately how many primes are there less than some given number x?

Notation

We write $f(x) \sim g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

A "good" answer to the question will mean finding a simple, smooth function $g(x)$ such that $\pi(x) \sim g(x)$.

The Prime Number Theorem

- Legendre conjectured that $\pi(x) \sim x/\ln x$.
- Gauss made a more precise conjecture:

$$\pi(x) \sim \mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

This is consistent with Legendre since $\mathrm{li}(x) \sim x/\ln x$.

Let's see a graph of these two functions alongside $\pi(x)$ . . .

- In 1859, Riemann wrote a groundbreaking memoir describing how he thought the question could be settled. His plan was gradually realized by many researchers, ending with Hadamard and de la Vallée Poussin independently in 1898. They didn't prove the most exact version of Riemann's argument, but they did prove Gauss's conjecture.

The Prime Number Theorem

- Legendre conjectured that $\pi(x) \sim x/\ln x$.
- Gauss made a more precise conjecture:

$$\pi(x) \sim \mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

This is consistent with Legendre since $\mathrm{li}(x) \sim x/\ln x$.

Let's see a graph of these two functions alongside $\pi(x)$ . . .

- In 1859, Riemann wrote a groundbreaking memoir describing how he thought the question could be settled. His plan was gradually realized by many researchers, ending with Hadamard and de la Vallée Poussin independently in 1898. They didn't prove the most exact version of Riemann's argument, but they did prove Gauss's conjecture.

The Prime Number Theorem

- Legendre conjectured that $\pi(x) \sim x/\ln x$.
- Gauss made a more precise conjecture:

$$\pi(x) \sim \mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

This is consistent with Legendre since $\mathrm{li}(x) \sim x/\ln x$.

Let's see a graph of these two functions alongside $\pi(x)$ . . .

- In 1859, Riemann wrote a groundbreaking memoir describing how he thought the question could be settled. His plan was gradually realized by many researchers, ending with Hadamard and de la Vallée Poussin independently in 1898. They didn't prove the most exact version of Riemann's argument, but they did prove Gauss's conjecture.

The Prime Number Theorem

- Legendre conjectured that $\pi(x) \sim x/\ln x$.
- Gauss made a more precise conjecture:

$$\pi(x) \sim \operatorname{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

This is consistent with Legendre since $\operatorname{li}(x) \sim x/\ln x$.

Let's see a graph of these two functions alongside $\pi(x)$ ...

- In 1859, Riemann wrote a groundbreaking memoir describing how he thought the question could be settled. His plan was gradually realized by many researchers, ending with Hadamard and de la Vallée Poussin independently in 1898. They didn't prove the most exact version of Riemann's argument, but they did prove Gauss's conjecture.

The Prime Number Theorem

- Legendre conjectured that $\pi(x) \sim x/\ln x$.
- Gauss made a more precise conjecture:

$$\pi(x) \sim \mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

This is consistent with Legendre since $\mathrm{li}(x) \sim x/\ln x$.

Let's see a graph of these two functions alongside $\pi(x)$ . . .

- In 1859, Riemann wrote a groundbreaking memoir describing how he thought the question could be settled. His plan was gradually realized by many researchers, ending with Hadamard and de la Vallée Poussin independently in 1898. They didn't prove the most exact version of Riemann's argument, but they did prove Gauss's conjecture.

The Prime Number Theorem

- Legendre conjectured that $\pi(x) \sim x/\ln x$.
- Gauss made a more precise conjecture:

$$\pi(x) \sim \mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

This is consistent with Legendre since $\mathrm{li}(x) \sim x/\ln x$.

Let's see a graph of these two functions alongside $\pi(x)$ . . .

- In 1859, Riemann wrote a groundbreaking memoir describing how he thought the question could be settled. His plan was gradually realized by many researchers, ending with Hadamard and de la Vallée Poussin independently in 1898. They didn't prove the most exact version of Riemann's argument, but they did prove Gauss's conjecture.

Riemann's magic plan

Riemann's plan for proving the Prime Number Theorem was to study the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for complex numbers s. (This formula works when $\Re s > 1$; other formulas define $\zeta(s)$ for all complex numbers s except for $s = 1$.)

Notation

ρ will denote a nontrivial zero of $\zeta(s)$, that is, a complex number with real part between 0 and 1 such that $\zeta(\rho) = 0$. Any sum written $\sum_\rho$ denotes a sum over all such nontrivial zeros.

Riemann's magic plan

Riemann's plan for proving the Prime Number Theorem was to study the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for complex numbers s. (This formula works when $\Re s > 1$; other formulas define $\zeta(s)$ for all complex numbers s except for $s = 1$.)

Notation

ρ will denote a nontrivial zero of $\zeta(s)$, that is, a complex number with real part between 0 and 1 such that $\zeta(\rho) = 0$. Any sum written $\sum_\rho$ denotes a sum over all such nontrivial zeros.

Riemann's magic plan

Riemann's plan for proving the Prime Number Theorem was to study the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for complex numbers s. (This formula works when $\Re s > 1$; other formulas define $\zeta(s)$ for all complex numbers s except for $s = 1$.)

Notation

ρ will denote a nontrivial zero of $\zeta(s)$, that is, a complex number with real part between 0 and 1 such that $\zeta(\rho) = 0$. Any sum written $\sum_\rho$ denotes a sum over all such nontrivial zeros.

Riemann's magic plan

Riemann's plan for proving the Prime Number Theorem was to study the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for complex numbers s. (This formula works when $\Re s > 1$; other formulas define $\zeta(s)$ for all complex numbers s except for $s = 1$.)

Notation

ρ will denote a nontrivial zero of $\zeta(s)$, that is, a complex number with real part between 0 and 1 such that $\zeta(\rho) = 0$. Any sum written $\sum_\rho$ denotes a sum over all such nontrivial zeros.

Riemann's magic formula

Riemann established a technically complicated formula that, from a modern perspective, can be written in the following form:

Riemann's magic formula (modernized)

Define $\psi(x) = \ln\big(\operatorname{lcm}[1,2,\dots,x]\big)$. Then

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \ln(2\pi) - \tfrac12 \ln(1-1/x^2).$$

(The last two terms aren't worth paying attention to.)

Let's look more closely at the left-hand side and the right-hand side of this magic formula.

Riemann's magic formula

Riemann established a technically complicated formula that, from a modern perspective, can be written in the following form:

Riemann's magic formula (modernized)

Define $\psi(x) = \ln\big(\mathrm{lcm}[1,2,\dots,x]\big)$. Then

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \ln(2\pi) - \tfrac{1}{2}\ln(1-1/x^2).$$

(The last two terms aren't worth paying attention to.)

Let's look more closely at the left-hand side and the right-hand side of this magic formula.

Riemann's magic formula

Riemann established a technically complicated formula that, from a modern perspective, can be written in the following form:

Riemann's magic formula (modernized)

Define $\psi(x) = \ln\big(\operatorname{lcm}[1, 2, \ldots, x]\big)$. Then

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \ln(2\pi) - \tfrac{1}{2}\ln(1 - 1/x^2).$$

(The last two terms aren't worth paying attention to.)

Let's look more closely at the left-hand side and the right-hand side of this magic formula.

Riemann's magic formula

Riemann established a technically complicated formula that, from a modern perspective, can be written in the following form:

Riemann's magic formula (modernized)

Define $\psi(x) = \ln\big(\operatorname{lcm}[1,2,\dots,x]\big)$. Then

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \ln(2\pi) - \tfrac{1}{2}\ln(1 - 1/x^2).$$

(The last two terms aren't worth paying attention to.)

Let's look more closely at the left-hand side and the right-hand side of this magic formula.

Riemann's magic formula

Riemann established a technically complicated formula that, from a modern perspective, can be written in the following form:

Riemann's magic formula (modernized)

Define $\psi(x) = \ln\big(\operatorname{lcm}[1,2,\dots,x]\big)$. Then

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \ln(2\pi) - \tfrac12 \ln(1-1/x^2).$$

(The last two terms aren't worth paying attention to.)

Let's look more closely at the left-hand side and the right-hand side of this magic formula.

The left-hand side

Notice that $\mathrm{lcm}[1,2,\dots,x] = \prod_{\text{primes } p\le x} p^{k(p)}$, where $k(p)$ is the power such that $p^{k(p)} \le x < p^{k(p)+1}$. Therefore:

$$\begin{aligned}\psi(x) = \ln\big(\mathrm{lcm}[1,2,\dots,x]\big) &= \sum_{p\le x} k(p)\ln p \\ &= \sum_{p\le x}\ln p + \sum_{p\le x^{1/2}}\ln p + \sum_{p\le x^{1/3}}\ln p + \cdots \\ &\approx \sum_{p\le x}\ln x + \sum_{p\le x^{1/2}}\ln(x^{1/2}) + \sum_{p\le x^{1/3}}\ln(x^{1/3}) + \cdots.\end{aligned}$$

Rule of thumb

$\psi(x)/\ln x$ acts like $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

Humans count primes; Nature counts primes and their powers.

The left-hand side

Notice that $\operatorname{lcm}[1,2,\ldots,x] = \prod_{\text{primes } p\le x} p^{k(p)}$, where $k(p)$ is the power such that $p^{k(p)} \le x < p^{k(p)+1}$. Therefore:

$$\psi(x) = \ln\big(\operatorname{lcm}[1,2,\ldots,x]\big) = \sum_{p\le x} k(p)\ln p$$

$$= \sum_{p\le x}\ln p + \sum_{p\le x^{1/2}}\ln p + \sum_{p\le x^{1/3}}\ln p + \cdots$$

$$\approx \sum_{p\le x}\ln x + \sum_{p\le x^{1/2}}\ln(x^{1/2}) + \sum_{p\le x^{1/3}}\ln(x^{1/3}) + \cdots.$$

Rule of thumb

$\psi(x)/\ln x$ acts like $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

Humans count primes; Nature counts primes and their powers.

The left-hand side

Notice that $\operatorname{lcm}[1,2,\dots,x] = \prod_{\text{primes } p\le x} p^{k(p)}$, where $k(p)$ is the power such that $p^{k(p)} \le x < p^{k(p)+1}$. Therefore:

$$\begin{aligned}\psi(x) &= \ln\big(\operatorname{lcm}[1,2,\dots,x]\big) = \sum_{p\le x} k(p)\ln p \\ &= \sum_{p\le x}\ln p + \sum_{p\le x^{1/2}}\ln p + \sum_{p\le x^{1/3}}\ln p + \cdots \\ &\approx \sum_{p\le x}\ln x + \sum_{p\le x^{1/2}}\ln(x^{1/2}) + \sum_{p\le x^{1/3}}\ln(x^{1/3}) + \cdots.\end{aligned}$$

Rule of thumb

$\psi(x)/\ln x$ acts like $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

Humans count primes; Nature counts primes and their powers.

The left-hand side

Notice that $\mathrm{lcm}[1,2,\ldots,x] = \prod_{\text{primes } p\le x} p^{k(p)}$, where $k(p)$ is the power such that $p^{k(p)} \le x < p^{k(p)+1}$. Therefore:

$$\begin{aligned}\psi(x) &= \ln\big(\mathrm{lcm}[1,2,\ldots,x]\big) = \sum_{p\le x} k(p)\ln p \\ &= \sum_{p\le x}\ln p + \sum_{p\le x^{1/2}}\ln p + \sum_{p\le x^{1/3}}\ln p + \cdots \\ &\approx \sum_{p\le x}\ln x + \sum_{p\le x^{1/2}}\ln(x^{1/2}) + \sum_{p\le x^{1/3}}\ln(x^{1/3}) + \cdots.\end{aligned}$$

Rule of thumb

$\psi(x)/\ln x$ acts like $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

Humans count primes; Nature counts primes and their powers.

The left-hand side

Notice that $\operatorname{lcm}[1,2,\dots,x] = \prod_{\text{primes } p \le x} p^{k(p)}$, where $k(p)$ is the power such that $p^{k(p)} \le x < p^{k(p)+1}$. Therefore:

$$\begin{aligned}\psi(x) &= \ln\big(\operatorname{lcm}[1,2,\dots,x]\big) = \sum_{p\le x} k(p)\ln p \\ &= \sum_{p\le x}\ln p + \sum_{p\le x^{1/2}}\ln p + \sum_{p\le x^{1/3}}\ln p + \cdots \\ &\approx \sum_{p\le x}\ln x + \sum_{p\le x^{1/2}}\ln(x^{1/2}) + \sum_{p\le x^{1/3}}\ln(x^{1/3}) + \cdots.\end{aligned}$$

Rule of thumb

$\psi(x)/\ln x$ acts like $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

Humans count primes; Nature counts primes and their powers.

The left-hand side

Notice that $\operatorname{lcm}[1,2,\ldots,x] = \prod_{\text{primes } p\le x} p^{k(p)}$, where $k(p)$ is the power such that $p^{k(p)} \le x < p^{k(p)+1}$. Therefore:

$$\begin{aligned}\psi(x) &= \ln\big(\operatorname{lcm}[1,2,\ldots,x]\big) = \sum_{p\le x} k(p)\ln p \\ &= \sum_{p\le x}\ln p + \sum_{p\le x^{1/2}}\ln p + \sum_{p\le x^{1/3}}\ln p + \cdots \\ &\approx \sum_{p\le x}\ln x + \sum_{p\le x^{1/2}}\ln(x^{1/2}) + \sum_{p\le x^{1/3}}\ln(x^{1/3}) + \cdots.\end{aligned}$$

Rule of thumb

$\psi(x)/\ln x$ acts like $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

Humans count primes; Nature counts primes and their powers.

The right-hand side

Write $\rho = \beta + i\gamma$. Note that

$$x^\rho = x^\beta x^{i\gamma} = x^\beta e^{i\gamma \ln x} = x^\beta \big(\cos(\gamma \ln x) + i \sin(\gamma \ln x) \big).$$

de la Vallée Poussin proved that β can't be very close to 1 if γ is small, which was enough to prove the Prime Number Theorem. But Riemann conjectured something much stronger:

Riemann Hypothesis

All nontrivial zeros ρ of $\zeta(s)$ have real part $\beta = 1/2$.

Assuming the Riemann Hypothesis, we obtain:

$$\Big(\pi(x) + \tfrac12 \pi(x^{1/2}) + \tfrac13 \pi(x^{1/3}) + \cdots \Big) \ln x$$
$$\approx x - \sum_\rho \frac{x^\rho}{\rho} = x - \sqrt{x} \sum_{\substack{\gamma \in \mathcal{R} \\ \zeta(1/2+i\gamma)=0}} \frac{\cos(\gamma \ln x) + i \sin(\gamma \ln x)}{1/2 + i\gamma}.$$

The right-hand side

Write $\rho = \beta + i\gamma$. Note that

$$x^\rho = x^\beta x^{i\gamma} = x^\beta e^{i\gamma \ln x} = x^\beta \big(\cos(\gamma \ln x) + i \sin(\gamma \ln x) \big).$$

de la Vallée Poussin proved that β can't be very close to 1 if γ is small, which was enough to prove the Prime Number Theorem. But Riemann conjectured something much stronger:

Riemann Hypothesis

All nontrivial zeros ρ of $\zeta(s)$ have real part $\beta = 1/2$.

Assuming the Riemann Hypothesis, we obtain:

$$\Big(\pi(x) + \tfrac12 \pi(x^{1/2}) + \tfrac13 \pi(x^{1/3}) + \cdots \Big) \ln x$$

$$\approx x - \sum_\rho \frac{x^\rho}{\rho} = x - \sqrt{x} \sum_{\substack{\gamma \in \mathcal{R} \\ \zeta(1/2+i\gamma)=0}} \frac{\cos(\gamma \ln x) + i \sin(\gamma \ln x)}{1/2 + i\gamma}.$$

The right-hand side

Write $\rho = \beta + i\gamma$. Note that

$$x^\rho = x^\beta x^{i\gamma} = x^\beta e^{i\gamma \ln x} = x^\beta \big(\cos(\gamma \ln x) + i \sin(\gamma \ln x) \big).$$

de la Vallée Poussin proved that β can't be very close to 1 if γ is small, which was enough to prove the Prime Number Theorem. But Riemann conjectured something much stronger:

Riemann Hypothesis

All nontrivial zeros ρ of $\zeta(s)$ have real part $\beta = 1/2$.

Assuming the Riemann Hypothesis, we obtain:

$$\big(\pi(x) + \tfrac12 \pi(x^{1/2}) + \tfrac13 \pi(x^{1/3}) + \cdots \big) \ln x$$
$$\approx x - \sum_\rho \frac{x^\rho}{\rho} = x - \sqrt{x} \sum_{\substack{\gamma \in \mathcal{R} \\ \zeta(1/2+i\gamma)=0}} \frac{\cos(\gamma \ln x) + i \sin(\gamma \ln x)}{1/2 + i\gamma}.$$

The right-hand side

Write $\rho = \beta + i\gamma$. Note that

$$x^\rho = x^\beta x^{i\gamma} = x^\beta e^{i\gamma \ln x} = x^\beta \big(\cos(\gamma \ln x) + i \sin(\gamma \ln x) \big).$$

de la Vallée Poussin proved that β can't be very close to 1 if γ is small, which was enough to prove the Prime Number Theorem. But Riemann conjectured something much stronger:

Riemann Hypothesis

All nontrivial zeros ρ of $\zeta(s)$ have real part $\beta = 1/2$.

Assuming the Riemann Hypothesis, we obtain:

$$\Big(\pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \tfrac{1}{3}\pi(x^{1/3}) + \cdots \Big) \ln x$$
$$\approx x - \sum_\rho \frac{x^\rho}{\rho} = x - \sqrt{x} \sum_{\substack{\gamma \in \mathcal{R} \\ \zeta(1/2+i\gamma)=0}} \frac{\cos(\gamma \ln x) + i \sin(\gamma \ln x)}{1/2 + i\gamma}.$$

Riemann's magic formula, transformed

Combining these observations, moving terms around, and tweaking yields the following formula:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4 + \gamma^2} + \frac{\cos(\gamma \ln x)}{1/2 + 2\gamma^2} \right).$$

The following approximation is easier to grasp:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \approx 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \frac{\sin(\gamma t)}{\gamma} \qquad \text{where } t = \ln x.$$

- This 1 arises from the term $\frac{1}{2}\pi(x^{1/2}) \sim \sqrt{x}/\ln x$.

Let's see two graphs showing this magic formula in action ...

Riemann's magic formula, transformed

Combining these observations, moving terms around, and tweaking yields the following formula:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4 + \gamma^2} + \frac{\cos(\gamma \ln x)}{1/2 + 2\gamma^2} \right).$$

The following approximation is easier to grasp:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \approx 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \frac{\sin(\gamma t)}{\gamma} \qquad \text{where } t = \ln x.$$

- This 1 arises from the term $\frac{1}{2}\pi(x^{1/2}) \sim \sqrt{x}/\ln x$.

Let's see two graphs showing this magic formula in action …

Riemann's magic formula, transformed

Combining these observations, moving terms around, and tweaking yields the following formula:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4 + \gamma^2} + \frac{\cos(\gamma \ln x)}{1/2 + 2\gamma^2} \right).$$

The following approximation is easier to grasp:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \approx 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \frac{\sin(\gamma t)}{\gamma} \qquad \text{where } t = \ln x.$$

- This 1 arises from the term $\frac{1}{2}\pi(x^{1/2}) \sim \sqrt{x}/\ln x$.

Let's see two graphs showing this magic formula in action . . .

Riemann's magic formula, transformed

Combining these observations, moving terms around, and tweaking yields the following formula:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4 + \gamma^2} + \frac{\cos(\gamma \ln x)}{1/2 + 2\gamma^2} \right).$$

The following approximation is easier to grasp:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \approx 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \frac{\sin(\gamma t)}{\gamma} \qquad \text{where } t = \ln x.$$

- This 1 arises from the term $\frac{1}{2}\pi(x^{1/2}) \sim \sqrt{x}/\ln x$.

Let's see two graphs showing this magic formula in action …

Riemann's magic formula, transformed

Combining these observations, moving terms around, and tweaking yields the following formula:

$$\frac{\operatorname{li}(x)-\pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4+\gamma^2} + \frac{\cos(\gamma \ln x)}{1/2+2\gamma^2} \right).$$

The following approximation is easier to grasp:

$$\frac{\operatorname{li}(x)-\pi(x)}{\sqrt{x}/\ln x} \approx 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \frac{\sin(\gamma t)}{\gamma} \qquad \text{where } t = \ln x.$$

- This 1 arises from the term $\frac{1}{2}\pi(x^{1/2}) \sim \sqrt{x}/\ln x$.

Let's see two graphs showing this magic formula in action . . .

The race between $\pi(x)$ and $\mathrm{li}(x)$

- We've seen there is a bias (due to the squares of primes) that causes $\mathrm{li}(x) > \pi(x)$ to be more likely.
- However, Littlewood proved that occasionally, the terms in the explicit formula can cooperate enough to force $\pi(x) > \mathrm{li}(x)$.
- We don't know a specific x for which $\pi(x) > \mathrm{li}(x)$, but we know it happens before 1.4×10^{316} (and that might be the first time it ever happens).
- Under assumptions on the zeros of $\zeta(s)$, including the Riemann hypothesis, Rubinstein and Sarnak found a way to define the "proportion of time" that $\mathrm{li}(x) > \pi(x)$: it happens approximately 99.999973% of the time.

The race between $\pi(x)$ and $\mathrm{li}(x)$

- We've seen there is a bias (due to the squares of primes) that causes $\mathrm{li}(x) > \pi(x)$ to be more likely.
- However, Littlewood proved that occasionally, the terms in the explicit formula can cooperate enough to force $\pi(x) > \mathrm{li}(x)$.
- We don't know a specific x for which $\pi(x) > \mathrm{li}(x)$, but we know it happens before 1.4×10^{316} (and that might be the first time it ever happens).
- Under assumptions on the zeros of $\zeta(s)$, including the Riemann hypothesis, Rubinstein and Sarnak found a way to define the "proportion of time" that $\mathrm{li}(x) > \pi(x)$: it happens approximately 99.999973% of the time.

The race between $\pi(x)$ and $\mathrm{li}(x)$

- We've seen there is a bias (due to the squares of primes) that causes $\mathrm{li}(x) > \pi(x)$ to be more likely.
- However, Littlewood proved that occasionally, the terms in the explicit formula can cooperate enough to force $\pi(x) > \mathrm{li}(x)$.
- We don't know a specific x for which $\pi(x) > \mathrm{li}(x)$, but we know it happens before 1.4×10^{316} (and that might be the first time it ever happens).
- Under assumptions on the zeros of $\zeta(s)$, including the Riemann hypothesis, Rubinstein and Sarnak found a way to define the "proportion of time" that $\mathrm{li}(x) > \pi(x)$: it happens approximately 99.999973% of the time.

The race between $\pi(x)$ and $\mathrm{li}(x)$

- We've seen there is a bias (due to the squares of primes) that causes $\mathrm{li}(x) > \pi(x)$ to be more likely.
- However, Littlewood proved that occasionally, the terms in the explicit formula can cooperate enough to force $\pi(x) > \mathrm{li}(x)$.
- We don't know a specific x for which $\pi(x) > \mathrm{li}(x)$, but we know it happens before 1.4×10^{316} (and that might be the first time it ever happens).
- Under assumptions on the zeros of $\zeta(s)$, including the Riemann hypothesis, Rubinstein and Sarnak found a way to define the "proportion of time" that $\mathrm{li}(x) > \pi(x)$: it happens approximately 99.999973% of the time.

Dirichlet characters

Definition

A Dirichlet character modulo q is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ satisfying:

1. χ is periodic with period q;
2. $\chi(n) = 0$ if $\gcd(n, q) > 1$;
3. χ is totally multiplicative: $\chi(mn) = \chi(m)\chi(n)$

There are always $\phi(q)$ Dirichlet characters modulo q, and their orthogonality can be used to pick out particular arithmetic progressions: for any a with $\gcd(a, q) = 1$,

$$\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi(a)}\chi(n) = \begin{cases} 1, & \text{if } n \equiv a \pmod{q}, \\ 0, & \text{if } n \not\equiv a \pmod{q} \end{cases}$$

as functions of n.

Dirichlet characters

Definition

A Dirichlet character modulo q is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ satisfying:

1. χ is periodic with period q;
2. $\chi(n) = 0$ if $\gcd(n,q) > 1$;
3. χ is totally multiplicative: $\chi(mn) = \chi(m)\chi(n)$

There are always $\phi(q)$ Dirichlet characters modulo q, and their orthogonality can be used to pick out particular arithmetic progressions: for any a with $\gcd(a,q) = 1$,

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) = \begin{cases} 1, & \text{if } n \equiv a \pmod q, \\ 0, & \text{if } n \not\equiv a \pmod q \end{cases}$$

as functions of n.

Dirichlet characters

Definition

A Dirichlet character modulo q is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ satisfying:

1. χ is periodic with period q;
2. $\chi(n) = 0$ if $\gcd(n,q) > 1$;
3. χ is totally multiplicative: $\chi(mn) = \chi(m)\chi(n)$

There are always $\phi(q)$ Dirichlet characters modulo q, and their orthogonality can be used to pick out particular arithmetic progressions: for any a with $\gcd(a,q) = 1$,

$$\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi(a)}\chi(n) = \begin{cases} 1, & \text{if } n \equiv a \pmod{q}, \\ 0, & \text{if } n \not\equiv a \pmod{q} \end{cases}$$

as functions of n.

Dirichlet characters

Definition

A Dirichlet character modulo q is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ satisfying:

1. χ is periodic with period q;
2. $\chi(n) = 0$ if $\gcd(n,q) > 1$;
3. χ is totally multiplicative: $\chi(mn) = \chi(m)\chi(n)$

There are always $\phi(q)$ Dirichlet characters modulo q, and their orthogonality can be used to pick out particular arithmetic progressions: for any a with $\gcd(a,q) = 1$,

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) = \begin{cases} 1, & \text{if } n \equiv a \pmod q, \\ 0, & \text{if } n \not\equiv a \pmod q) \end{cases}$$

as functions of n.

Dirichlet characters

Definition

A Dirichlet character modulo q is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ satisfying:

1. χ is periodic with period q;
2. $\chi(n) = 0$ if $\gcd(n, q) > 1$;
3. χ is totally multiplicative: $\chi(mn) = \chi(m)\chi(n)$

There are always $\phi(q)$ Dirichlet characters modulo q, and their orthogonality can be used to pick out particular arithmetic progressions: for any a with $\gcd(a, q) = 1$,

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) = \begin{cases} 1, & \text{if } n \equiv a \pmod q, \\ 0, & \text{if } n \not\equiv a \pmod q \end{cases}$$

as functions of n.

Dirichlet characters

Examples of Dirichlet characters

- The principal character modulo q:

$$\chi_0(n) = \begin{cases} 1, & \text{if } \gcd(n,q) = 1, \\ 0, & \text{if } \gcd(n,q) > 1. \end{cases}$$

- The only nonprincipal character modulo 4 has values

$$1,\ 0,\ -1,\ 0;\ 1,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 10, with values

$$1,\ 0,\ i,\ 0,\ 0,\ 0,\ -i,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 7, with values

$$1,\ -\tfrac12 + \tfrac{i\sqrt3}{2},\ \tfrac12 + \tfrac{i\sqrt3}{2},\ -\tfrac12 - \tfrac{i\sqrt3}{2},\ \tfrac12 - \tfrac{i\sqrt3}{2},\ -1,\ 0;\ \dots.$$

Dirichlet characters

Examples of Dirichlet characters

- The principal character modulo q:

$$\chi_0(n) = \begin{cases} 1, & \text{if } \gcd(n,q) = 1, \\ 0, & \text{if } \gcd(n,q) > 1. \end{cases}$$

- The only nonprincipal character modulo 4 has values

$$1,\ 0,\ -1,\ 0;\ 1,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 10, with values

$$1,\ 0,\ i,\ 0,\ 0,\ 0,\ -i,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 7, with values

$$1,\ -\tfrac{1}{2}+\tfrac{i\sqrt{3}}{2},\ \tfrac{1}{2}+\tfrac{i\sqrt{3}}{2},\ -\tfrac{1}{2}-\tfrac{i\sqrt{3}}{2},\ \tfrac{1}{2}-\tfrac{i\sqrt{3}}{2},\ -1,\ 0;\ \dots.$$

Dirichlet characters

Examples of Dirichlet characters

- The principal character modulo q:

$$\chi_0(n) = \begin{cases} 1, & \text{if } \gcd(n,q) = 1, \\ 0, & \text{if } \gcd(n,q) > 1. \end{cases}$$

- The only nonprincipal character modulo 4 has values

$$1,\ 0,\ -1,\ 0;\ 1,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 10, with values

$$1,\ 0,\ i,\ 0,\ 0,\ 0,\ -i,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 7, with values

$$1,\ -\tfrac{1}{2}+\tfrac{i\sqrt{3}}{2},\ \tfrac{1}{2}+\tfrac{i\sqrt{3}}{2},\ -\tfrac{1}{2}-\tfrac{i\sqrt{3}}{2},\ \tfrac{1}{2}-\tfrac{i\sqrt{3}}{2},\ -1,\ 0;\ \dots.$$

Dirichlet characters

Examples of Dirichlet characters

- The principal character modulo q:

$$\chi_0(n) = \begin{cases} 1, & \text{if } \gcd(n,q) = 1, \\ 0, & \text{if } \gcd(n,q) > 1. \end{cases}$$

- The only nonprincipal character modulo 4 has values

$$1,\ 0,\ -1,\ 0;\ 1,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 10, with values

$$1,\ 0,\ i,\ 0,\ 0,\ 0,\ -i,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 7, with values

$$1,\ -\tfrac12 + \tfrac{i\sqrt3}{2},\ \tfrac12 + \tfrac{i\sqrt3}{2},\ -\tfrac12 - \tfrac{i\sqrt3}{2},\ \tfrac12 - \tfrac{i\sqrt3}{2},\ -1,\ 0;\ \dots.$$

Dirichlet L-functions

Each Dirichlet character χ gives rise to a Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Example

When $\chi = \chi_0$ is the principal character (mod q),

$$L(s,\chi_0) = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

- By showing $\lim_{s\to1} L(s,\chi)$ exists and is nonzero for every nonprincipal character χ, Dirichlet proved that there are infinitely many primes $p \equiv a \pmod q$ when $\gcd(a,q) = 1$.
- If we incorporate these Dirichlet L-functions, Riemann's plan adapts to counting primes in arithmetic progressions.

Dirichlet L-functions

Each Dirichlet character χ gives rise to a Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Example

When $\chi = \chi_0$ is the principal character (mod q),

$$L(s,\chi_0) = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

- By showing $\lim_{s \to 1} L(s,\chi)$ exists and is nonzero for every nonprincipal character χ, Dirichlet proved that there are infinitely many primes $p \equiv a \pmod{q}$ when $\gcd(a,q) = 1$.
- If we incorporate these Dirichlet L-functions, Riemann's plan adapts to counting primes in arithmetic progressions.

Dirichlet L-functions

Each Dirichlet character χ gives rise to a Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Example

When $\chi = \chi_0$ is the principal character (mod q),

$$L(s,\chi_0) = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

- By showing $\lim_{s\to1} L(s,\chi)$ exists and is nonzero for every nonprincipal character χ, Dirichlet proved that there are infinitely many primes $p \equiv a \pmod q$ when $\gcd(a,q) = 1$.
- If we incorporate these Dirichlet L-functions, Riemann's plan adapts to counting primes in arithmetic progressions.

Dirichlet L-functions

Each Dirichlet character χ gives rise to a Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Example

When $\chi = \chi_0$ is the principal character (mod q),

$$L(s,\chi_0) = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

- By showing $\lim_{s\to 1} L(s,\chi)$ exists and is nonzero for every nonprincipal character χ, Dirichlet proved that there are infinitely many primes $p \equiv a \pmod{q}$ when $\gcd(a,q) = 1$.
- If we incorporate these Dirichlet L-functions, Riemann's plan adapts to counting primes in arithmetic progressions.

Dirichlet L-functions

Each Dirichlet character χ gives rise to a Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Example

When $\chi = \chi_0$ is the principal character (mod q),

$$L(s,\chi_0) = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

- By showing $\lim_{s\to 1} L(s,\chi)$ exists and is nonzero for every nonprincipal character χ, Dirichlet proved that there are infinitely many primes $p \equiv a \pmod{q}$ when $\gcd(a,q) = 1$.
- If we incorporate these Dirichlet L-functions, Riemann's plan adapts to counting primes in arithmetic progressions.

Primes in arithmetic progressions

A combination of Dirichlet L-functions, from orthogonality

$$\sum_{n\equiv a \pmod q} n^{-s} = \sum_{n=1}^{\infty} \left(\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) \right) n^{-s}$$

$$= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} L(s,\chi)$$

Prime number theorem for arithmetic progressions

If $\gcd(a,q)=1$, then $\pi(x;q,a) \sim \frac{1}{\phi(q)} \mathrm{li}(x)$.

In other words, all $\phi(q)$ eligible arithmetic progressions contain, asymptotically, about the same number of primes.

How is this asymptotic equity compatible with the “winners” and “losers” we saw earlier in the prime number races?

Primes in arithmetic progressions

A combination of Dirichlet L-functions, from orthogonality

$$\sum_{n\equiv a \pmod q} n^{-s} = \sum_{n=1}^{\infty} \left(\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) \right) n^{-s}$$

$$= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} L(s,\chi)$$

Prime number theorem for arithmetic progressions

If $\gcd(a,q)=1$, then $\pi(x;q,a) \sim \frac{1}{\phi(q)} \operatorname{li}(x)$.

In other words, all $\phi(q)$ eligible arithmetic progressions contain, asymptotically, about the same number of primes.

How is this asymptotic equity compatible with the “winners” and “losers” we saw earlier in the prime number races?

Primes in arithmetic progressions

A combination of Dirichlet L-functions, from orthogonality

$$\sum_{n\equiv a \pmod q} n^{-s} = \sum_{n=1}^{\infty} \left(\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) \right) n^{-s}$$

$$= \frac{1}{\phi(q)} \left(\zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) + \sum_{\chi \ne \chi_0} \overline{\chi(a)} L(s,\chi) \right)$$

Prime number theorem for arithmetic progressions

If $\gcd(a,q)=1$, then $\pi(x;q,a) \sim \dfrac{1}{\phi(q)} \operatorname{li}(x)$.

In other words, all $\phi(q)$ eligible arithmetic progressions contain, asymptotically, about the same number of primes.

How is this asymptotic equity compatible with the “winners” and “losers” we saw earlier in the prime number races?

Primes in arithmetic progressions

A combination of Dirichlet L-functions, from orthogonality

$$\sum_{n \equiv a \pmod{q}} n^{-s} = \sum_{n=1}^{\infty} \left(\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi(a)} \chi(n) \right) n^{-s}$$

$$= \frac{1}{\phi(q)} \left(\zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right) + \sum_{\chi \neq \chi_0} \overline{\chi(a)} L(s, \chi) \right)$$

Prime number theorem for arithmetic progressions

If $\gcd(a,q) = 1$, then $\pi(x;q,a) \sim \dfrac{1}{\phi(q)} \operatorname{li}(x)$.

In other words, all $\phi(q)$ eligible arithmetic progressions contain, asymptotically, about the same number of primes.

How is this asymptotic equity compatible with the “winners” and “losers” we saw earlier in the prime number races?

Primes in arithmetic progressions

A combination of Dirichlet L-functions, from orthogonality

$$\sum_{n\equiv a \pmod q} n^{-s} = \sum_{n=1}^{\infty} \left(\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) \right) n^{-s}$$
$$= \frac{1}{\phi(q)} \left(\zeta(s) \prod_{p|q} \left(1 - \tfrac{1}{p^s}\right) + \sum_{\chi \ne \chi_0} \overline{\chi(a)} L(s,\chi) \right)$$

Prime number theorem for arithmetic progressions

If $\gcd(a,q) = 1$, then $\pi(x;q,a) \sim \dfrac{1}{\phi(q)} \operatorname{li}(x)$.

In other words, all $\phi(q)$ eligible arithmetic progressions contain, asymptotically, about the same number of primes.

How is this asymptotic equity compatible with the “winners” and “losers” we saw earlier in the prime number races?

Primes in arithmetic progressions

A combination of Dirichlet L-functions, from orthogonality

$$\sum_{n\equiv a \pmod q} n^{-s} = \sum_{n=1}^{\infty} \left(\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) \right) n^{-s}$$

$$= \frac{1}{\phi(q)} \left(\zeta(s) \prod_{p|q} \left(1 - \tfrac{1}{p^s}\right) + \sum_{\chi\neq\chi_0} \overline{\chi(a)} L(s,\chi) \right)$$

Prime number theorem for arithmetic progressions

If $\gcd(a,q)=1$, then $\pi(x;q,a) \sim \dfrac{1}{\phi(q)} \operatorname{li}(x)$.

In other words, all $\phi(q)$ eligible arithmetic progressions contain, asymptotically, about the same number of primes.

How is this asymptotic equity compatible with the “winners” and “losers” we saw earlier in the prime number races?

More magic: the mod 4 race

A closer analysis reveals subtle differences among the functions $\pi(x; q, a)$.

Example ($q = 4$)

Let χ be the nonprincipal character modulo 4, so that

$$L(s,\chi) = 1 + 0 - \frac{1}{3^s} + 0 + \frac{1}{5^s} + 0 - \frac{1}{7^s} + \cdots .$$

Assuming the Riemann hypothesis for $\zeta(s)$:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4+\gamma^2} + \frac{\cos(\gamma \ln x)}{1/2+2\gamma^2} \right).$$

[Of course, from the difference $\pi(x;4,3) - \pi(x;4,1)$ and the sum $\pi(x;4,3) + \pi(x;4,1) = \pi(x) - 1$, we can recover the functions $\pi(x;4,3)$ and $\pi(x;4,1)$ individually.]

More magic: the mod 4 race

A closer analysis reveals subtle differences among the functions $\pi(x; q, a)$.

Example ($q = 4$)

Let χ be the nonprincipal character modulo 4, so that

$$L(s,\chi) = 1 + 0 - \frac{1}{3^s} + 0 + \frac{1}{5^s} + 0 - \frac{1}{7^s} + \cdots .$$

Assuming the Riemann hypothesis for $\zeta(s)$:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4+\gamma^2} + \frac{\cos(\gamma \ln x)}{1/2+2\gamma^2} \right).$$

[Of course, from the difference $\pi(x;4,3) - \pi(x;4,1)$ and the sum $\pi(x;4,3) + \pi(x;4,1) = \pi(x) - 1$, we can recover the functions $\pi(x;4,3)$ and $\pi(x;4,1)$ individually.]

More magic: the mod 4 race

A closer analysis reveals subtle differences among the functions $\pi(x; q, a)$.

Example ($q = 4$)

Let χ be the nonprincipal character modulo 4, so that

$$L(s, \chi) = 1 + 0 - \frac{1}{3^s} + 0 + \frac{1}{5^s} + 0 - \frac{1}{7^s} + \cdots .$$

Assuming the Riemann hypothesis for $\zeta(s)$:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4 + \gamma^2} + \frac{\cos(\gamma \ln x)}{1/2 + 2\gamma^2} \right).$$

[Of course, from the difference $\pi(x; 4, 3) - \pi(x; 4, 1)$ and the sum $\pi(x; 4, 3) + \pi(x; 4, 1) = \pi(x) - 1$, we can recover the functions $\pi(x; 4, 3)$ and $\pi(x; 4, 1)$ individually.]

More magic: the mod 4 race

A closer analysis reveals subtle differences among the functions $\pi(x; q, a)$.

Example ($q = 4$)

Let χ be the nonprincipal character modulo 4, so that

$$L(s, \chi) = 1 + 0 - \frac{1}{3^s} + 0 + \frac{1}{5^s} + 0 - \frac{1}{7^s} + \cdots .$$

Assuming the Riemann hypothesis for $L(s, \chi)$:

$$\frac{\pi(x; 4, 3) - \pi(x; 4, 1)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma > 0 \\ L(1/2+i\gamma, \chi)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4 + \gamma^2} + \frac{\cos(\gamma \ln x)}{1/2 + 2\gamma^2} \right).$$

[Of course, from the difference $\pi(x; 4, 3) - \pi(x; 4, 1)$ and the sum $\pi(x; 4, 3) + \pi(x; 4, 1) = \pi(x) - 1$, we can recover the functions $\pi(x; 4, 3)$ and $\pi(x; 4, 1)$ individually.]

More magic: the mod 4 race

A closer analysis reveals subtle differences among the functions $\pi(x; q, a)$.

Example ($q = 4$)

Let χ be the nonprincipal character modulo 4, so that

$$L(s,\chi) = 1 + 0 - \frac{1}{3^s} + 0 + \frac{1}{5^s} + 0 - \frac{1}{7^s} + \cdots .$$

Assuming the Riemann hypothesis for $L(s,\chi)$:

$$\frac{\pi(x;4,3) - \pi(x;4,1)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4+\gamma^2} + \frac{\cos(\gamma \ln x)}{1/2+2\gamma^2} \right).$$

[Of course, from the difference $\pi(x;4,3) - \pi(x;4,1)$ and the sum $\pi(x;4,3) + \pi(x;4,1) = \pi(x) - 1$, we can recover the functions $\pi(x;4,3)$ and $\pi(x;4,1)$ individually.]

Nonsquares beat squares

Humans count primes; Nature counts primes and their powers.

- In general each $\pi(x; q, a)$, suitably normalized, can be expressed (assuming a generalized Riemann Hypothesis) as a sum of terms of the form $\sin(\gamma \ln x)/\gamma$, where some Dirichlet L-function corresponding to a Dirichlet character modulo q has a zero at the point $1/2 + i\gamma$.
- The difference is: some residue classes a (mod q) contain squares of primes. For these, the formula has an additional negative term (like the 1 in the formula involving $\mathrm{li}(x) - \pi(x)$), while for the nonsquares it's absent.

These squares of primes are what causes all the biases in the prime number races we've seen!

Nonsquares beat squares

Humans count primes; Nature counts primes and their powers.

- In general each $\pi(x; q, a)$, suitably normalized, can be expressed (assuming a generalized Riemann Hypothesis) as a sum of terms of the form $\sin(\gamma \ln x)/\gamma$, where some Dirichlet L-function corresponding to a Dirichlet character modulo q has a zero at the point $1/2 + i\gamma$.
- The difference is: some residue classes a (mod q) contain squares of primes. For these, the formula has an additional negative term (like the 1 in the formula involving $\mathrm{li}(x) - \pi(x)$), while for the nonsquares it's absent.

These squares of primes are what causes all the biases in the prime number races we've seen!

Nonsquares beat squares

Humans count primes; Nature counts primes and their powers.

- In general each $\pi(x; q, a)$, suitably normalized, can be expressed (assuming a generalized Riemann Hypothesis) as a sum of terms of the form $\sin(\gamma \ln x)/\gamma$, where some Dirichlet L-function corresponding to a Dirichlet character modulo q has a zero at the point $1/2 + i\gamma$.
- The difference is: some residue classes $a \pmod q$ contain squares of primes. For these, the formula has an additional negative term (like the 1 in the formula involving $\mathrm{li}(x) - \pi(x)$), while for the nonsquares it's absent.

These squares of primes are what causes all the biases in the prime number races we've seen!

Nonsquares beat squares

Humans count primes; Nature counts primes and their powers.

- In general each $\pi(x;q,a)$, suitably normalized, can be expressed (assuming a generalized Riemann Hypothesis) as a sum of terms of the form $\sin(\gamma \ln x)/\gamma$, where some Dirichlet L-function corresponding to a Dirichlet character modulo q has a zero at the point $1/2 + i\gamma$.
- The difference is: some residue classes a (mod q) contain squares of primes. For these, the formula has an additional negative term (like the 1 in the formula involving $\mathrm{li}(x) - \pi(x)$), while for the nonsquares it's absent.

These squares of primes are what causes all the biases in the prime number races we've seen!

Nonsquares beat squares

Humans count primes; Nature counts primes and their powers.

- In general each $\pi(x;q,a)$, suitably normalized, can be expressed (assuming a generalized Riemann Hypothesis) as a sum of terms of the form $\sin(\gamma \ln x)/\gamma$, where some Dirichlet L-function corresponding to a Dirichlet character modulo q has a zero at the point $1/2 + i\gamma$.
- The difference is: some residue classes a (mod q) contain squares of primes. For these, the formula has an additional negative term (like the 1 in the formula involving $\mathrm{li}(x) - \pi(x)$), while for the nonsquares it's absent.

These squares of primes are what causes all the biases in the prime number races we've seen!

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are 5, 7, or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are $5, 7,$ or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$
- Primes that are 3 or $7 \pmod{10}$ over primes that are 1 or $9 \pmod{10}$
- Primes that are 5, 7, or $11 \pmod{12}$ over primes that are $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

How badly are these races skewed?

Using these “sums of waves” also allows us to calculate the proportion of time one contestant is ahead of the other.

We still have to make assumptions about the zeros of Dirichlet L-functions, such as the generalized Riemann hypothesis; and we have to define “proportion of time” very carefully.

Mod 4 race

$\pi(x; 4, 3) > \pi(x; 4, 1)$ about 99.59% of the time.

Mod 3 race

$\pi(x; 3, 2) > \pi(x; 3, 1)$ about 99.90% of the time.

How badly are these races skewed?

Using these “sums of waves” also allows us to calculate the proportion of time one contestant is ahead of the other.

We still have to make assumptions about the zeros of Dirichlet L-functions, such as the generalized Riemann hypothesis; and we have to define “proportion of time” very carefully.

Mod 4 race

$\pi(x;4,3) > \pi(x;4,1)$ about 99.59% of the time.

Mod 3 race

$\pi(x;3,2) > \pi(x;3,1)$ about 99.90% of the time.

How badly are these races skewed?

Using these "sums of waves" also allows us to calculate the proportion of time one contestant is ahead of the other.

We still have to make assumptions about the zeros of Dirichlet L-functions, such as the generalized Riemann hypothesis; and we have to define "proportion of time" very carefully.

Mod 4 race

$\pi(x; 4, 3) > \pi(x; 4, 1)$ about 99.59% of the time.

Mod 3 race

$\pi(x; 3, 2) > \pi(x; 3, 1)$ about 99.90% of the time.

How badly are these races skewed?

Using these "sums of waves" also allows us to calculate the proportion of time one contestant is ahead of the other.

We still have to make assumptions about the zeros of Dirichlet L-functions, such as the generalized Riemann hypothesis; and we have to define "proportion of time" very carefully.

Mod 4 race

$\pi(x;4,3) > \pi(x;4,1)$ about 99.59% of the time.

Mod 3 race

$\pi(x;3,2) > \pi(x;3,1)$ about 99.90% of the time.

How badly are these races skewed?

Mod 10 race

- $\pi(x;10,1) > \pi(x;10,9)$ and $\pi(x;10,3) > \pi(x;10,7)$ exactly 50% of the time.
- $\pi(x;10,3 \text{ or } 7) > \pi(x;10,1 \text{ or } 9)$ about 95.21% of the time.

Mod 8 race

- Any two of $\pi(x;8,3)$, $\pi(x;8,5)$, $\pi(x;8,7)$ make a 50%–50% race.
- $\pi(x;8,5) > \pi(x;8,1)$ about 99.74% of the time.
- $\pi(x;8,7) > \pi(x;8,1)$ about 99.89% of the time.
- $\pi(x;8,3) > \pi(x;8,1)$ about 99.95% of the time.

How badly are these races skewed?

Mod 10 race

- $\pi(x;10,1) > \pi(x;10,9)$ and $\pi(x;10,3) > \pi(x;10,7)$ exactly 50% of the time.
- $\pi(x;10,3 \text{ or } 7) > \pi(x;10,1 \text{ or } 9)$ about 95.21% of the time.

Mod 8 race

- Any two of $\pi(x;8,3)$, $\pi(x;8,5)$, $\pi(x;8,7)$ make a 50%–50% race.
- $\pi(x;8,5) > \pi(x;8,1)$ about 99.74% of the time.
- $\pi(x;8,7) > \pi(x;8,1)$ about 99.89% of the time.
- $\pi(x;8,3) > \pi(x;8,1)$ about 99.95% of the time.

How badly are these races skewed?

Mod 10 race

- $\pi(x;10,1) > \pi(x;10,9)$ and $\pi(x;10,3) > \pi(x;10,7)$ exactly 50% of the time.
- $\pi(x;10,3 \text{ or } 7) > \pi(x;10,1 \text{ or } 9)$ about 95.21% of the time.

Mod 8 race

- Any two of $\pi(x;8,3)$, $\pi(x;8,5)$, $\pi(x;8,7)$ make a 50%–50% race.
- $\pi(x;8,5) > \pi(x;8,1)$ about 99.74% of the time.
- $\pi(x;8,7) > \pi(x;8,1)$ about 99.89% of the time.
- $\pi(x;8,3) > \pi(x;8,1)$ about 99.95% of the time.

How badly are these races skewed?

Mod 10 race

- $\pi(x; 10, 1) > \pi(x; 10, 9)$ and $\pi(x; 10, 3) > \pi(x; 10, 7)$ exactly 50% of the time.
- $\pi(x; 10, 3 \text{ or } 7) > \pi(x; 10, 1 \text{ or } 9)$ about 95.21% of the time.

Mod 8 race

- Any two of $\pi(x; 8, 3)$, $\pi(x; 8, 5)$, $\pi(x; 8, 7)$ make a 50%–50% race.
- $\pi(x; 8, 5) > \pi(x; 8, 1)$ about 99.74% of the time.
- $\pi(x; 8, 7) > \pi(x; 8, 1)$ about 99.89% of the time.
- $\pi(x; 8, 3) > \pi(x; 8, 1)$ about 99.95% of the time.

How badly are these races skewed?

Mod 10 race

- $\pi(x;10,1) > \pi(x;10,9)$ and $\pi(x;10,3) > \pi(x;10,7)$ exactly 50% of the time.
- $\pi(x;10,3 \text{ or } 7) > \pi(x;10,1 \text{ or } 9)$ about 95.21% of the time.

Mod 8 race

- Any two of $\pi(x;8,3)$, $\pi(x;8,5)$, $\pi(x;8,7)$ make a 50%–50% race.
- $\pi(x;8,5) > \pi(x;8,1)$ about 99.74% of the time.
- $\pi(x;8,7) > \pi(x;8,1)$ about 99.89% of the time.
- $\pi(x;8,3) > \pi(x;8,1)$ about 99.95% of the time.

How badly are these races skewed?

Mod 10 race

- $\pi(x;10,1) > \pi(x;10,9)$ and $\pi(x;10,3) > \pi(x;10,7)$ exactly 50% of the time.
- $\pi(x;10,3 \text{ or } 7) > \pi(x;10,1 \text{ or } 9)$ about 95.21% of the time.

Mod 8 race

- Any two of $\pi(x;8,3)$, $\pi(x;8,5)$, $\pi(x;8,7)$ make a 50%–50% race.
- $\pi(x;8,5) > \pi(x;8,1)$ about 99.74% of the time.
- $\pi(x;8,7) > \pi(x;8,1)$ about 99.89% of the time.
- $\pi(x;8,3) > \pi(x;8,1)$ about 99.95% of the time.

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x; 12, 5)$, $\pi(x; 12, 7)$, $\pi(x; 12, 11)$ make a 50%–50% race.
- $\pi(x; 12, 7) > \pi(x; 12, 1)$ about 99.86% of the time.
- $\pi(x; 12, 5) > \pi(x; 12, 1)$ about 99.92% of the time.
- $\pi(x; 12, 11) > \pi(x; 12, 1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x; 12, 5) > \pi(x; 12, 7) > \pi(x; 12, 11)$ about 19.8% of the time (and same for $\pi(x; 12, 11) > \pi(x; 12, 7) > \pi(x; 12, 5)$).
- $\pi(x; 12, 7) > \pi(x; 12, 5) > \pi(x; 12, 11)$ about 18.0% of the time (and same for $\pi(x; 12, 11) > \pi(x; 12, 5) > \pi(x; 12, 7)$).
- $\pi(x; 12, 5) > \pi(x; 12, 11) > \pi(x; 12, 7)$ about 12.2% of the time (and same for $\pi(x; 12, 7) > \pi(x; 12, 11) > \pi(x; 12, 5)$).

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x;12,5)$, $\pi(x;12,7)$, $\pi(x;12,11)$ make a 50%–50% race.
- $\pi(x;12,7) > \pi(x;12,1)$ about 99.86% of the time.
- $\pi(x;12,5) > \pi(x;12,1)$ about 99.92% of the time.
- $\pi(x;12,11) > \pi(x;12,1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x;12,5) > \pi(x;12,7) > \pi(x;12,11)$ about 19.8% of the time (and same for $\pi(x;12,11) > \pi(x;12,7) > \pi(x;12,5)$).
- $\pi(x;12,7) > \pi(x;12,5) > \pi(x;12,11)$ about 18.0% of the time (and same for $\pi(x;12,11) > \pi(x;12,5) > \pi(x;12,7)$).
- $\pi(x;12,5) > \pi(x;12,11) > \pi(x;12,7)$ about 12.2% of the time (and same for $\pi(x;12,7) > \pi(x;12,11) > \pi(x;12,5)$).

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x;12,5)$, $\pi(x;12,7)$, $\pi(x;12,11)$ make a 50%–50% race.
- $\pi(x;12,7) > \pi(x;12,1)$ about 99.86% of the time.
- $\pi(x;12,5) > \pi(x;12,1)$ about 99.92% of the time.
- $\pi(x;12,11) > \pi(x;12,1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x;12,5) > \pi(x;12,7) > \pi(x;12,11)$ about 19.8% of the time (and same for $\pi(x;12,11) > \pi(x;12,7) > \pi(x;12,5)$).
- $\pi(x;12,7) > \pi(x;12,5) > \pi(x;12,11)$ about 18.0% of the time (and same for $\pi(x;12,11) > \pi(x;12,5) > \pi(x;12,7)$).
- $\pi(x;12,5) > \pi(x;12,11) > \pi(x;12,7)$ about 12.2% of the time (and same for $\pi(x;12,7) > \pi(x;12,11) > \pi(x;12,5)$).

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x;12,5)$, $\pi(x;12,7)$, $\pi(x;12,11)$ make a 50%–50% race.
- $\pi(x;12,7) > \pi(x;12,1)$ about 99.86% of the time.
- $\pi(x;12,5) > \pi(x;12,1)$ about 99.92% of the time.
- $\pi(x;12,11) > \pi(x;12,1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x;12,5) > \pi(x;12,7) > \pi(x;12,11)$ about 19.8% of the time (and same for $\pi(x;12,11) > \pi(x;12,7) > \pi(x;12,5)$).
- $\pi(x;12,7) > \pi(x;12,5) > \pi(x;12,11)$ about 18.0% of the time (and same for $\pi(x;12,11) > \pi(x;12,5) > \pi(x;12,7)$).
- $\pi(x;12,5) > \pi(x;12,11) > \pi(x;12,7)$ about 12.2% of the time (and same for $\pi(x;12,7) > \pi(x;12,11) > \pi(x;12,5)$).

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x; 12, 5)$, $\pi(x; 12, 7)$, $\pi(x; 12, 11)$ make a 50%–50% race.
- $\pi(x; 12, 7) > \pi(x; 12, 1)$ about 99.86% of the time.
- $\pi(x; 12, 5) > \pi(x; 12, 1)$ about 99.92% of the time.
- $\pi(x; 12, 11) > \pi(x; 12, 1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x; 12, 5) > \pi(x; 12, 7) > \pi(x; 12, 11)$ about 19.8% of the time (and same for $\pi(x; 12, 11) > \pi(x; 12, 7) > \pi(x; 12, 5)$).
- $\pi(x; 12, 7) > \pi(x; 12, 5) > \pi(x; 12, 11)$ about 18.0% of the time (and same for $\pi(x; 12, 11) > \pi(x; 12, 5) > \pi(x; 12, 7)$).
- $\pi(x; 12, 5) > \pi(x; 12, 11) > \pi(x; 12, 7)$ about 12.2% of the time (and same for $\pi(x; 12, 7) > \pi(x; 12, 11) > \pi(x; 12, 5)$).

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x;12,5)$, $\pi(x;12,7)$, $\pi(x;12,11)$ make a 50%–50% race.
- $\pi(x;12,7) > \pi(x;12,1)$ about 99.86% of the time.
- $\pi(x;12,5) > \pi(x;12,1)$ about 99.92% of the time.
- $\pi(x;12,11) > \pi(x;12,1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x;12,5) > \pi(x;12,7) > \pi(x;12,11)$ about 19.8% of the time (and same for $\pi(x;12,11) > \pi(x;12,7) > \pi(x;12,5)$).
- $\pi(x;12,7) > \pi(x;12,5) > \pi(x;12,11)$ about 18.0% of the time (and same for $\pi(x;12,11) > \pi(x;12,5) > \pi(x;12,7)$).
- $\pi(x;12,5) > \pi(x;12,11) > \pi(x;12,7)$ about 12.2% of the time (and same for $\pi(x;12,7) > \pi(x;12,11) > \pi(x;12,5)$).

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x;12,5)$, $\pi(x;12,7)$, $\pi(x;12,11)$ make a 50%–50% race.
- $\pi(x;12,7) > \pi(x;12,1)$ about 99.86% of the time.
- $\pi(x;12,5) > \pi(x;12,1)$ about 99.92% of the time.
- $\pi(x;12,11) > \pi(x;12,1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x;12,5) > \pi(x;12,7) > \pi(x;12,11)$ about 19.8% of the time (and same for $\pi(x;12,11) > \pi(x;12,7) > \pi(x;12,5)$).
- $\pi(x;12,7) > \pi(x;12,5) > \pi(x;12,11)$ about 18.0% of the time (and same for $\pi(x;12,11) > \pi(x;12,5) > \pi(x;12,7)$).
- $\pi(x;12,5) > \pi(x;12,11) > \pi(x;12,7)$ about 12.2% of the time (and same for $\pi(x;12,7) > \pi(x;12,11) > \pi(x;12,5)$).

How badly are these races skewed?

Mod 12 race

- Any two of $\pi(x;12,5)$, $\pi(x;12,7)$, $\pi(x;12,11)$ make a 50%–50% race.
- $\pi(x;12,7) > \pi(x;12,1)$ about 99.86% of the time.
- $\pi(x;12,5) > \pi(x;12,1)$ about 99.92% of the time.
- $\pi(x;12,11) > \pi(x;12,1)$ about 99.998% of the time.

An asymmetrical three-way race

- $\pi(x;12,5) > \pi(x;12,7) > \pi(x;12,11)$ about 19.8% of the time (and same for $\pi(x;12,11) > \pi(x;12,7) > \pi(x;12,5)$).
- $\pi(x;12,7) > \pi(x;12,5) > \pi(x;12,11)$ about 18.0% of the time (and same for $\pi(x;12,11) > \pi(x;12,5) > \pi(x;12,7)$).
- $\pi(x;12,5) > \pi(x;12,11) > \pi(x;12,7)$ about 12.2% of the time (and same for $\pi(x;12,7) > \pi(x;12,11) > \pi(x;12,5)$).

The end

These slides

www.math.ubc.ca/~gerg/index.shtml?slides

My survey article with Andrew Granville, "Prime number races"

www.math.ubc.ca/~gerg/index.shtml?abstract=PNR

My article with Andrey Feuerverger, "Biases in the Shanks–Rényi prime number race" (numerical computations)

www.math.ubc.ca/~gerg/index.shtml?abstract=BSRPNR