Prime number races

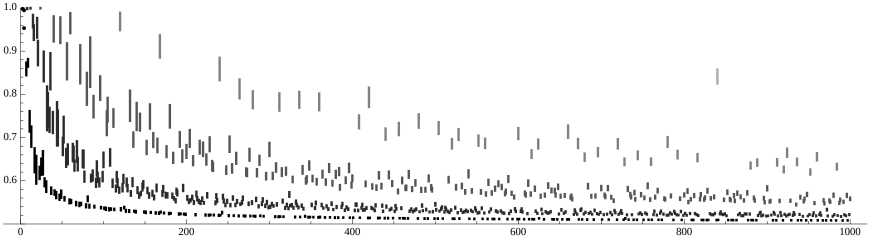
An asymptotic formula for the densities

Greg Martin
University of British Columbia

joint work with Daniel Fiorilli
Université de Montréal

Analytic and Combinatorial Number Theory
Institute of Mathematical Sciences
Chennai, India
August 31, 2010

Please turn off your cell phones and beepy things.



Outline

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement:

There is a notable difference in the splitting of the prime numbers between the two forms $4n+3$, $4n+1$: the first form contains a lot more than the second.

Since then, "notable differences" have been observed among primes of various forms $qn+a$.

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement:

There is a notable difference in the splitting of the prime numbers between the two forms $4n+3$, $4n+1$: the first form contains a lot more than the second.

Since then, "notable differences" have been observed among primes of various forms $qn+a$.

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement:

> *There is a notable difference in the splitting of the prime numbers between the two forms* $4n+3$, $4n+1$*: the first form contains a lot more than the second.*

Since then, "notable differences" have been observed among primes of various forms $qn+a$.

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 2 or $3 \pmod 5$ over primes that are 1 or $4 \pmod 5$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$; and 5, 7, or $11 \pmod{12}$ over $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 2 or $3 \pmod 5$ over primes that are 1 or $4 \pmod 5$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$; and 5, 7, or $11 \pmod{12}$ over $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 2 or $3 \pmod 5$ over primes that are 1 or $4 \pmod 5$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$; and $5, 7,$ or $11 \pmod{12}$ over $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 2 or $3 \pmod 5$ over primes that are 1 or $4 \pmod 5$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$; and $5, 7,$ or $11 \pmod{12}$ over $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 2 or $3 \pmod 5$ over primes that are 1 or $4 \pmod 5$
- Primes that are $3, 5,$ or $6 \pmod 7$ over primes that are $1, 2,$ or $4 \pmod 7$
- Primes that are $3, 5,$ or $7 \pmod 8$ over primes that are $1 \pmod 8$; and $5, 7,$ or $11 \pmod{12}$ over $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 2 or $3 \pmod 5$ over primes that are 1 or $4 \pmod 5$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$; and 5, 7, or $11 \pmod{12}$ over $1 \pmod{12}$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

Past results: computational

Notation

$\pi(x; q, a) = \{\text{number of primes } p \le x \text{ such that } p \equiv a \pmod q\}$

Further computation (1950s and beyond) reveals that there are occasional periods of triumph for the square residue classes over nonsquare residue classes:

- $\pi(x; 4, 1) > \pi(x; 4, 3)$ for the first time at $x = 26{,}861$, but $\pi(x; 4, 3) = \pi(x; 4, 1)$ again at $x = 26{,}863$; then $\pi(x; 4, 1) > \pi(x; 4, 3)$ for the second time at $x = 616{,}481$
- $\pi(x; 8, 1) > \pi(x; 8, 5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x; 8, 1)$ still lags behind $\pi(x; 8, 3)$ and $\pi(x; 8, 7)$
- $\pi(x; 3, 1) > \pi(x; 3, 2)$ for the first time at $x = 608{,}981{,}813{,}029$

Past results: computational

Notation

$\pi(x;q,a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod q\}$

Further computation (1950s and beyond) reveals that there are occasional periods of triumph for the square residue classes over nonsquare residue classes:

- $\pi(x;4,1) > \pi(x;4,3)$ for the first time at $x = 26{,}861$, but $\pi(x;4,3) = \pi(x;4,1)$ again at $x = 26{,}863$; then $\pi(x;4,1) > \pi(x;4,3)$ for the second time at $x = 616{,}481$
- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
- $\pi(x;3,1) > \pi(x;3,2)$ for the first time at $x = 608{,}981{,}813{,}029$

Past results: computational

Notation

$\pi(x;q,a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod q\}$

Further computation (1950s and beyond) reveals that there are occasional periods of triumph for the square residue classes over nonsquare residue classes:

- $\pi(x;4,1) > \pi(x;4,3)$ for the first time at $x = 26{,}861$, but $\pi(x;4,3) = \pi(x;4,1)$ again at $x = 26{,}863$; then $\pi(x;4,1) > \pi(x;4,3)$ for the second time at $x = 616{,}481$
- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
- $\pi(x;3,1) > \pi(x;3,2)$ for the first time at $x = 608{,}981{,}813{,}029$

Past results: computational

Notation

$\pi(x;q,a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod{q}\}$

Further computation (1950s and beyond) reveals that there are occasional periods of triumph for the square residue classes over nonsquare residue classes:

- $\pi(x;4,1) > \pi(x;4,3)$ for the first time at $x = 26{,}861$, but $\pi(x;4,3) = \pi(x;4,1)$ again at $x = 26{,}863$; then $\pi(x;4,1) > \pi(x;4,3)$ for the second time at $x = 616{,}481$
- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
- $\pi(x;3,1) > \pi(x;3,2)$ for the first time at $x = 608{,}981{,}813{,}029$

Past results: computational

Notation

$\pi(x; q, a) = \{$number of primes $p \le x$ such that $p \equiv a \pmod q\}$

Further computation (1950s and beyond) reveals that there are occasional periods of triumph for the square residue classes over nonsquare residue classes:

- $\pi(x; 4, 1) > \pi(x; 4, 3)$ for the first time at $x = 26{,}861$, but $\pi(x; 4, 3) = \pi(x; 4, 1)$ again at $x = 26{,}863$; then $\pi(x; 4, 1) > \pi(x; 4, 3)$ for the second time at $x = 616{,}481$
- $\pi(x; 8, 1) > \pi(x; 8, 5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x; 8, 1)$ still lags behind $\pi(x; 8, 3)$ and $\pi(x; 8, 7)$
- $\pi(x; 3, 1) > \pi(x; 3, 2)$ for the first time at $x = 608{,}981{,}813{,}029$

Past results: theoretical

And theoretical results as well:

- The prime number theorem for arithmetic progressions (1900 + $O(1)$): $\pi(x; q, a) \sim \pi(x; q, b)$
- Littlewood (1910s): each of $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$ is ahead of the other for arbitrarily large x, and similarly for $\pi(x; 3, 1)$ and $\pi(x; 3, 2)$
- Turán and Knapowski (1960s): for many pairs a, b of residue classes, $\pi(x; q, a)$ is ahead of $\pi(x; q, b)$ for arbitrarily large x. However, assumptions on the locations of zeros of Dirichlet L-functions are necessary.
- Kaczorowski (1990s): further results in this vein and also for “3-way races”, “4-way races”, etc.

Past results: theoretical

And theoretical results as well:

- The prime number theorem for arithmetic progressions (1900 $+ O(1)$): $\pi(x;q,a) \sim \pi(x;q,b)$
- Littlewood (1910s): each of $\pi(x;4,1)$ and $\pi(x;4,3)$ is ahead of the other for arbitrarily large x, and similarly for $\pi(x;3,1)$ and $\pi(x;3,2)$
- Turán and Knapowski (1960s): for many pairs a, b of residue classes, $\pi(x;q,a)$ is ahead of $\pi(x;q,b)$ for arbitrarily large x. However, assumptions on the locations of zeros of Dirichlet L-functions are necessary.
- Kaczorowski (1990s): further results in this vein and also for "3-way races", "4-way races", etc.

Past results: theoretical

And theoretical results as well:

- The prime number theorem for arithmetic progressions ($1900 + O(1)$): $\pi(x;q,a) \sim \pi(x;q,b)$
- Littlewood (1910s): each of $\pi(x;4,1)$ and $\pi(x;4,3)$ is ahead of the other for arbitrarily large x, and similarly for $\pi(x;3,1)$ and $\pi(x;3,2)$
- Turán and Knapowski (1960s): for many pairs a, b of residue classes, $\pi(x;q,a)$ is ahead of $\pi(x;q,b)$ for arbitrarily large x. However, assumptions on the locations of zeros of Dirichlet L-functions are necessary.
- Kaczorowski (1990s): further results in this vein and also for "3-way races", "4-way races", etc.

Past results: theoretical

And theoretical results as well:

- The prime number theorem for arithmetic progressions (1900 $+ O(1)$): $\pi(x; q, a) \sim \pi(x; q, b)$
- Littlewood (1910s): each of $\pi(x; 4, 1)$ and $\pi(x; 4, 3)$ is ahead of the other for arbitrarily large x, and similarly for $\pi(x; 3, 1)$ and $\pi(x; 3, 2)$
- Turán and Knapowski (1960s): for many pairs a, b of residue classes, $\pi(x; q, a)$ is ahead of $\pi(x; q, b)$ for arbitrarily large x. However, assumptions on the locations of zeros of Dirichlet L-functions are necessary.
- Kaczorowski (1990s): further results in this vein and also for “3-way races”, “4-way races”, etc.

Defining delta

The central question

How often is $\pi(x;q,a)$ ahead of $\pi(x;q,b)$?

Definition

Define $\delta_{q;a,b}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying $\pi(x;q,a) > \pi(x;q,b)$. More explicity,

$$\delta_{q;a,b} = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1\le x\le T \\ \pi(x;q,a)>\pi(x;q,b)}} \frac{dx}{x} \right).$$

$\delta_{q;a,b}$ is the limiting "probability" that when a "random" real number x is chosen, there are more primes that are congruent to $a \pmod q$ up to x then there are congruent to $b \pmod q$.

Defining delta

The central question

How often is $\pi(x;q,a)$ ahead of $\pi(x;q,b)$?

Definition

Define $\delta_{q;a,b}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying $\pi(x;q,a) > \pi(x;q,b)$. More explicity,

$$\delta_{q;a,b} = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1\le x\le T \\ \pi(x;q,a)>\pi(x;q,b)}} \frac{dx}{x} \right).$$

$\delta_{q;a,b}$ is the limiting “probability” that when a “random” real number x is chosen, there are more primes that are congruent to $a \pmod q$ up to x then there are congruent to $b \pmod q$.

Defining delta

The central question

How often is $\pi(x;q,a)$ ahead of $\pi(x;q,b)$?

Definition

Define $\delta_{q;a,b}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying $\pi(x;q,a) > \pi(x;q,b)$. More explicity,

$$\delta_{q;a,b} = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1\le x\le T \\ \pi(x;q,a)>\pi(x;q,b)}} \frac{dx}{x} \right).$$

$\delta_{q;a,b}$ is the limiting "probability" that when a "random" real number x is chosen, there are more primes that are congruent to $a \pmod q$ up to x then there are congruent to $b \pmod q$.

Defining delta

The central question

How often is $\pi(x;q,a)$ ahead of $\pi(x;q,b)$?

Definition

Define $\delta_{q;a,b}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying $\pi(x;q,a) > \pi(x;q,b)$. More explicity,

$$\delta_{q;a,b} = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1\le x\le T \\ \pi(x;q,a)>\pi(x;q,b)}} \frac{dx}{x} \right).$$

$\delta_{q;a,b}$ is the limiting “probability” that when a “random” real number x is chosen, there are more primes that are congruent to $a \pmod{q}$ up to x then there are congruent to $b \pmod{q}$.

Two hypotheses

Rubinstein and Sarnak (1994) investigated these densities $\delta_{q;a,b}$ under the following two hypotheses:

- The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet L-functions have real part equal to $\frac{1}{2}$
- A linear independence hypothesis (LI): the nonnegative imaginary parts of these nontrivial zeros are linearly independent over the rationals

- Work of Ford and Konyagin (2002) shows that certain hypothetical violations of GRH do actually lead to pathological behavior in prime number races.
- LI is somewhat analogous to a "nonsingularity" hypothesis: with precise information about any linear dependences that might exist, we could probably still work out the answer. . . .

Two hypotheses

Rubinstein and Sarnak (1994) investigated these densities $\delta_{q;a,b}$ under the following two hypotheses:

- The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet L-functions have real part equal to $\frac{1}{2}$
- A linear independence hypothesis (LI): the nonnegative imaginary parts of these nontrivial zeros are linearly independent over the rationals

- Work of Ford and Konyagin (2002) shows that certain hypothetical violations of GRH do actually lead to pathological behavior in prime number races.
- LI is somewhat analogous to a "nonsingularity" hypothesis: with precise information about any linear dependences that might exist, we could probably still work out the answer. . . .

Two hypotheses

Rubinstein and Sarnak (1994) investigated these densities $\delta_{q;a,b}$ under the following two hypotheses:

- The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet L-functions have real part equal to $\frac{1}{2}$
- A linear independence hypothesis (LI): the nonnegative imaginary parts of these nontrivial zeros are linearly independent over the rationals

- Work of Ford and Konyagin (2002) shows that certain hypothetical violations of GRH do actually lead to pathological behavior in prime number races.
- LI is somewhat analogous to a "nonsingularity" hypothesis: with precise information about any linear dependences that might exist, we could probably still work out the answer. . . .

Two hypotheses

Rubinstein and Sarnak (1994) investigated these densities $\delta_{q;a,b}$ under the following two hypotheses:

- The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet L-functions have real part equal to $\frac{1}{2}$
- A linear independence hypothesis (LI): the nonnegative imaginary parts of these nontrivial zeros are linearly independent over the rationals

- Work of Ford and Konyagin (2002) shows that certain hypothetical violations of GRH do actually lead to pathological behavior in prime number races.
- LI is somewhat analogous to a “nonsingularity” hypothesis: with precise information about any linear dependences that might exist, we could probably still work out the answer. . . .

Rubinstein and Sarnak's results

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Under these two hypotheses GRH and LI, Rubinstein and Sarnak proved (1994):

- $\delta_{q;a,b}$ always exists and is strictly between 0 and 1
- $\delta_{q;a,b} + \delta_{q;b,a} = 1 \dots$ that is, $\delta(\text{"tie"}) = 0$
- "Chebyshev's bias": $\delta_{q;a,b} > \frac{1}{2}$ if and only if a is a nonsquare $\pmod q$ and b is a square $\pmod q$
- if a and b are distinct squares $\pmod q$ or distinct nonsquares $\pmod q$, then $\delta_{q;a,b} = \delta_{q;b,a} = \frac{1}{2}$
- $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, uniformly for all pairs a, b of distinct reduced residues $\pmod q$.

Rubinstein and Sarnak's results

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Under these two hypotheses GRH and LI, Rubinstein and Sarnak proved (1994):

- $\delta_{q;a,b}$ always exists and is strictly between 0 and 1
- $\delta_{q;a,b} + \delta_{q;b,a} = 1$ ... that is, $\delta(\text{"tie"}) = 0$
- "Chebyshev's bias": $\delta_{q;a,b} > \frac{1}{2}$ if and only if a is a nonsquare $\pmod{q}$ and b is a square $\pmod{q}$
- if a and b are distinct squares $\pmod{q}$ or distinct nonsquares $\pmod{q}$, then $\delta_{q;a,b} = \delta_{q;b,a} = \frac{1}{2}$
- $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, uniformly for all pairs a, b of distinct reduced residues $\pmod{q}$.

Rubinstein and Sarnak's results

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Under these two hypotheses GRH and LI, Rubinstein and Sarnak proved (1994):

- $\delta_{q;a,b}$ always exists and is strictly between 0 and 1
- $\delta_{q;a,b} + \delta_{q;b,a} = 1$ ... that is, $\delta(\text{"tie"}) = 0$
- "Chebyshev's bias": $\delta_{q;a,b} > \frac{1}{2}$ if and only if a is a nonsquare $\pmod{q}$ and b is a square $\pmod{q}$
- if a and b are distinct squares $\pmod{q}$ or distinct nonsquares $\pmod{q}$, then $\delta_{q;a,b} = \delta_{q;b,a} = \frac{1}{2}$
- $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, uniformly for all pairs a, b of distinct reduced residues $\pmod{q}$.

Rubinstein and Sarnak's results

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Under these two hypotheses GRH and LI, Rubinstein and Sarnak proved (1994):

- $\delta_{q;a,b}$ always exists and is strictly between 0 and 1
- $\delta_{q;a,b} + \delta_{q;b,a} = 1$ ... that is, $\delta(\text{"tie"}) = 0$
- "Chebyshev's bias": $\delta_{q;a,b} > \frac{1}{2}$ if and only if a is a nonsquare $\pmod q$ and b is a square $\pmod q$
- if a and b are distinct squares $\pmod q$ or distinct nonsquares $\pmod q$, then $\delta_{q;a,b} = \delta_{q;b,a} = \frac{1}{2}$
- $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, uniformly for all pairs a, b of distinct reduced residues $\pmod q$.

Comparisons of the densities $\delta_{q;a,b}$

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Feuerverger and M. (2000) generalized Rubinstein and Sarnak's approach in several directions.

We calculated (assuming, as usual, GRH and LI) many examples of the densities $\delta_{q;a,b}$.

- The calculations required numerical evaluation of complicated integrals, which involved many explicitly computed zeros of Dirichlet L-functions.
- One significant discovery is that even with q fixed, the values of $\delta_{q;a,b}$ vary significantly as a and b vary over squares and nonsquares $(\mathrm{mod}\ q)$.

Comparisons of the densities $\delta_{q;a,b}$

$\delta_{q;a,b}$: the “probability” that $\pi(x;q,a) > \pi(x;q,b)$

Feuerverger and M. (2000) generalized Rubinstein and Sarnak’s approach in several directions.

We calculated (assuming, as usual, GRH and LI) many examples of the densities $\delta_{q;a,b}$.

- The calculations required numerical evaluation of complicated integrals, which involved many explicitly computed zeros of Dirichlet L-functions.
- One significant discovery is that even with q fixed, the values of $\delta_{q;a,b}$ vary significantly as a and b vary over squares and nonsquares $(\text{mod } q)$.

Comparisons of the densities $\delta_{q;a,b}$

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Feuerverger and M. (2000) generalized Rubinstein and Sarnak's approach in several directions.

We calculated (assuming, as usual, GRH and LI) many examples of the densities $\delta_{q;a,b}$.

- The calculations required numerical evaluation of complicated integrals, which involved many explicitly computed zeros of Dirichlet L-functions.
- One significant discovery is that even with q fixed, the values of $\delta_{q;a,b}$ vary significantly as a and b vary over squares and nonsquares $(\mathrm{mod}\ q)$.

Comparisons of the densities $\delta_{q;a,b}$

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Feuerverger and M. (2000) generalized Rubinstein and Sarnak's approach in several directions.

We calculated (assuming, as usual, GRH and LI) many examples of the densities $\delta_{q;a,b}$.

- The calculations required numerical evaluation of complicated integrals, which involved many explicitly computed zeros of Dirichlet L-functions.
- One significant discovery is that even with q fixed, the values of $\delta_{q;a,b}$ vary significantly as a and b vary over squares and nonsquares $(\mathrm{mod}\ q)$.

Example: races modulo 24

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 24

a	$\delta_{24;a,1}$
5	0.999987
11	0.999983
23	0.999889
7	0.999833
19	0.999719
17	0.999125
13	0.998722

Note that 1 is the only square $\pmod{24}$. The density in any race between two non-squares, such as $\delta_{24;5,7}$, equals $\frac{1}{2}$.

Example: races modulo 24

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 24

a	$\delta_{24;a,1}$
5	0.999987
11	0.999983
23	0.999889
7	0.999833
19	0.999719
17	0.999125
13	0.998722

Note that 1 is the only square $\pmod{24}$. The density in any race between two non-squares, such as $\delta_{24;5,7}$, equals $\frac{1}{2}$.

Example: races modulo 24

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 24

a	$\delta_{24;a,1}$
5	0.999987
11	0.999983
23	0.999889
7	0.999833
19	0.999719
17	0.999125
13	0.998722

Note that 1 is the only square $\pmod{24}$. The density in any race between two non-squares, such as $\delta_{24;5,7}$, equals $\frac{1}{2}$.

Example: races modulo 24

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 24

a	$\delta_{24;a,1}$
5	0.999987
11	0.999983
23	0.999889
7	0.999833
19	0.999719
17	0.999125
13	0.998722

Note that 1 is the only square $\pmod{24}$. The density in any race between two non-squares, such as $\delta_{24;5,7}$, equals $\frac{1}{2}$.

Example: races modulo 43

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

- $\delta_{q;a,b} = \delta_{q;ab^{-1},1}$ for any square $b \pmod{q}$. Thus it suffices to calculate only the values of $\delta_{q;a,1}$ for nonsquares $a \pmod{q}$.
- $\delta_{q;a,1} = \delta_{q;a^{-1},1}$ for any $a \pmod{q}$.

Example: races modulo 43

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

- $\delta_{q;a,b} = \delta_{q;ab^{-1},1}$ for any square $b \pmod q$. Thus it suffices to calculate only the values of $\delta_{q;a,1}$ for nonsquares $a \pmod q$.
- $\delta_{q;a,1} = \delta_{q;a^{-1},1}$ for any $a \pmod q$.

Example: races modulo 43

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

- $\delta_{q;a,b} = \delta_{q;ab^{-1},1}$ for any square $b \pmod{q}$. Thus it suffices to calculate only the values of $\delta_{q;a,1}$ for nonsquares $a \pmod{q}$.
- $\delta_{q;a,1} = \delta_{q;a^{-1},1}$ for any $a \pmod{q}$.

Example: races modulo 43

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

- $\delta_{q;a,b} = \delta_{q;ab^{-1},1}$ for any square $b \pmod{q}$. Thus it suffices to calculate only the values of $\delta_{q;a,1}$ for nonsquares $a \pmod{q}$.
- $\delta_{q;a,1} = \delta_{q;a^{-1},1}$ for any $a \pmod{q}$.

Example: races modulo 43

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

- $\delta_{q;a,b} = \delta_{q;ab^{-1},1}$ for any square $b \pmod{q}$. Thus it suffices to calculate only the values of $\delta_{q;a,1}$ for nonsquares $a \pmod{q}$.
- $\delta_{q;a,1} = \delta_{q;a^{-1},1}$ for any $a \pmod{q}$.

Current goals

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Current goals

- A more precise understanding of the sizes of $\delta_{q;a,b}$. Recalling that $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, for example, we would like an asymptotic formula for $\delta_{q;a,b} - \frac{1}{2}$.
- A way to decide which $\delta_{q;a,b}$ are likely to be larger than others as a and b vary (with q fixed), based on elementary criteria rather than laborious numerical calculation.

These goals are the subject of *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, which is joint work with Daniel Fiorilli (Montréal).

Current goals

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Current goals

- A more precise understanding of the sizes of $\delta_{q;a,b}$. Recalling that $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, for example, we would like an asymptotic formula for $\delta_{q;a,b} - \frac{1}{2}$.
- A way to decide which $\delta_{q;a,b}$ are likely to be larger than others as a and b vary (with q fixed), based on elementary criteria rather than laborious numerical calculation.

These goals are the subject of *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, which is joint work with Daniel Fiorilli (Montréal).

Current goals

$\delta_{q;a,b}$: the "probability" that $\pi(x; q, a) > \pi(x; q, b)$

Current goals

- A more precise understanding of the sizes of $\delta_{q;a,b}$. Recalling that $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, for example, we would like an asymptotic formula for $\delta_{q;a,b} - \frac{1}{2}$.
- A way to decide which $\delta_{q;a,b}$ are likely to be larger than others as a and b vary (with q fixed), based on elementary criteria rather than laborious numerical calculation.

These goals are the subject of *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, which is joint work with Daniel Fiorilli (Montréal).

Current goals

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Current goals

- A more precise understanding of the sizes of $\delta_{q;a,b}$. Recalling that $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, for example, we would like an asymptotic formula for $\delta_{q;a,b} - \frac{1}{2}$.
- A way to decide which $\delta_{q;a,b}$ are likely to be larger than others as a and b vary (with q fixed), based on elementary criteria rather than laborious numerical calculation.

These goals are the subject of *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, which is joint work with Daniel Fiorilli (Montréal).

Current goals

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Current goals

- A more precise understanding of the sizes of $\delta_{q;a,b}$. Recalling that $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, for example, we would like an asymptotic formula for $\delta_{q;a,b} - \frac{1}{2}$.
- A way to decide which $\delta_{q;a,b}$ are likely to be larger than others as a and b vary (with q fixed), based on elementary criteria rather than laborious numerical calculation.

These goals are the subject of *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, which is joint work with Daniel Fiorilli (Montréal).

Asymptotic formula, version I

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi}\,(\phi(q)\log q)^{1/2}} + O\left(\frac{\rho(q)\log\log q}{\phi(q)^{1/2}(\log q)^{3/2}}\right).$$

In particular, $\delta_{q;a,b} = \frac{1}{2} + O_\varepsilon\left(q^{-1/2+\varepsilon}\right)$ for any $\varepsilon > 0$.

$\rho(q) =$ the number of square roots of 1 $\pmod q$

$= 2^{\#\text{number of odd prime factors of } q} \times \{1, 2, \text{or } 4\}$

Asymptotic formula, version I

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi}\,(\phi(q)\log q)^{1/2}} + O\left(\frac{\rho(q)\log\log q}{\phi(q)^{1/2}(\log q)^{3/2}}\right).$$

In particular, $\delta_{q;a,b} = \frac{1}{2} + O_\varepsilon\big(q^{-1/2+\varepsilon}\big)$ for any $\varepsilon > 0$.

$$\rho(q) = \text{the number of square roots of 1} \pmod q$$
$$= 2^{\#\text{number of odd prime factors of } q} \times \{1, 2, \text{or } 4\}$$

Asymptotic formula, version I

$\delta_{q;a,b}$: the “probability” that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi}\,(\phi(q)\log q)^{1/2}} + O\left(\frac{\rho(q)\log\log q}{\phi(q)^{1/2}(\log q)^{3/2}}\right).$$

In particular, $\delta_{q;a,b} = \frac{1}{2} + O_\varepsilon\left(q^{-1/2+\varepsilon}\right)$ for any $\varepsilon > 0$.

$$\rho(q) = \text{the number of square roots of 1} \pmod q$$
$$= 2^{\#\text{number of odd prime factors of } q} \times \{1, 2, \text{or } 4\}$$

Asymptotic formula, version I

$\delta_{q;a,b}$: the “probability” that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi}\,(\phi(q)\log q)^{1/2}} + O\left(\frac{\rho(q)\log\log q}{\phi(q)^{1/2}(\log q)^{3/2}}\right).$$

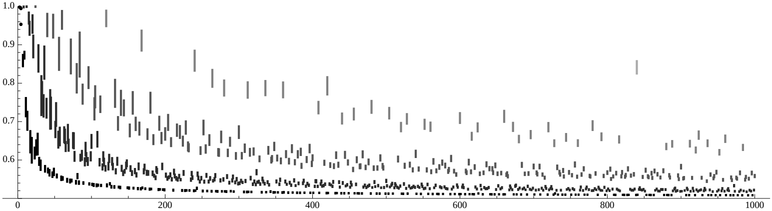
In particular, $\delta_{q;a,b} = \frac{1}{2} + O_\varepsilon\left(q^{-1/2+\varepsilon}\right)$ for any $\varepsilon > 0$.

$$\rho(q) = \text{the number of square roots of 1} \pmod q$$
$$= 2^{\#\text{number of odd prime factors of } q} \times \{1, 2, \text{or } 4\}$$

Graph of the densities

We have a full asymptotic series for $\delta(q;a,b)$, allowing us to compute the densities rapidly for $\phi(q) > 80$, say (which is when the numerical integration technique becomes worse).

Figure: All densities $\delta_{q;a,b}$ with $q \le 1000$

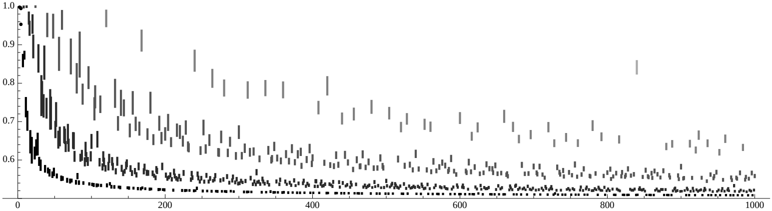


$$\delta_{q;a,b} \approx \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi\phi(q)\log q}}$$

Graph of the densities

We have a full asymptotic series for $\delta(q; a, b)$, allowing us to compute the densities rapidly for $\phi(q) > 80$, say (which is when the numerical integration technique becomes worse).

Figure: All densities $\delta_{q;a,b}$ with $q \le 1000$



$$\delta_{q;a,b} \approx \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi\phi(q)\log q}}$$

Our approach

A central limit theorem for differences of prime counts

- For given q, a, b, the normalized difference

$$\Delta_{q;a,b}(x) = \left(\frac{\phi(q)}{2\log q}\right)^{1/2} \frac{\pi(x;q,a)-\pi(x;q,b)}{x^{1/2}/\log x}$$

has a limiting distribution function

$$F_{q;a,b}(u) = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1<x<T \\ \Delta_{q;a,b}(x)<u}} \frac{dx}{x}\right).$$

- Rubinstein–Sarnak: As $q \to \infty$, these distribution functions converge to the standard normal distribution with variance 1.

A quantitative formula for the rate of convergence to the normal distribution allows us to asymptotically evaluate the density $\delta_{q;a,b}$ in terms of the variance $V(q;a,b)$ of $F_{q;a,b}$.

Our approach

A central limit theorem for differences of prime counts

- For given q, a, b, the normalized difference

$$\Delta_{q;a,b}(x) = \left(\frac{\phi(q)}{2\log q}\right)^{1/2} \frac{\pi(x;q,a)-\pi(x;q,b)}{x^{1/2}/\log x}$$

has a limiting distribution function

$$F_{q;a,b}(u) = \lim_{T\to\infty} \left(\frac{1}{\log T} \int_{\substack{1<x<T \\ \Delta_{q;a,b}(x)<u}} \frac{dx}{x}\right).$$

- Rubinstein–Sarnak: As $q \to \infty$, these distribution functions converge to the standard normal distribution with variance 1.

A quantitative formula for the rate of convergence to the normal distribution allows us to asymptotically evaluate the density $\delta_{q;a,b}$ in terms of the variance $V(q;a,b)$ of $F_{q;a,b}$.

Our approach

A central limit theorem for differences of prime counts

- For given q, a, b, the normalized difference

$$\Delta_{q;a,b}(x) = \left(\frac{\phi(q)}{2\log q}\right)^{1/2} \frac{\pi(x;q,a)-\pi(x;q,b)}{x^{1/2}/\log x}$$

  has a limiting distribution function

$$F_{q;a,b}(u) = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1<x<T \\ \Delta_{q;a,b}(x)<u}} \frac{dx}{x}\right).$$

- Rubinstein–Sarnak: As $q \to \infty$, these distribution functions converge to the standard normal distribution with variance 1.

A quantitative formula for the rate of convergence to the normal distribution allows us to asymptotically evaluate the density $\delta_{q;a,b}$ in terms of the variance $V(q;a,b)$ of $F_{q;a,b}$.

Our approach

A central limit theorem for differences of prime counts

- For given q, a, b, the normalized difference
$$\Delta_{q;a,b}(x) = \left(\frac{\phi(q)}{2\log q}\right)^{1/2} \frac{\pi(x;q,a)-\pi(x;q,b)}{x^{1/2}/\log x}$$
has a limiting distribution function
$$F_{q;a,b}(u) = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1<x<T \\ \Delta_{q;a,b}(x)<u}} \frac{dx}{x}\right).$$
- Rubinstein–Sarnak: As $q \to \infty$, these distribution functions converge to the standard normal distribution with variance 1.

A quantitative formula for the rate of convergence to the normal distribution allows us to asymptotically evaluate the density $\delta_{q;a,b}$ in terms of the variance $V(q;a,b)$ of $F_{q;a,b}$.

A few ugly details

Proposition (Feuerverger–M., *a la* Rubinstein–Sarnak)

The density $\delta_{q;a,b}$, in terms of the Fourier transform of $F_{q;a,b}$:

$$\delta_{q;a,b} - \frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bigg\{ \frac{\sin(\rho(q)t)}{t} \times \prod_{\chi \ (\mathrm{mod}\ q)} \prod_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} J_0\bigg(\frac{2|\chi(a)-\chi(b)|t}{\sqrt{1/4+\gamma^2}} \bigg)\, dt \bigg\}$$

Truncate the right-hand side and rewrite as

$$\sim \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \exp\big(-V(q;a,b)t^2/2\big) \exp\big(V_4t^4 + V_6t^6 + \cdots\big) \frac{\sin(\rho(q)t)}{t}\, dt.$$

Expand everything but the first factor into power series and integrate term by term to obtain an asymptotic series.

A few ugly details

Proposition (Feuerverger–M., *a la* Rubinstein–Sarnak)

The density $\delta_{q;a,b}$, in terms of the Fourier transform of $F_{q;a,b}$:

$$\delta_{q;a,b} - \frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{\sin(\rho(q)t)}{t} \times \prod_{\chi \pmod q} \prod_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} J_0\left(\frac{2|\chi(a)-\chi(b)|t}{\sqrt{1/4+\gamma^2}} \right) dt \right\}$$

Truncate the right-hand side and rewrite as

$$\sim \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \exp\left(-V(q;a,b)t^2/2\right) \exp\left(V_4 t^4 + V_6 t^6 + \cdots\right) \frac{\sin(\rho(q)t)}{t}\, dt.$$

Expand everything but the first factor into power series and integrate term by term to obtain an asymptotic series.

A few ugly details

Proposition (Feuerverger–M., *a la* Rubinstein–Sarnak)

The density $\delta_{q;a,b}$, in terms of the Fourier transform of $F_{q;a,b}$:

$$\delta_{q;a,b} - \frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{\sin(\rho(q)t)}{t} \times \prod_{\chi \pmod q} \prod_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} J_0\left(\frac{2|\chi(a)-\chi(b)|t}{\sqrt{1/4+\gamma^2}} \right) dt \right\}$$

Truncate the right-hand side and rewrite as

$$\sim \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \exp\left(-V(q;a,b)t^2/2\right) \exp\left(V_4 t^4 + V_6 t^6 + \cdots\right) \frac{\sin(\rho(q)t)}{t}\, dt.$$

Expand everything but the first factor into power series and integrate term by term to obtain an asymptotic series.

A few ugly details

Proposition (Feuerverger–M., *a la* Rubinstein–Sarnak)

The density $\delta_{q;a,b}$, in terms of the Fourier transform of $F_{q;a,b}$:

$$\delta_{q;a,b} - \frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \biggl\{ \frac{\sin(\rho(q)t)}{t} \\ \times \prod_{\chi \pmod q} \prod_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} J_0\biggl(\frac{2|\chi(a)-\chi(b)|t}{\sqrt{1/4+\gamma^2}} \biggr)\, dt \biggr\}$$

Truncate the right-hand side and rewrite as

$$\sim \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \exp\bigl(-V(q;a,b)t^2/2\bigr) \exp\bigl(V_4t^4 + V_6t^6 + \cdots\bigr) \frac{\sin(\rho(q)t)}{t}\, dt.$$

Expand everything but the first factor into power series and integrate term by term to obtain an asymptotic series.

Asymptotic formula, version II

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\biggl(\frac{1}{\phi(q)\log q}\biggr),$$

where

$$V(q;a,b) = 2 \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(b) - \chi(a)|^2 \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14 + \gamma^2}.$$

Asymptotic formula, version II

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\biggl(\frac{1}{\phi(q)\log q}\biggr),$$

where

$$V(q;a,b) = 2 \sum_{\substack{\chi \pmod q \\ \chi\ne\chi_0}} |\chi(b)-\chi(a)|^2 \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14+\gamma^2}.$$

Asymptotic formula, version III

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod{q}$ and b is a square $\pmod{q}$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{\phi(q)\log q}\right), \text{ where}$$

$$V(q;a,b) = 2\sum_{\substack{\chi \pmod{q} \\ \chi\ne\chi_0}} |\chi(b)-\chi(a)|^2 \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14+\gamma^2}$$

$$= 2\phi(q)\left(\log q - \sum_{p\mid q}\frac{\log p}{p-1} - (\gamma_0+\log 2\pi) + R_q(a-b)\right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

Asymptotic formula, version III

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\biggl(\frac{1}{\phi(q)\log q}\biggr), \text{ where}$$

$$V(q;a,b) = 2 \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(b) - \chi(a)|^2 \sum_{\substack{\gamma > 0 \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{1}{\frac{1}{4}+\gamma^2}$$

$$= 2\phi(q)\biggl(\log q - \sum_{p \mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b)\biggr)$$

$$+ (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

Asymptotic formula, version III

Theorem (Fiorilli and M., 2010+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{\phi(q)\log q}\right),$$ *where*

$$V(q;a,b) = 2 \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(b)-\chi(a)|^2 \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14+\gamma^2}$$

$$= 2\phi(q)\left(\log q - \sum_{p\mid q}\frac{\log p}{p-1} - (\gamma_0+\log 2\pi) + R_q(a-b)\right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p|q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- γ_0 = Euler's constant: $\lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n} - \log x \right) \approx 0.577216$

There are three terms in this formula for the variance $V(q;a,b)$ that depend on a and b. Whenever any of the three is bigger than normal, the variance increases, causing the density $\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{\phi(q)\log q}\right)$ to decrease.

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p\mid q}\frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b)\right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- γ_0 = Euler's constant: $\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right) \approx 0.577216$

There are three terms in this formula for the variance $V(q;a,b)$ that depend on a and b. Whenever any of the three is bigger than normal, the variance increases, causing the density $\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{\phi(q)\log q}\right)$ to decrease.

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p\mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- $\gamma_0 =$ Euler's constant: $\lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n} - \log x \right) \approx 0.577216$

There are three terms in this formula for the variance $V(q;a,b)$ that depend on a and b. Whenever any of the three is bigger than normal, the variance increases, causing the density $\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{\phi(q)\log q}\right)$ to decrease.

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p\mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- $\gamma_0 =$ Euler's constant: $\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right) \approx 0.577216$

There are three terms in this formula for the variance $V(q;a,b)$ that depend on a and b. Whenever any of the three is bigger than normal, the variance increases, causing the density $\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{\phi(q)\log q}\right)$ to decrease.

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\bigg(\log q - \sum_{p\mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \bigg) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- $R_q(n) = \dfrac{\Lambda(q/(q,n))}{\phi(q/(q,n))}$
- $\iota_q(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod q, \\ 0, & \text{if } n \not\equiv 1 \pmod q \end{cases}$
- If χ^* is the primitive character that induces χ, then

$$M(q;a,b) = \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$$

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p \mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- $R_q(n) = \dfrac{\Lambda(q/(q,n))}{\phi(q/(q,n))}$
- $\iota_q(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{q}, \\ 0, & \text{if } n \not\equiv 1 \pmod{q} \end{cases}$
- If χ^* is the primitive character that induces χ, then

$$M(q;a,b) = \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$$

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p\mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- $R_q(n) = \dfrac{\Lambda(q/(q,n))}{\phi(q/(q,n))}$
- $\iota_q(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{q}, \\ 0, & \text{if } n \not\equiv 1 \pmod{q} \end{cases}$
- If χ^* is the primitive character that induces χ, then

$$M(q;a,b) = \sum_{\substack{\chi \pmod{q} \\ \chi \ne \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$$

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p\mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- $R_q(n) = \dfrac{\Lambda(q/(q,n))}{\phi(q/(q,n))}$
- $\iota_q(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{q}, \\ 0, & \text{if } n \not\equiv 1 \pmod{q} \end{cases}$
- If χ^* is the primitive character that induces χ, then

$$M(q;a,b) = \sum_{\substack{\chi \pmod{q} \\ \chi \ne \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$$

The effect of R_q

$$R_q(a-b) = \frac{\Lambda(q/(q,a-b))}{\phi(q/(q,a-b))}$$

provides extra variance (reducing the corresponding density $\delta_{q;a,b}$) if a is congruent to b modulo suitable large divisors of q.

a	$\delta_{24;a,1}$	$24/(24,a-1)$	$R_{24}(a-1)$
5	0.999987	6	0
11	0.999983	12	0
23	0.999889	12	0
7	0.999833	4	$(\log 2)/2$
19	0.999719	4	$(\log 2)/2$
17	0.999125	3	$(\log 3)/2$
13	0.998722	2	$\log 2$

The effect of R_q

$$R_q(a-b) = \frac{\Lambda(q/(q,a-b))}{\phi(q/(q,a-b))}$$

provides extra variance (reducing the corresponding density $\delta_{q;a,b}$) if a is congruent to b modulo suitable large divisors of q.

a	$\delta_{24;a,1}$	$24/(24,a-1)$	$R_{24}(a-1)$
5	0.999987	6	0
11	0.999983	12	0
23	0.999889	12	0
7	0.999833	4	$(\log 2)/2$
19	0.999719	4	$(\log 2)/2$
17	0.999125	3	$(\log 3)/2$
13	0.998722	2	$\log 2$

The effect of R_q

$$R_q(a-b) = \frac{\Lambda(q/(q,a-b))}{\phi(q/(q,a-b))}$$

provides extra variance (reducing the corresponding density $\delta_{q;a,b}$) if a is congruent to b modulo suitable large divisors of q.

a	$\delta_{24;a,1}$	$24/(24,a-1)$	$R_{24}(a-1)$
5	0.999987	6	0
11	0.999983	12	0
23	0.999889	12	0
7	0.999833	4	$(\log 2)/2$
19	0.999719	4	$(\log 2)/2$
17	0.999125	3	$(\log 3)/2$
13	0.998722	2	$\log 2$

The effect of ι_q and M

- $\iota_q(-ab^{-1}) = \begin{cases} 1, & \text{if } -ab^{-1} \equiv 1 \pmod{q}, \\ 0, & \text{if } -ab^{-1} \not\equiv 1 \pmod{q} \end{cases}$
- $M(q;a,b) = \sum_{\substack{\chi \pmod{q} \\ \chi \ne \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$

- $\iota_q(-ab^{-1})$ provides extra variance exactly when $a \equiv -b \pmod{q}$.
- It can be shown that $M(q;a,b)$ tends to provide extra variance when there are small prime powers congruent to ab^{-1} or ba^{-1} modulo q. (Note: it's a bit more complicated to state when q is not an odd prime power.)

The effect of ι_q and M

- $$\iota_q(-ab^{-1}) = \begin{cases} 1, & \text{if } -ab^{-1} \equiv 1 \pmod q, \\ 0, & \text{if } -ab^{-1} \not\equiv 1 \pmod q \end{cases}$$
- $$M(q;a,b) = \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$$

- $\iota_q(-ab^{-1})$ provides extra variance exactly when $a \equiv -b \pmod q$.
- It can be shown that $M(q;a,b)$ tends to provide extra variance when there are small prime powers congruent to ab^{-1} or ba^{-1} modulo q. (Note: it's a bit more complicated to state when q is not an odd prime power.)

The effect of ι_q and M

- $\iota_q(-ab^{-1}) = \begin{cases} 1, & \text{if } -ab^{-1} \equiv 1 \pmod q, \\ 0, & \text{if } -ab^{-1} \not\equiv 1 \pmod q \end{cases}$
- $M(q;a,b) = \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$

- $\iota_q(-ab^{-1})$ provides extra variance exactly when $a \equiv -b \pmod q$.
- It can be shown that $M(q;a,b)$ tends to provide extra variance when there are small prime powers congruent to ab^{-1} or ba^{-1} modulo q. (Note: it's a bit more complicated to state when q is not an odd prime power.)

The effect of ι_q and M

- $\iota_q(-a)$ provides extra variance (hence decreases $\delta_{q;a,1}$) exactly when $a \equiv -1 \pmod q$.
- It can be shown that $M(q;a,1)$ tends to provide extra variance (hence decreases $\delta_{q;a,1}$) when there are small prime powers congruent to a or a^{-1} modulo q.

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

The effect of ι_q and M

- $\iota_q(-a)$ provides extra variance (hence decreases $\delta_{q;a,1}$) exactly when $a \equiv -1 \pmod{q}$.
- It can be shown that $M(q; a, 1)$ tends to provide extra variance (hence decreases $\delta_{q;a,1}$) when there are small prime powers congruent to a or a^{-1} modulo q.

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

The effect of ι_q and M

- $\iota_q(-a)$ provides extra variance (hence decreases $\delta_{q;a,1}$) exactly when $a \equiv -1 \pmod{q}$.
- It can be shown that $M(q;a,1)$ tends to provide extra variance (hence decreases $\delta_{q;a,1}$) when there are small prime powers congruent to a or a^{-1} modulo q.

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

The effect of ι_q and M

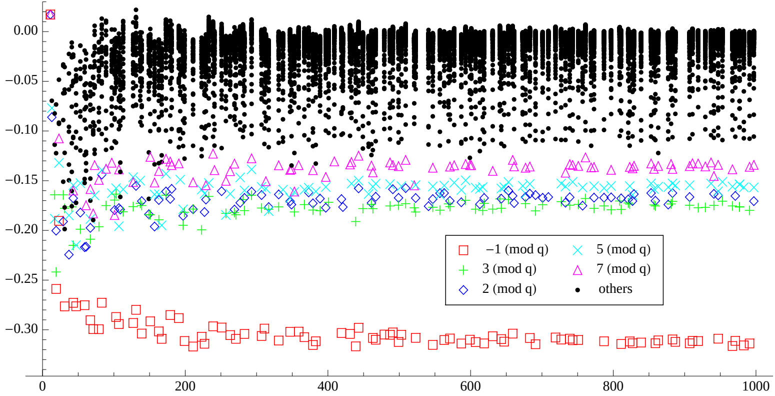
- $\iota_q(-a)$ provides extra variance (hence decreases $\delta_{q;a,1}$) exactly when $a \equiv -1 \pmod q$.
- It can be shown that $M(q; a, 1)$ tends to provide extra variance (hence decreases $\delta_{q;a,1}$) when there are small prime powers congruent to a or a^{-1} modulo q.

Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

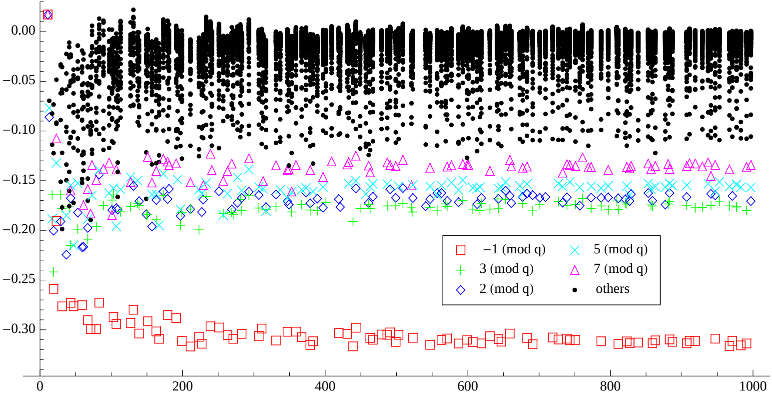
Graph of normalized densities

Figure: Densities $\delta_{q;a,1}$ for primes q, after a normalization to display them at the same scale



Graph of normalized densities

Figure: Densities $\delta_{q;a,1}$ for primes q, after a normalization to display them at the same scale



Note: $\frac{\Lambda(3)}{3} > \frac{\Lambda(2)}{2} > \frac{\Lambda(5)}{5} > \frac{\Lambda(7)}{7} > \cdots$

Top Ten List

Top 10 Most Unfair Races

Modulus q	Winner a	Loser b	Proportion $\delta_{q;a,b}$
24	5	1	99.9987%
24	11	1	99.9982%
12	11	1	99.9976%
24	23	1	99.9888%
24	7	1	99.9833%
24	19	1	99.9718%
8	3	1	99.9568%
12	5	1	99.9206%
24	17	1	99.9124%
3	2	1	99.9064%

There are 117 distinct densities greater than $9/10$ (the last one is $\delta_{56;37,1} = 0.900863$).

Top Ten List

Top 10 Most Unfair Races

Modulus q	Winner a	Loser b	Proportion $\delta_{q;a,b}$
24	5	1	99.9987%
24	11	1	99.9982%
12	11	1	99.9976%
24	23	1	99.9888%
24	7	1	99.9833%
24	19	1	99.9718%
8	3	1	99.9568%
12	5	1	99.9206%
24	17	1	99.9124%
3	2	1	99.9064%

There are 117 distinct densities greater than $9/10$ (the last one is $\delta_{56;37,1} = 0.900863$).

Top Ten List

Top 10 Most Unfair Races

Modulus q	Winner a	Loser b	Proportion $\delta_{q;a,b}$
24	5	1	99.9987%
24	11	1	99.9982%
12	11	1	99.9976%
24	23	1	99.9888%
24	7	1	99.9833%
24	19	1	99.9718%
8	3	1	99.9568%
12	5	1	99.9206%
24	17	1	99.9124%
3	2	1	99.9064%

There are 117 distinct densities greater than $9/10$ (the last one is $\delta_{56;37,1} = 0.900863$).

Races with more than two contestants

Definition

Define $\delta_{q;a_1,\dots,a_r}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying

$$\pi(x;q,a_1) > \pi(x;q,a_2) > \cdots > \pi(x;q,a_r). \qquad (\star)$$

Under GRH and LI, Rubinstein and Sarnak proved:

- $\delta_{q;a_1,\dots,a_r}$ always exists and is strictly between 0 and 1
- $\delta_{q;a_1,\dots,a_r}$ tends to $\frac{1}{r!}$ as q tends to infinity, uniformly for all distinct residue classes $a_1, \dots, a_r$.

Races with more than two contestants

Definition

Define $\delta_{q;a_1,\dots,a_r}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying

$$\pi(x;q,a_1) > \pi(x;q,a_2) > \cdots > \pi(x;q,a_r). \qquad (\star)$$

Under GRH and LI, Rubinstein and Sarnak proved:

- $\delta_{q;a_1,\dots,a_r}$ always exists and is strictly between 0 and 1
- $\delta_{q;a_1,\dots,a_r}$ tends to $\frac{1}{r!}$ as q tends to infinity, uniformly for all distinct residue classes $a_1,\dots,a_r$.

Races with more than two contestants

Definition

Define $\delta_{q;a_1,\dots,a_r}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying

$$\pi(x;q,a_1) > \pi(x;q,a_2) > \cdots > \pi(x;q,a_r). \qquad (\star)$$

Definition

The prime number race among r distinct residue classes $A \pmod q$ is *inclusive* if, for any permutation $(a_1,\dots,a_r)$ of A, the simultaneous inequalities $(\star)$ hold for a set of positive logarithmic density.

In particular, for inclusive prime number races, the inequalities $(\star)$ will hold on an unbounded set of real numbers x for any permutation of $(a_1,\dots,a_r)$.

Races with more than two contestants

Definition

Define $\delta_{q;a_1,\dots,a_r}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying

$$\pi(x;q,a_1) > \pi(x;q,a_2) > \cdots > \pi(x;q,a_r). \qquad (\star)$$

Definition

The prime number race among r distinct residue classes $A \pmod{q}$ is *inclusive* if, for any permutation $(a_1,\dots,a_r)$ of A, the simultaneous inequalities $(\star)$ hold for a set of positive logarithmic density.

In particular, for inclusive prime number races, the inequalities $(\star)$ will hold on an unbounded set of real numbers x for any permutation of $(a_1,\dots,a_r)$.

Weakening the linear independence assumption

Rubinstein and Sarnak: assuming GRH and LI, every prime number race is inclusive.

Theorem (M. and Ng, 2010+)

Assume GRH. Every prime number race $\pmod q$ is inclusive if every nonprincipal Dirichlet L-function $\pmod q$ has $\gg T/\log T$ zeros up to height T (as $T \to \infty$) that are not involved in any linear relations among the zeros.

- We only need $\sum \frac{1}{\gamma}$, summed over such zeros for a given L-function, to diverge.
- We only need this for "enough" characters χ_j to distinguish the contestants $a_1, \dots, a_r$, that is, $\mathbb{C}^r$ must be spanned by

$$\big\{\big(\chi_j(a_1), \dots, \chi_j(a_r)\big)\big\} \cup (1, \dots, 1).$$

Weakening the linear independence assumption

Rubinstein and Sarnak: assuming GRH and LI, every prime number race is inclusive.

Theorem (M. and Ng, 2010+)

Assume GRH. Every prime number race $\pmod q$ is inclusive if every nonprincipal Dirichlet L-function $\pmod q$ has $\gg T/\log T$ zeros up to height T (as $T \to \infty$) that are not involved in any linear relations among the zeros.

- We only need $\sum \frac{1}{\gamma}$, summed over such zeros for a given L-function, to diverge.
- We only need this for "enough" characters χ_j to distinguish the contestants $a_1, \dots, a_r$, that is, $\mathbb{C}^r$ must be spanned by

$$\{(\chi_j(a_1), \dots, \chi_j(a_r))\} \cup (1, \dots, 1).$$

Weakening the linear independence assumption

Rubinstein and Sarnak: assuming GRH and LI, every prime number race is inclusive.

Theorem (M. and Ng, 2010+)

Assume GRH. Every prime number race $\pmod q$ is inclusive if every nonprincipal Dirichlet L-function $\pmod q$ has $\gg T/\log T$ zeros up to height T (as $T \to \infty$) that are not involved in any linear relations among the zeros.

- We only need $\sum \frac{1}{\gamma}$, summed over such zeros for a given L-function, to diverge.
- We only need this for "enough" characters χ_j to distinguish the contestants $a_1, \dots, a_r$, that is, $\mathbb{C}^r$ must be spanned by

$$\big\{ \big(\chi_j(a_1), \dots, \chi_j(a_r) \big) \big\} \cup (1, \dots, 1).$$

Weakening the linear independence assumption

Rubinstein and Sarnak: assuming GRH and LI, every prime number race is inclusive.

Theorem (M. and Ng, 2010+)

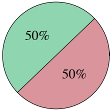
Assume GRH. Every prime number race $\pmod q$ is inclusive if every nonprincipal Dirichlet L-function $\pmod q$ has $\gg T/\log T$ zeros up to height T (as $T \to \infty$) that are not involved in any linear relations among the zeros.

- We only need $\sum \frac{1}{\gamma}$, summed over such zeros for a given L-function, to diverge.
- We only need this for "enough" characters χ_j to distinguish the contestants $a_1, \dots, a_r$, that is, $\mathbb{C}^r$ must be spanned by

$$\{(\chi_j(a_1), \dots, \chi_j(a_r))\} \cup (1, \dots, 1).$$

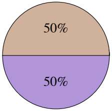
Asymmetries in three-way races

$\pi(x;12,11) > \pi(x;12,7)$



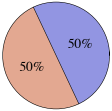
$\pi(x;12,7) > \pi(x;12,11)$

$\pi(x;12,11) > \pi(x;12,5)$



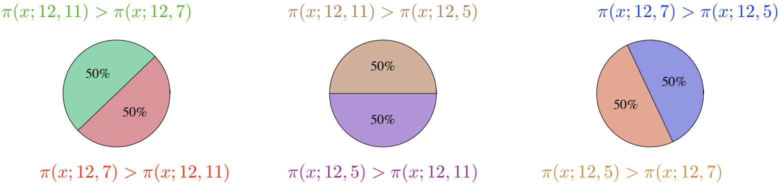
$\pi(x;12,5) > \pi(x;12,11)$

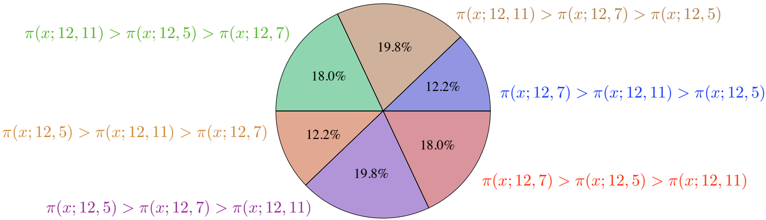
$\pi(x;12,7) > \pi(x;12,5)$



$\pi(x;12,5) > \pi(x;12,7)$

Asymmetries in three-way races





The end

The survey article *Prime number races*, with Andrew Granville

www.math.ubc.ca/∼gerg/index.shtml?abstract=PNR

My paper with Daniel (and pointers to other papers)

www.math.ubc.ca/∼gerg/
index.shtml?abstract=ISRPNRAFD

These slides

www.math.ubc.ca/∼gerg/index.shtml?slides