Inequities in the Shanks-Rényi prime number race

Greg Martin
University of British Columbia

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Outline

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement:

> *There is a notable difference in the splitting of the prime numbers between the two forms $4n+3$, $4n+1$: the first form contains a lot more than the second.*

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Who has the advantage?

Races where such advantages are observed:

- Primes that are $2 \pmod 3$ over primes that are $1 \pmod 3$
- Primes that are $3 \pmod 4$ over primes that are $1 \pmod 4$
- Primes that are 2 or $3 \pmod 5$ over primes that are 1 or $4 \pmod 5$
- Primes that are 3, 5, or $6 \pmod 7$ over primes that are 1, 2, or $4 \pmod 7$
- Primes that are 3, 5, or $7 \pmod 8$ over primes that are $1 \pmod 8$

The general pattern

Primes that are nonsquares $\pmod q$ over primes that are squares $\pmod q$

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Past results: computational

Notation

$\pi(x;q,a) = \{\text{number of primes } p \le x \text{ such that } p \equiv a \pmod q\}$

Further computation (1950s and beyond) reveals that there are occasional periods of triumph for the square residue classes over nonsquare residue classes:

- $\pi(x;4,1) > \pi(x;4,3)$ for the first time at $x = 26{,}861$, but $\pi(x;4,3) = \pi(x;4,1)$ again at $x = 26{,}863$; then $\pi(x;4,1) > \pi(x;4,3)$ for the second time at $x = 616{,}481$
- $\pi(x;8,1) > \pi(x;8,5)$ for the first time at $x = 588{,}067{,}889$ —although $\pi(x;8,1)$ still lags behind $\pi(x;8,3)$ and $\pi(x;8,7)$
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And theoretical results as well:

- The prime number theorem for arithmetic progressions (1900 + $O(1)$): $\pi(x;q,a) \sim \pi(x;q,b)$
- Littlewood (1910s): each of $\pi(x;4,1)$ and $\pi(x;4,3)$ is ahead of the other for arbitrarily large x, and similarly for $\pi(x;3,1)$ and $\pi(x;3,2)$
- Turán and Knapowski (1960s): for many pairs a,b of residue classes, $\pi(x;q,a)$ is ahead of $\pi(x;q,b)$ for arbitrarily large x. However, assumptions on the locations of zeros of Dirichlet L-functions are necessary.
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Defining delta

The central question

How often is $\pi(x;q,a)$ ahead of $\pi(x;q,b)$?

Definition

Define $\delta_{q;a,b}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying $\pi(x;q,a) > \pi(x;q,b)$. More explicity,

$$\delta_{q;a,b} = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1\le x\le T \\ \pi(x;q,a)>\pi(x;q,b)}} \frac{dx}{x} \right).$$

$\delta_{q;a,b}$ is the limiting "probability" that when a "random" real number x is chosen, there are more primes that are congruent to $a \pmod q$ up to x then there are congruent to $b \pmod q$.

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Two hypotheses

Rubinstein and Sarnak (1994) investigated these densities $\delta_{q;a,b}$ under the following two hypotheses:

- The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet L-functions have real part equal to $\frac{1}{2}$
- A linear independence hypothesis (LI): the nonnegative imaginary parts of these nontrivial zeros are linearly independent over the rationals

- Recent work of Ford and Konyagin shows that certain hypothetical violations of GRH do actually lead to pathological behavior in prime number races.
- LI is somewhat analogous to a “nonsingularity” hypothesis: with precise information about any linear dependences that might exist, we could probably still work out the answer. . . .

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Rubinstein and Sarnak's results

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Under these two hypotheses GRH and LI, Rubinstein and Sarnak proved (1994):

- $\delta_{q;a,b}$ always exists and is strictly between 0 and 1
- $\delta_{q;a,b} + \delta_{q;b,a} = 1$
- "Chebyshev's bias": $\delta_{q;a,b} > \frac{1}{2}$ if and only if a is a nonsquare $\pmod q$ and b is a square $\pmod q$
- if a and b are distinct squares $\pmod q$ or distinct nonsquares $\pmod q$, then $\delta_{q;a,b} = \delta_{q;b,a} = \frac{1}{2}$
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Comparisons of the densities $\delta_{q;a,b}$

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Feuerverger and M. (2000) generalized Rubinstein and Sarnak's approach in several directions.

We calculated (assuming, as usual, GRH and LI) many examples of the densities $\delta_{q;a,b}$.

- The calculations required numerical evaluation of complicated integrals, which involved many explicitly computed zeros of Dirichlet L-functions.
- One significant discovery is that even with q fixed, the values of $\delta_{q;a,b}$ vary significantly as a and b vary over squares and nonsquares $\pmod q$.

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Example: races modulo 24

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Example: races modulo 24

a	$\delta_{24;a,1}$
5	0.999987
11	0.999983
23	0.999889
7	0.999833
19	0.999719
17	0.999125
13	0.998722

Note that 1 is the only square $\pmod{24}$. The density in any race between two non-squares, such as $\delta_{24;5,7}$, equals $\frac{1}{2}$.

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a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

- $\delta_{q;a,b} = \delta_{q;ab^{-1},1}$ for any square $b \pmod{q}$. Thus it suffices to calculate only the values of $\delta_{q;a,1}$ for nonsquares $a \pmod{q}$.
- $\delta_{q;a,1} = \delta_{q;a^{-1},1}$ for any $a \pmod{q}$.

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Example: races modulo 43

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a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
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Current goals

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

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- A more precise understanding of the sizes of $\delta_{q;a,b}$. Recalling that $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, for example, we would like an asymptotic formula for $\delta_{q;a,b} - \frac{1}{2}$.
- A way to decide which $\delta_{q;a,b}$ are likely to be larger than others as a and b vary (with q fixed), based on elementary criteria rather than laborious numerical calculation.

These goals are the subject of *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, which is joint work in progress with Daniel Fiorilli (Montréal).

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Our approach

A central limit theorem for differences of prime counts

- For given q, a, b, the normalized difference

$$\Delta_{q;a,b}(x) = \left(\frac{\phi(q)}{2\log q}\right)^{1/2} \frac{\pi(x;q,a) - \pi(x;q,b)}{x^{1/2}(\log x)^{-1}}$$

has a limiting distribution function

$$F_{q;a,b}(u) = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1\le x\le T \\ \Delta_{q;a,b}(x)<u}} \frac{dx}{x}\right).$$

- As $q \to \infty$, these distribution functions converge to the standard normal distribution with variance 1.

- A quantitative formula for the rate of convergence to the normal distribution allows us to asymptotically evaluate the densities $\delta_{q;a,b}$ in terms of the variance of $F_{q;a,b}$.

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Asymptotic formula, version I

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2009)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi}\,(\phi(q)\log q)^{1/2}} + O\left(\frac{\rho(q)\log\log q}{\phi(q)^{1/2}(\log q)^{3/2}}\right).$$

In particular, $\delta_{q;a,b} = \frac{1}{2} + O_\varepsilon\big(q^{-1/2+\varepsilon}\big)$ for any $\varepsilon > 0$.

$\rho(q)$ = the number of square roots of 1 $\pmod q$

$= 2^{\#\text{number of odd prime factors of } q} \times \{1, 2, \text{or } 4\}$

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Asymptotic formula, version II

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

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Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

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where

$$V(q;a,b) = 2 \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(b) - \chi(a)|^2 \sum_{\substack{\gamma > 0 \\ L(\frac12 + i\gamma, \chi) = 0}} \frac{1}{\frac14 + \gamma^2}.$$

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Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2\phi(q)\left(\log q - \sum_{p|q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \right) + (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

- γ_0 = Euler's constant: $\lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n} - \log x \right) \approx 0.577216$

There are three terms in this formula for the variance $V(q;a,b)$ that depend on a and b. Whenever any of the three is bigger than normal, the variance increases, causing the density $\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{\phi(q)\log q}\right)$ to decrease.

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- $R_q(n) = \dfrac{\Lambda(q/(q,n))}{\phi(q/(q,n))}$
- $\iota_q(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{q}, \\ 0, & \text{if } n \not\equiv 1 \pmod{q} \end{cases}$
- If χ^* is the primitive character that induces χ, then

$$M(q;a,b) = \sum_{\substack{\chi \pmod{q} \\ \chi \ne \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$$

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The effect of R_q

$$R_q(a-b) = \frac{\Lambda(q/(q,a-b))}{\phi(q/(q,a-b))}$$

provides extra variance (reducing the corresponding density $\delta_{q;a,b}$) if a is congruent to b modulo suitable large divisors of q.

a	$\delta_{24;a,1}$	$24/(24,a-1)$	$R_{24}(a-1)$
5	0.999987	6	0
11	0.999983	12	0
23	0.999889	12	0
7	0.999833	4	$(\log 2)/2$
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The effect of ι_q and M

- $\iota_q(-ab^{-1}) = \begin{cases} 1, & \text{if } -ab^{-1} \equiv 1 \pmod{q}, \\ 0, & \text{if } -ab^{-1} \not\equiv 1 \pmod{q} \end{cases}$
- $M(q;a,b) = \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} |\chi(a) - \chi(b)|^2 \frac{L'(1,\chi^*)}{L(1,\chi^*)}$

- $\iota_q(-ab^{-1})$ provides extra variance exactly when $a \equiv -b \pmod{q}$.
- It can be shown that $M(q;a,b)$ tends to provide extra variance when there are small prime powers congruent to ab^{-1} or ba^{-1} modulo q. (Note: it's a bit more complicated to state when q is not an odd prime power.)

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- $\iota_q(-a)$ provides extra variance (hence decreases $\delta_{q;a,1}$) exactly when $a \equiv -1 \pmod{q}$.
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Example: races modulo 43

a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$	a	$a^{-1} \pmod{43}$	$\delta_{43;a,1}$
32	39	0.5743	5	26	0.5672
30	33	0.5742	7	37	0.5670
12	18	0.5729	2	22	0.5663
20	28	0.5728	3	29	0.5639
19	34	0.5700	42	42	0.5607
8	27	0.5700			

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Top Ten List

Top 10 Most Unfair Races

Modulus q	Winner a	Loser b	Proportion $\delta_{q;a,b}$
24	5	1	99.9987%
24	11	1	99.9982%
12	11	1	99.9976%
24	23	1	99.9888%
24	7	1	99.9833%
24	19	1	99.9718%
8	3	1	99.9568%
12	5	1	99.9206%
24	17	1	99.9124%
3	2	1	99.9064%

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The end

The survey article *Prime number races*, with Andrew Granville

www.math.ubc.ca/~gerg/index.shtml?abstract=PNR

My research on prime number races

www.math.ubc.ca/~gerg/index.shtml?abstract=ISRPNR

These slides

www.math.ubc.ca/~gerg/index.shtml?slides