Friable values of polynomials

How often do the values of a polynomial
have only small prime factors?

Greg Martin
University of British Columbia

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notes to be placed on web page:
www.math.ubc.ca/∼gerg/talks.html

Outline

1. Introduction
2. Bounds for friable values of polynomials
3. Conjecture for prime values of polynomials
4. Conjecture for friable values of polynomials

Friable integers

Definition

$\Psi(x, y)$ is the number of integers up to x whose prime factors are all at most y:

$$\Psi(x, y) = \#\{n \leq x : p \mid n \implies p \leq y\}$$

Asymptotics: For a large range of x and y,

$$\Psi(x, y) \sim x\rho\left(\frac{\log x}{\log y}\right),$$

where $\rho(u)$ is the "Dickman–de Bruijn rho-function".

Interpretation: A "randomly chosen" integer of size X has probability $\rho(u)$ of being $X^{1/u}$-friable.

In this talk: Think of $u = \log x / \log y$ as being bounded above, that is, $y \geq x^{\varepsilon}$ for some $\varepsilon > 0$.

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The Dickman–de Bruijn ρ-function

Definition

$\rho(u)$ is the unique continuous solution of the differential-difference equation $u\rho'(u) = -\rho(u-1)$ for $u \geq 1$ that satisfies the initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$.

Example

For $1 \leq u \leq 2$,

$$\rho'(u) = -\frac{\rho(u-1)}{u} = -\frac{1}{u} \implies \rho(u) = C - \log u.$$

Since $\rho(u) = 1$, we have $\rho(u) = 1 - \log u$ for $1 \leq u \leq 2$.

Consequence: Note that $\rho(u) = \frac{1}{2}$ when $u = \sqrt{e}$. Therefore the "median size" of the largest prime factor of n is $n^{1/\sqrt{e}}$.

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Friable numbers among values of polynomials

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- When $F(x)$ is a linear polynomial (friable numbers in arithmetic progressions), we have the same asymptotic $\Psi(F;x,y) \sim \rho\left(\frac{\log x}{\log y}\right)$.
- Knowing the size of $\Psi(F;x,y)$ has applications to analyzing the running time of modern factoring algorithms (quadratic sieve, number field sieve).
- A basic sort of question in number theory: are two arithmetic properties (in this case, friability and being the value of a polynomial) independent?

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How friable can values of special polynomials be?

- For binomials, there's a nice trick which yields:

Theorem (Schinzel, 1967)

For any nonzero integers A and B, any positive integer d, and any $\varepsilon > 0$, there are infinitely many numbers n for which $An^d + B$ is n^ε-friable.

- Balog and Wooley (1998), building on an idea of Eggleton and Selfridge, extended this result to products of binomials

$$\prod_{j=1}^{L}(A_j n^{d_j} + B_j).$$

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Proof for an explicit binomial

Example

For any $\varepsilon > 0$, there are infinitely many numbers n for which $F(n) = 3n^5 + 7$ is n^ε-friable.

Define $n_k = 3^{8k-1}7^{2k}$. Then

$$F(n_k) = 3^{5(8k-1)+1}7^{5(2k)} + 7 = -7\big((-3^4 7)^{10k-1} - 1\big)$$

factors into values of cyclotomic polynomials:

$$F(n_k) = -7 \prod_{m|(10k-1)} \Phi_m(-3^4 7).$$

- $\Phi_m(x) = \prod_{\substack{1\le r\le m \\ (r,m)=1}} (x - e^{2\pi i r/m})$
- Φ_m has integer coefficients and degree $\phi(m)$

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From the last slide

- $F(n) = 3n^5 + 7$
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- primes dividing $F(n_k)$ are $\le \max_{m|(10k-1)} |\Phi_m(-3^4 7)|$
- $\Phi_m(x)$ is roughly $x^{\phi(m)} \le x^{\phi(10k-1)}$
- n_k is roughly $(3^4 7)^{4k}$, but the largest prime factor of $F(n_k)$ is bounded by roughly $(3^4 7)^{\phi(10k-1)}$
- infinitely many k with $\phi(10k-1)/4k < \varepsilon$

How many such friable values? $\gg_{F,\varepsilon} \log x$, for $n \le x$

ε can be made quantitative $n^{c_F / \log\log\log n}$-friable values

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Polynomial factorizations

Example

The polynomial $F(x + F(x))$ is always divisible by $F(x)$. In particular, if $\deg F = d$, then $F(x + F(x))$ is roughly x^{d^2} yet is automatically roughly x^{d^2-d}-friable.

Mnemonic

$$x + F(x) \equiv x \pmod{F(x)}$$

Special case:

- If $F(x)$ is quadratic with lead coefficient a, then

$$F(x + F(x)) = F(x) \cdot aF\left(x + \tfrac{1}{a}\right).$$

- In particular, if $F(x) = x^2 + bx + c$, then

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A refinement of Schinzel

- Idea: use the reciprocal polynomial $x^d F(1/x)$.
- Restrict to $F(x) = x^d + a_2 x^{d-2} + \dots$ for simplicity.

Proposition

Let $h(x)$ be a polynomial such that $xh(x) - 1$ is divisible by $x^d F(1/x)$. Then $F(h(x))$ is divisible by $x^d F(1/x)$. In particular, we can take $\deg h = d - 1$, in which case $F(h(x))$ is roughly x^{d^2-d} yet is automatically roughly x^{d^2-2d}-friable.

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Recursively use Schinzel's construction

D_m: an unspecified polynomial of degree m

Example

$\deg F(x) = 4$. Use Schinzel's construction repeatedly:

$D_{12} = F(D_3) = D_4 D_8$ "score" = 8/3

$D_{84} = F(D_{21}) = D_{28} D_8 D_{48}$ "score" = 16/7

$D_{3984} = F(D_{987}) = D_{1316} D_{376} D_{48} D_{2208}$ "score" = 736/329

- For $\deg F = 2$, begin with $F(D_4) = D_2 D_2 D_4$. Specifically,

$$F(x + F(x) + F(x + F(x))) = F(x) \cdot aF\left(x + \tfrac{1}{a}\right) \cdot D_4.$$

- For $\deg F = 3$, begin with $F(D_4) = D_3 D_3 D_6$.

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How friable can values of general polynomials be?

- $d \ge 4$: define $s(d) = d \prod_{j=1}^{\infty} \left(1 - \frac{1}{u_j(d)}\right)$, where $u_1(d) = d - 1$ and $u_{j+1}(d) = u_j(d)^2 - 2$
- $s(2) = s(4)/4$ and $s(3) = s(6)/4$

Theorem

(Schinzel, 1967) Given a polynomial $F(x)$ of degree $d \ge 2$, there are infinitely many numbers n for which $F(n)$ is $n^{s(d)}$-friable.

$F(n)$	can be $n^{?}$-friable	$F(n)$	can be $n^{?}$-friable
degree 1	ε	degree 5	3.46410
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Polynomial substitution yields small lower bounds

Special case

Given a quadratic polynomial $F(x)$, there are infinitely many numbers n for which $F(n)$ is $n^{0.55902}$-friable.

Example

To obtain n for which $F(n)$ is $n^{0.56}$-friable:

$$D_{168} = F(D_{84}) = D_{42} D_{42} D_{28} D_8 D_{48} \qquad \text{“score”} = 4/7 > 0.56$$

$$D_{7896} = F(D_{3948}) = D_{1974} D_{1974} D_{1316} D_{376} D_{48} D_{2208} \qquad \text{“score”} = 92/329 < 0.56$$

The counting function of such n is about $x^{1/3948}$.

“Improvement” Balog, M., Wooley can get $x^{2/3948}$ and an analogous improvement for $\deg F = 3$.

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$$D_{168} = F(D_{84}) = D_{42} D_{42} D_{28} D_8 D_{48} \qquad \text{“score”} = 4/7 > 0.56$$

$$D_{7896} = F(D_{3948}) = D_{1974} D_{1974} D_{1316} D_{376} D_{48} D_{2208} \qquad \text{“score”} = 92/329 < 0.56$$

The counting function of such n is about $x^{1/3948}$.

“Improvement” Balog, M., Wooley can get $x^{2/3948}$ and an analogous improvement for $\deg F = 3$.

Polynomial substitution yields small lower bounds

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Given a quadratic polynomial $F(x)$, there are infinitely many numbers n for which $F(n)$ is $n^{0.55902}$-friable.

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Can we have lots of friable values?

Our expectation

For any $\varepsilon > 0$, a positive proportion of values $F(n)$ are n^ε-friable.

We know this for:

- linear polynomials (arithmetic progressions)
- Hildebrand, then Balog and Ruzsa: $F(n) = n(an + b)$, values n^ε-friable for any $\varepsilon > 0$
- Hildebrand: $F(n) = (n + 1) \cdots (n + L)$, values n^β-friable for any $\beta > e^{-1/(L-1)}$

  Note: $\rho(e^{-1/L}) = 1 - \frac{1}{L}$, so $\beta > e^{-1/L}$ is trivial
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Theorem (Dartyge, M., Tenenbaum, 2001)

Let $F(x)$ be any polynomial, let d be the highest degree of any irreducible factor of F, and let F have exactly K distinct irreducible factors of degree d. Then for any $\varepsilon > 0$, a positive proportion of values $F(n)$ are $n^{d-1/K+\varepsilon}$-friable.

Remark: for friability of level n^{d-1} or higher, only irreducible factors of degree $\geq d$ matter

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2. Bounds for friable values of polynomials
3. Conjecture for prime values of polynomials
   - Schinzel's "Hypothesis H" (Bateman–Horn conjecture)
   - A uniform version of Hypothesis H
4. Conjecture for friable values of polynomials

Schinzel's "Hypothesis H" (Bateman–Horn conjecture)

Definition

$\pi(F;x) = \#\{n \le x :$
$f(n)$ is prime for each irreducible factor f of $F\}$

Conjecture: $\pi(F;x)$ is asymptotic to $H(F)\,\mathrm{li}(F;x)$, where:

- $$\mathrm{li}(F;x) = \int\limits_{\substack{0<t<x \\ \min\{|F_1(t)|,\dots,|F_L(t)|\}\ge 2}} \frac{dt}{\log|F_1(t)|\dots\log|F_L(t)|}.$$
- $$H(F) = \prod_p \left(1-\frac{1}{p}\right)^{-L}\left(1-\frac{\sigma(F;p)}{p}\right).$$

L: the number of distinct irreducible factors of F

$\sigma(F;n)$: the number of solutions of $F(a) \equiv 0 \pmod{n}$

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A uniform version of Hypothesis H

Hypothesis UH

$$\pi(F;t) - H(F)\,\mathrm{li}(F;t) \ll_{d,B} 1 + \frac{H(F)t}{(\log t)^{L+1}}$$

uniformly for all polynomials F of degree d with L distinct irreducible factors, each of which has coefficients bounded by t^B in absolute value.

- $\mathrm{li}(F;t)$ is asymptotic to $\dfrac{t}{(\log t)^L}$ for fixed F
- For $d = K = 1$, equivalent to expected number of primes, in an interval of length $y = x^\varepsilon$ near x, in an arithmetic progression to a modulus $q \le y^{1-\varepsilon}$
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4. Conjecture for friable values of polynomials
   - Statement of the conjecture
   - Reduction to convenient polynomials
   - Translation into prime values of polynomials
   - Shepherding the local factors
   - Sums of multiplicative functions

What would we expect on probablistic grounds?

Let $F(x) = f_1(x) \cdots f_L(x)$, where $\deg f_j(x) = d_j$. Let $u > 0$.

- $f_j(n)$ is roughly n^{d_j}, and integers of that size are $n^{1/u}$-friable with probability $\rho(d_j u)$.
- Are the friabilities of the various factors $f_j(n)$ independent? This would lead to a prediction involving

$$x \prod_{j=1}^{L} \rho(d_j u).$$

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Conjecture for friable values of polynomials

Conjecture

Let $F(x)$ be any polynomial, let $f_1, \dots, f_L$ be its distinct irreducible factors, and let $d_1, \dots, d_L$ be their degrees. Then

$$\Psi(F; x, x^{1/u}) = x \prod_{j=1}^{L} \rho(d_j u) + O\left(\frac{x}{\log x}\right)$$

for all $0 < u$.

If F irreducible: $\Psi(F; x, x^{1/u}) = x\rho(du) + O(x/\log x)$ for $0 < u$.

Remark: Rather more controversial than Hypothesis H.

Conjecture for friable values of polynomials

Theorem (M., 2002)

Assume Hypothesis UH. Let $F(x)$ be any polynomial, let $f_1, \dots, f_L$ be its distinct irreducible factors, and let $d_1, \dots, d_L$ be their degrees. Let $d = \max\{d_1, \dots, d_L\}$, and let F have exactly K distinct irreducible factors of degree d. Then

$$\Psi(F; x, x^{1/u}) = x \prod_{j=1}^{L} \rho(d_j u) + O\left(\frac{x}{\log x}\right)$$

for all $0 < u < 1/(d - 1/K)$.

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Trivial: $0 < u < 1/d$.

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Reduction to convenient polynomials

Without loss of generality, we may assume:

1. $F(x)$ is the product of distinct irreducible polynomials $f_1(x), \dots, f_K(x)$, all of the same degree d.
2. $F(x)$ takes at least one nonzero value modulo every prime.
3. No two distinct irreducible factors $f_i(x)$, $f_j(x)$ of $F(x)$ have a common zero modulo any prime.

- (1) is acceptable since the friability level exceeds x^{d-1}.
- (2) is *not* a necessary condition to have friable values (as it is to have prime values). Nevertheless, we can still reduce to this case.
- Both (2) and (3) are achieved by looking at values of $F(x)$ on suitable arithmetic progressions $F(Qx + R)$ separately.

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Under (1), we want to prove that

$$\Psi(F; x, x^{1/u}) = x\rho(du)^K + O\left(\frac{x}{\log x}\right)$$

for all $0 < u < 1/(d - 1/K)$.

Inclusion-exclusion on irreducible factors

Proposition

Let F be a primitive polynomial, and let $F_1, \dots, F_K$ denote the distinct irreducible factors of F. Then for $x \ge y \ge 1$,

$$\Psi(F; x, y) = \lfloor x \rfloor + \sum_{1 \le k \le K} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le K} M(F_{i_1} \dots F_{i_k}; x, y).$$

Definition

$M(f; x, y) = \#\{1 \le n \le x : \text{for each irreducible factor } g \text{ of } f,$ there exists a prime $p > y$ such that $p \mid g(n)\}$.

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If we knew that $M(F_{i_1} \dots F_{i_k}; x, x^{1/u}) \sim x(\log du)^k$, then

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$$= x\left(1 + \sum_{1 \le k \le K} \binom{K}{k} (-\log du)^k\right)$$

$$= x(1 - \log du)^K = x\rho(du)^K.$$

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We want to prove $M(F_{i_1} \dots F_{i_k}; x, x^{1/u}) \sim x(\log du)^k$. To do this, we sort by the values $n_j = F_{i_j}(n)/p_j$, among those n counted by $M(F_{i_1} \dots F_{i_k}; x, x^{1/u})$.

Proposition

For $f = f_1 \dots f_k$ and x and y sufficiently large,

$$M(f;x,y) = \sum_{\substack{n_1 \le \xi_1/y \\ (n_i,n_j)=1\ (1 \le i < j \le k)}} \cdots \sum_{n_k \le \xi_k/y} \sum_{b \in \mathcal{R}(f;n_1,\dots,n_k)} \left(\pi\left(f_{n_1\cdots n_k,b}; \frac{x-b}{n_1 \cdots n_k} \right) - \pi\left(f_{n_1\cdots n_k,b}; \eta_{n_1,\dots,n_k} \right) \right).$$

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DON'T PANIC

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not important

$$\xi_j = f_j(x) \approx x^d$$

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$$\eta_{n_1,\dots,n_k} \approx (y \max\{n_1,\dots,n_k\})^{1/d} (n_1\cdots n_k)^{-1}$$

It's here only because the large primes dividing $f_j(n)$ had to exceed y. (Later we'll take $y = x^{1/u}$.)

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fairly important

$$\mathcal{R}(f;n_1,\dots,n_k) = \{ b \pmod{n_1\cdots n_k} : n_1 \mid f_1(b),\ n_2 \mid f_2(b),\ \dots,\ n_k \mid f_k(b) \}$$

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rather important

$$f_{n_1\cdots n_k,b}(t) = \frac{f(n_1\cdots n_k t + b)}{n_1\cdots n_k} \in \mathbb{Z}[x]$$

In fact, a good understanding of the family $f_{n_1\cdots n_k,b}$ is necessary even to treat error terms. However, we'll only include the details when treating the main term.

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$$f_{n_1\cdots n_k,b}(t) = \frac{f(n_1\cdots n_k t + b)}{n_1\cdots n_k} \in \mathbb{Z}[x]$$

In fact, a good understanding of the family $f_{n_1\cdots n_k,b}$ is necessary even to treat error terms. However, we'll only include the details when treating the main term.

Proposition

For $f = f_1 \dots f_k$ and x and y sufficiently large,

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First: concentrate on

$$\pi\left(f_{n_1\cdots n_k,b}; \frac{x-b}{n_1\cdots n_k}\right) - \pi\left(f_{n_1\cdots n_k,b}; \eta_{n_1,\dots,n_k}\right)$$

Understanding $M(f; x, y)$ inside out

- Look at $\pi\left(f_{n_1\cdots n_k,b}; \frac{x-b}{n_1\cdots n_k}\right) - \pi(f_{n_1\cdots n_k,b}; \eta_{n_1,\dots,n_k})$
- Upper bound sieve (Brun, Selberg):

$$\pi\left(f_{n_1\cdots n_k,b}; \frac{x-b}{n_1\cdots n_k}\right) + O\left(\frac{H(f_{n_1\cdots n_k,b})x/n_1\cdots n_k}{(\log x)^{k+1}}\right)$$

- Main term for $\pi(f; x)$ (we use Hypothesis UH here!):

$$H(f_{n_1\cdots n_k,b})\,\mathrm{li}\left(f_{n_1\cdots n_k,b}; \frac{x-b}{n_1\cdots n_k}\right) + O\left(\frac{H(f_{n_1\cdots n_k,b})x}{n_1\cdots n_k(\log x)^{k+1}}\right)$$

- li is a pretty smooth function:

$$\frac{H(f_{n_1\cdots n_k,b})x/n_1\cdots n_k}{\log(\xi_1/n_1)\cdots\log(\xi_k/n_k)} + O\left(\frac{H(f_{n_1\cdots n_k,b})x}{n_1\cdots n_k(\log x)^{k+1}}\right)$$

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For $f = f_1 \dots f_k$ and x and y sufficiently large,

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$$= \sum_{\substack{n_1 \le \xi_1/y \\ (n_i,n_j)=1\ (1\le i<j\le k)}} \cdots \sum_{n_k \le \xi_k/y} \left(\sum_{b \in \mathcal{R}(f;n_1,\dots,n_k)} H(f_{n_1\cdots n_k,b}) \right) \times \frac{x/n_1\cdots n_k}{\log(\xi_1/n_1)\cdots\log(\xi_k/n_k)} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

- Now we have:

$$\frac{H(f_{n_1\cdots n_k,b})x/n_1\cdots n_k}{\log(\xi_1/n_1)\cdots\log(\xi_k/n_k)} + O\left(\frac{H(f_{n_1\cdots n_k,b})x}{n_1\cdots n_k(\log x)^{k+1}}\right)$$

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Next: concentrate on $\displaystyle\sum_{b \in \mathcal{R}(f;n_1,\dots,n_k)} H(f_{n_1\cdots n_k,b})$

Nice sums over local solutions

Recall

$$H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f;p)}{p}\right)$$

Recall

$$\sigma(f;p) = \{a \pmod p : f(a) \equiv 0 \pmod p\}$$

Proposition

$$\sum_{b \in \mathcal{R}(f;n_1,\dots,n_k)} H(f_{n_1 \cdots n_k, b}) = H(f) g_1(n_1) \cdots g_k(n_k), \text{ where}$$

$$g_j(n_j) = \prod_{p^\nu \| n_j} \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} \left(\sigma(f_j;p^\nu) - \frac{\sigma(f_j;p^{\nu+1})}{p}\right).$$

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Proving this proposition . . .

. . . is fun, actually, involving the Chinese remainder theorem, counting lifts of local solutions (Hensel's lemma), and so on.

Proposition

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$$M(f;x,y) = \sum_{\substack{n_1 \le \xi_1/y \\ (n_i,n_j)=1}} \cdots \sum_{\substack{n_k \le \xi_k/y \\ (1 \le i<j \le k)}} \Bigg(\sum_{b \in \mathcal{R}(f;n_1,\dots,n_k)} H(f_{n_1\cdots n_k,b}) \Bigg) \times \frac{x/n_1\cdots n_k}{\log(\xi_1/n_1)\cdots\log(\xi_k/n_k)} \bigg(1 + O\Big(\frac{1}{\log x}\Big)\bigg)$$

$$= xH(f)\bigg(1+O\Big(\frac{1}{\log x}\Big)\bigg) \times \sum_{\substack{n_1 \le \xi_1/y \\ (n_i,n_j)=1}} \cdots \sum_{\substack{n_k \le \xi_k/y \\ (1 \le i<j \le k)}} \frac{g_1(n_1)\cdots g_k(n_k)/n_1\cdots n_k}{\log(\xi_1/n_1)\cdots\log(\xi_k/n_k)}.$$

Therefore: consider $\displaystyle \sum_{\substack{n_1 \le \xi_1/y \\ (n_i,n_j)=1}} \cdots \sum_{\substack{n_k \le \xi_k/y \\ (1 \le i<j \le k)}} \frac{g_1(n_1)\cdots g_k(n_k)}{n_1\cdots n_k}$

(take care of logarithms later, via partial summation)

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First: consider more general sums of multiplicative functions

Multiplicative functions: one-variable sums

Definition

Let's say a multiplicative function $g(n)$ is α **on average** if it takes nonnegative values and

$$\sum_{p \le w} \frac{g(p) \log p}{p} \sim \alpha \log w.$$

Note: we really need upper bounds on $g(p^\nu)$ as well ...

Lemma

If the multiplicative function $g(n)$ is α on average, then

$$\sum_{n \le t} \frac{g(n)}{n} \sim c(g)(\log t)^\alpha,$$

where $c(g) = \prod_p \Big(1 - \frac{1}{p}\Big)^\alpha \Big(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\Big).$

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Multiplicative functions: more variables

From previous slide

$$c(g) = \prod_p \left(1 - \frac{1}{p}\right)^\alpha \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)$$

By the lemma on the previous slide, we easily get:

Proposition

If the multiplicative functions $g_1(n), \dots, g_k(n)$ are each 1 on average, then

$$\sum_{n_1 \le t} \cdots \sum_{n_k \le t} \frac{g_1(n_1) \cdots g_k(n_k)}{n_1 \cdots n_k} \sim c(g_1) \cdots c(g_k)(\log t)^k.$$

However, we need the analogous sum with the coprimality condition $(n_i, n_j) = 1$. (This is where $K > 1$ makes life harder!)

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Never mind that $g_1 + \cdots + g_k$ isn't multiplicative!

Partial summation: return of the logs

The proposition on the previous slide:

…gives, after a k-fold partial summation argument:

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If the multiplicative functions $g_1(n), \dots, g_k(n)$ are each 1 on average, then

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For our functions, $g_j(p) = \big(1-\frac{\sigma(f;p)}{p}\big)^{-1}\big(\sigma(f_j;p)-\frac{\sigma(f_j;p^2)}{p}\big)$ $= \sigma(f_j;p)\big(1+O(\frac{1}{p})\big)$, and $\sigma(f_j;p)$ is indeed 1 on average by the prime ideal theorem.

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$$\sim c(g_1+\cdots+g_k)\prod_{j=1}^k \log\frac{\log \xi_j}{\log y}.$$

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Proposition

If the multiplicative functions $g_1(n), \dots, g_k(n)$ are each 1 on average, then

$$\sum_{\substack{n_1 \le \xi_1/y \\ (n_i,n_j)=1}} \cdots \sum_{\substack{n_k \le \xi_k/y \\ (1 \le i<j \le k)}} \frac{g_1(n_1)\cdots g_k(n_k)}{n_1\cdots n_k \log(\xi_1/n_1)\cdots\log(\xi_k/n_k)}$$

$$\sim c(g_1+\cdots+g_k)\prod_{j=1}^k \log\frac{\log \xi_j}{\log y}.$$

For our functions, $g_j(p) = \big(1-\frac{\sigma(f;p)}{p}\big)^{-1}\big(\sigma(f_j;p)-\frac{\sigma(f_j;p^2)}{p}\big)$ $=\sigma(f_j;p)\big(1+O(\frac{1}{p})\big)$, and $\sigma(f_j;p)$ is indeed 1 on average by the prime ideal theorem.

For $f = f_1 \dots f_k$ and x and y sufficiently large,

$$M(f;x,y) = xH(f)\left(1+O\left(\frac{1}{\log x}\right)\right)$$
$$\times \sum_{\substack{n_1 \le \xi_1/y \\ (n_i,n_j)=1\ (1\le i<j\le k)}} \cdots \sum_{n_k \le \xi_k/y} \frac{g_1(n_1)\cdots g_k(n_k)/n_1\cdots n_k}{\log(\xi_1/n_1)\cdots\log(\xi_k/n_k)}$$
$$= H(f)c(g_1+\cdots+g_k)$$
$$\times\ x\left(\prod_{j=1}^{k} \log\frac{\log \xi_j}{\log y}\right)\left(1+O\left(\frac{1}{\log x}\right)\right).$$

Recall: $\xi_j = f_j(x) \approx x^d$, and we care about $y = x^{1/u}$. Then $\log(\log \xi_j/\log y) \sim \log du$.

We have the order of magnitude $x(\log du)^k$ we wanted … but what about the local factors $H(f)c(g_1+\cdots+g_k)$?

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$$= H(f)c(g_1 + \cdots + g_k)$$

$$\times \; x\Big(\quad (\log du)^k \quad \Big)\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

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For $f = f_1 \dots f_k$ and x and y sufficiently large,

$$M(f;x,y) = xH(f)\biggl(1 + O\Bigl(\frac{1}{\log x}\Bigr)\biggr) \times \sum_{\substack{n_1 \le \xi_1/y}} \cdots \sum_{\substack{n_k \le \xi_k/y \\ (n_i,n_j)=1\ (1\le i<j\le k)}} \frac{g_1(n_1)\cdots g_k(n_k)/n_1\cdots n_k}{\log(\xi_1/n_1)\cdots\log(\xi_k/n_k)}$$

$$= H(f)c(g_1 + \cdots + g_k) \times x\biggl(\quad (\log du)^k \quad \biggr)\biggl(1 + O\Bigl(\frac{1}{\log x}\Bigr)\biggr).$$

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The magic moment for $H(f)c(g_1 + \cdots + g_k)$

- $H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f;p)}{p}\right)$
- $c(g) = \prod_p \left(1 - \frac{1}{p}\right)^{\alpha} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)$

We have $g_j(p^\nu) = \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} \left(\sigma(f_j;p^\nu) - \frac{\sigma(f_j;p^{\nu+1})}{p}\right)$,

and so $\frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$

$$= \frac{1}{p^\nu} \sum_{j=1}^{k} \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} \left(\frac{\sigma(f_j;p^\nu)}{p^\nu} - \frac{\sigma(f_j;p^{\nu+1})}{p^{\nu+1}}\right)$$

$$= \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} \left(\frac{\sigma(f;p^\nu)}{p^\nu} - \frac{\sigma(f;p^{\nu+1})}{p^{\nu+1}}\right)$$

since the f_j have no common roots modulo p.

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Therefore

$$1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$$

$$= 1 + \sum_{\nu=1}^{\infty} \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} \left(\frac{\sigma(f;p^\nu)}{p^\nu} - \frac{\sigma(f;p^{\nu+1})}{p^{\nu+1}}\right)$$

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Therefore

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This is a telescoping sum …

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This is a telescoping sum ... tada!

The magic moment for $H(f)c(g_1 + \cdots + g_k)$

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$$1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$$

$$= 1 + \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} \left(\frac{\sigma(f;p)}{p}\right) =$$

And this whole expression simplifies ...

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Therefore

$$1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$$

$$= 1 + \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} \left(\frac{\sigma(f;p)}{p}\right) = \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1}.$$

And this whole expression simplifies . . . nicely.

The magic moment for $H(f)c(g_1 + \cdots + g_k)$

- $H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f;p)}{p}\right)$
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We conclude that

$$H(f)c(g_1 + \cdots + g_k)$$
$$= H(f) \prod_p \left(1 - \frac{1}{p}\right)^{k} \left(1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}\right)$$
$$= H(f) \prod_p \left(1 - \frac{1}{p}\right)^{k} \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1}$$

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We conclude that

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$$= H(f) \prod_p \left(1 - \frac{1}{p}\right)^{k} \left(1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}\right)$$
$$= H(f) \prod_p \left(1 - \frac{1}{p}\right)^{k} \left(1 - \frac{\sigma(f;p)}{p}\right)^{-1} = 1$$

… amazing!

Summary

- There are lots of open problems concerning friable values of polynomials—and many possible improvements from a single clever new idea.
- The asymptotics for friable values of polynomials depends on the degrees of their irreducible factors—but shouldn't depend on the polynomial otherwise.

Notes to be placed on web page

www.math.ubc.ca/∼gerg/talks.html