Prime number races

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Outline

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss with the following statement:

There is a notable difference in the splitting of the prime numbers between the two forms $4n+3$, $4n+1$: the first form contains a lot more than the second.

Since then, "notable differences" have been observed between primes of various other forms $qn+a$, where q and a are constants.

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Since then, "notable differences" have been observed between primes of various other forms $qn+a$, where q and a are constants.

Notation

- $\pi(x;q,a)$ denotes the number of primes $p \le x$ such that $p \equiv a \pmod{q}$
- $\pi(x) = \pi(x;1,1)$ denotes the total number of primes $p \le x$
- $\phi(q)$ denotes the number of integers $1 \le a \le q$ such that $\gcd(a,q) = 1$

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Dirichlet's theorem

It was already known in Chebyshev's time that each contestant in these prime number races could run forever:

Theorem (Dirichlet, 1837)

If $\gcd(a, q) = 1$, then there are infinitely many primes $p \equiv a \pmod{q}$.

To prove this, Dirichlet used two innovations (both now named for him):

- Dirichlet characters modulo q
- a Dirichlet L-function for each Dirichlet character

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Dirichlet characters

A Dirichlet character modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying:

1. χ is periodic with period q;
2. $\chi(n) = 0$ if $\gcd(n, q) > 1$;
3. χ is totally multiplicative: $\chi(mn) = \chi(m)\chi(n)$

There are always $\phi(q)$ Dirichlet characters modulo q, and their orthogonality can be used to pick out particular arithmetic progressions: for any a with $\gcd(a, q) = 1$,

$$\sum_{\chi \pmod q} \overline{\chi(a)}\chi(n) = \begin{cases} \phi(q), & \text{if } n \equiv a \pmod q, \\ 0, & \text{if } n \not\equiv a \pmod q. \end{cases}$$

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Dirichlet characters

Examples of Dirichlet characters:

- The principal character modulo q:

$$\chi_0(n) = \begin{cases} 1, & \text{if } \gcd(n,q) = 1, \\ 0, & \text{if } \gcd(n,q) > 1. \end{cases}$$

- The only nonprincipal character modulo 4, whose values are

$$1,\ 0,\ -1,\ 0;\ 1,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 10, whose values are

$$1,\ 0,\ i,\ 0,\ 0,\ 0,\ -i,\ 0,\ -1,\ 0;\ \dots.$$

- A nonprincipal character modulo 7, whose values are

$$1,\ -\tfrac12 + \tfrac{i\sqrt3}{2},\ \tfrac12 + \tfrac{i\sqrt3}{2},\ -\tfrac12 - \tfrac{i\sqrt3}{2},\ \tfrac12 - \tfrac{i\sqrt3}{2},\ -1,\ 0;\ \dots.$$

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Dirichlet L-functions

Each Dirichlet character χ gives rise to a Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

By showing that $\lim_{s\to 1} L(s,\chi)$ exists and is nonzero for every nonprincipal character χ, Dirichlet could prove that there are infinitely many primes $p \equiv a \pmod{q}$ when $\gcd(a,q) = 1$.

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A few goals

Learning to “handicap” prime number races means understanding the following questions:

Question

When is $\pi(x;q,a)$ bigger than $\pi(x;q,b)$?

More fundamental question

Given q and a, how fast does $\pi(x;q,a)$ grow as a function of x?

Even more fundamental question

How fast does $\pi(x)$ grow as a function of x?

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How many primes?

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Approximately how many primes are there less than some given number x?

We write $f(x) \sim g(x)$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

A "good" answer to the question will mean finding a simple, smooth function $g(x)$ such that $\pi(x) \sim g(x)$.

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The Prime Number Theorem

- Legendre conjectured that $\pi(x) \sim x/\ln x$.
- Gauss made a more precise conjecture:

$$\pi(x) \sim \mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

  This is consistent with Legendre since $\mathrm{li}(x) \sim x/\ln x$.
- In 1859, Riemann wrote a groundbreaking memoir describing how he thought the question could be settled. His plan was gradually realized by many researchers, ending with Hadamard and de la Vallée-Poussin independently in 1898. They didn't prove the most exact version of Riemann's argument, but they did prove Gauss's conjecture.

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Riemann's magic formula

Riemann's plan for proving the Prime Number Theorem was to study the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes } p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for complex numbers s. (Note that $\zeta(s)$ is exactly a Dirichlet L-function corresponding to the principal character modulo 1!)

Notation

ρ will denote a nontrivial zero of $\zeta(s)$, that is, a complex number with real part between 0 and 1 such that $\zeta(\rho) = 0$. Any sum written $\sum_\rho$ denotes a sum over all such nontrivial zeros.

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Riemann established a technically complicated formula that, from a modern perspective, can be written in the following form:

Riemann's magic formula (modernized)

Define $\psi(x) = \ln\big(\operatorname{lcm}[1, 2, \dots, x]\big)$. Then

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \ln(2\pi) - \tfrac{1}{2}\ln(1 - 1/x^2).$$

(The last two terms aren't worth paying attention to.)

Let's look more closely at the left-hand side and the right-hand side of this magic formula.

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The left-hand side

Notice that $\mathrm{lcm}[1,2,\dots,x] = \prod_{\text{primes } p \le x} p^{k(p)}$, where $k(p)$ is the power such that $p^{k(p)} \le x < p^{k(p)+1}$. Therefore:

$$\psi(x) = \ln\big(\mathrm{lcm}[1,2,\dots,x]\big) = \sum_{\text{primes } p \le x} k(p) \ln p$$
$$= \sum_{\text{primes } p \le x} \ln p + \sum_{\text{primes } p \le x^{1/2}} \ln p + \sum_{\text{primes } p \le x^{1/3}} \ln p + \cdots .$$

Rule of thumb

$\psi(x)/\ln x$ acts like $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots$.

Humans count primes; Nature counts primes and their powers.

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The right-hand side

Write $\rho = \beta + i\gamma$. Note that

$$x^\rho = x^\beta x^{i\gamma} = x^\beta\big(\cos(\gamma \ln x) + i \sin(\gamma \ln x)\big).$$

Hadamard and de la Vallée-Poussin proved that β can't be very close to 1 if γ is small, which was enough to prove the Prime Number Theorem. But Riemann conjectured something much stronger:

Riemann Hypothesis

All nontrivial zeros ρ of $\zeta(s)$ have real part $\beta = 1/2$.

If we assume the Riemann Hypothesis, then the right-hand side becomes

$$x - \sqrt{x} \sum_{\gamma\colon \zeta(1/2+i\gamma)=0} \frac{\cos(\gamma \ln x) + i \sin(\gamma \ln x)}{1/2 + i\gamma} - \cdots.$$

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Riemann's magic formula, transformed

Combining these observations, moving terms around, and tweaking yields the following formula:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4 + \gamma^2} + \frac{\cos(\gamma \ln x)}{1/2 + 2\gamma^2} \right).$$

The following approximation is easier to grasp:

$$\frac{\mathrm{li}(x) - \pi(x)}{\sqrt{x}/\ln x} \approx 1 + 2 \sum_{\substack{\gamma>0 \\ \zeta(1/2+i\gamma)=0}} \frac{\sin(\gamma \ln x)}{\gamma}.$$

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Primes in arithmetic progressions

Riemann's plan can be adapted to counting primes in arithmetic progressions, if we incorporate Dirichlet L-functions as well.

Theorem

If $\gcd(a,q)=1$, *then*

$$\pi(x;q,a) \sim \frac{1}{\phi(q)}\,\mathrm{li}(x).$$

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More magic: the mod 4 race

A closer analysis reveals a subtle difference among the functions $\pi(x;q,a)$. For example, let χ be the nonprincipal character modulo 4, so that

$$L(s,\chi) = 1 + 0 - \frac{1}{3^s} + 0 + \frac{1}{5^s} + 0 - \frac{1}{7^s} + \cdots .$$

$$\frac{\pi(x;4,3) - \pi(x;4,1)}{\sqrt{x}/\ln x} \sim 1 + 2 \sum_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} \left(\frac{\gamma \sin(\gamma \ln x)}{1/4+\gamma^2} + \frac{\cos(\gamma \ln x)}{1/2+2\gamma^2} \right).$$

Of course, from $\pi(x;4,3) - \pi(x;4,1)$ and $\pi(x;4,3) + \pi(x;4,1) = \pi(x) - 1$, we can recover the functions $\pi(x;4,3)$ and $\pi(x;4,1)$ individually.

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Nonsquares beat squares

Humans count primes; Nature counts primes and their powers.

In general each $\pi(x; q, a)$, suitably normalized, can be expressed (assuming a generalized Riemann Hypothesis) as a sum of terms of the form $\sin(\gamma \ln x)/\gamma$, where some Dirichlet L-function corresponding to a character modulo q has a zero at the point $1/2 + i\gamma$.

The difference is: some residue classes a (mod q) contain squares of primes. For these, the formula has an additional -1, while for the nonsquares it's absent.

This is what causes all the biases in the prime number races we've seen!

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Mod 4 race

All odd squares are $1 \pmod 4$ . . . in the race, $3 \pmod 4$ is better than $1 \pmod 4$.

Mod 3 race

All squares are (0 or) $1 \pmod 3$ . . . in the race, $2 \pmod 3$ is better than $1 \pmod 3$.

Mod 10 race

All odd squares end in 1 or 9 (or 5) . . . in the race, 3 and 7 (mod 10) are better than 1 and $9 \pmod{10}$.

Mod 8 race

All odd squares are $1 \pmod 8$ . . . in the race, 3 and 5 and $7 \pmod 8$ are better than $1 \pmod 8$.

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How badly are these races skewed?

Using these "sums of waves" also allows us to calculate the proportion of time one contestant is ahead of the other.

We still have to make assumptions about the zeros of Dirichlet L-functions, such as the generalized Riemann Hypothesis; and we have to define "proportion of time" very carefully.

Mod 4 race

$\pi(x;4,3) > \pi(x;4,1)$ about 99.59% of the time.

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$\pi(x;3,2) > \pi(x;3,1)$ about 99.90% of the time.

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- $\pi(x; 10, 1) > \pi(x; 10, 9)$ and $\pi(x; 10, 3) > \pi(x; 10, 7)$ exactly 50% of the time.
- $\pi(x; 10, 3 \text{ or } 7) > \pi(x; 10, 1 \text{ or } 9)$ about 95.21% of the time.

Mod 8 race

- Any two of $\pi(x; 8, 3)$, $\pi(x; 8, 5)$, $\pi(x; 8, 7)$ make a 50%–50% race.
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Very recent work

Definition

Let $\delta(q;a,b)$ denote the "proportion of time" that $\pi(x;q,a)$ is ahead of $\pi(x;q,b)$.

- Rubinstein and Sarnak (1994) proved that $\delta(q;a,b) > \frac{1}{2}$ exactly when a is a nonsquare (mod q) and b is a square (mod q).
- They also proved that $\lim_{q\to\infty} \delta(q;a,b) = \frac{1}{2}$, uniformly in the choices of a and b.

Theorem (M.–Fiorilli, 2010)

When a is a nonsquare (mod q) and b is a square (mod q),

$$\delta(q;a,b) - \tfrac{1}{2} \sim \frac{\rho(q)}{2\sqrt{\pi\phi(q)\log q}},$$

where $\rho(q)$ is the number of solutions of $t^2 \equiv 1 \pmod q$.

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The end

These slides

www.math.ubc.ca/~gerg/index.shtml?slides

The survey article I wrote with Andrew Granville, “Prime number races”

www.math.ubc.ca/~gerg/index.shtml?abstract=PNR