Inclusive prime number races

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joint work with Nathan Ng
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slides can be found on my web page
www.math.ubc.ca/∼gerg/index.shtml?slides

Outline

1. What are inclusive prime number races, and do they exist?
2. Proving races inclusive under weaker hypotheses
3. Ideas that go into the proof

Inclusive prime number races

Notation

$$\pi(x;q,a) = \#\{\text{primes } p \le x\colon p \equiv a \pmod q\}$$

A prime number race is the study of the string of inequalities

$$\pi(x;q,a_1) > \pi(x;q,a_2) > \cdots > \pi(x;q,a_r).$$

(The a_j will always be relatively prime to q, so $r \le \phi(q)$.)

Definition

The prime number race among $a_1, \dots, a_r \pmod q$ is inclusive if these inequalities are satisfied by arbitrarily large values of x, no matter how the a_j are permuted.

For example, when $r = 2$, the prime number race is inclusive if and only if $\pi(x;q,a_1) - \pi(x;q,a_2)$ changes sign infinitely often.

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Past results (for more than 2 contestants)

Theorem (Kaczorowski, 1993)

- *Assume GRH. The full* 4*-way race* $\pmod 5$ *is inclusive.*
- *Assume GRH. The contestant* $\pi(x;q,1)$ *is in first place for arbitrarily large* x.

Theorem (Rubinstein/Sarnak, 1994)

Assume GRH and LI. Every prime number race, including the full $\phi(q)$*-way race* $\pmod q$, *is inclusive.*

A strong linear independence hypothesis

LI: all the nonnegative imaginary parts of zeros of Dirichlet L-functions are linearly independent over the rationals.

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Malicious configurations of zeros

Theorem (Ford/Konyagin, 2002)

Given a prime number race $a_1, \dots, a_r \pmod q$ with $r \ge 3$, there exist specific points off the critical line (with $\frac{1}{2} < \sigma < 1$) such that, if Dirichlet L-functions $\pmod q$ have zeros at those points, then the race is not inclusive.

Moral of the story: if one wanted to unconditionally prove prime number races to be inclusive, then one would have to at least rule out these malicious configurations of zeros. (Is this any easier than GRH?)

Theorem (Ford/Konyagin, 2003)

Given a prime number race $a_1, \dots, a_r \pmod q$ with $r \ge 3$, there exist malicious configurations of zeros that allow fewer than r^2 of the $r!$ possible contestant orderings to occur for large x.

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Narrowing the gap between theorems

Current situation

1. If we don't assume GRH, prime number races might not be inclusive (if there are malicious configurations of zeros).
2. If we assume GRH and LI, all prime number races can be proved to be inclusive.

One could:

(a) try to improve (1), by constructing malicious configurations of zeros on the critical line (which must not satisfy LI); or

(b) try to improve (2), by weakening the LI hypothesis.

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Naturally, we succeeded at doing (b).

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Self-sufficient zeros

Ordinates (y-coordinates) of zeros of Dirichet L-functions

$$Y(q) = \{\gamma \ge 0\colon \text{there exists } \chi \pmod q \text{ with } L(\tfrac12 + i\gamma, \chi) = 0\}$$

Definition

An ordinate $\gamma \in Y(q)$ is self-sufficient if γ is not in the $\mathbb{Q}$-span of $Y(q) \setminus \{\gamma\}$. In other words, γ is not involved in any integer linear relations with other ordinates of zeros of L-functions $\pmod q$.

Example

Let χ and χ' be characters $\pmod q$. Imagine that $L(\frac12 + i\pi, \chi) = L(\frac12 + i\pi^2, \chi) = \cdots = L(\frac12 + i\pi^k, \chi) = 0$, while $L(\frac12 + i(\pi + \pi^2 + \cdots + \pi^k), \chi') = 0$. Then any proper subset of $\{\pi, \pi^2, \dots, \pi^k, \pi + \pi^2 + \cdots + \pi^k\}$ is linearly independent—but none of those ordinates would be self-sufficient.

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Robust characters (weakening LI)

Definition

A character χ is robust if $\displaystyle\sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0 \\ \gamma \text{ self-sufficient}}} \frac{1}{\gamma}$ diverges.

Recall that the counting function of zeros of Dirichlet L-functions satisfies (assuming GRH)

$$\#\{0 \le \gamma \le T : L(\tfrac12 + i\gamma, \chi) = 0\} \sim \frac{T}{2\pi} \log T.$$

The character χ will be robust if

$$\#\{0 \le \gamma \le T : L(\tfrac12 + i\gamma, \chi) = 0,\ \gamma \text{ self-sufficient}\} \gg \frac{T}{\log T}$$

(or even slightly less).

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Special case 1: the full $\phi(q)$-way race

Theorem (M.–Ng, in progress)

Assume GRH. Suppose that every nonprincipal character $\pmod q$ is robust. Then the full $\phi(q)$-way prime number race $\pmod q$ is inclusive.

Why shouldn't we care about the principal character?

Recall the explicit formula (assuming GRH)

$$\pi(x;q,a) = \frac{\pi(x)}{\phi(q)} - (c_{q,a} + o(1))\frac{\sqrt{x}}{\log x} - \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} \frac{\bar\chi(a)}{\log x} \sum_{\substack{\gamma\in\mathbb{R} \\ L(1/2+i\gamma,\chi)=0}} \frac{x^{1/2+i\gamma}}{1/2+i\gamma}.$$

The contribution from the principal character χ_0 affects all contestants $\pi(x;q,a)$ equally, hence does not affect their order.

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Special case 2: two-way races

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Assume GRH. Let a, b be relatively prime to q. Suppose that there exists a robust character χ satisfying $\chi(a) \neq \chi(b)$. Then the 2-way race between $\pi(x; q, a)$ and $\pi(x; q, b)$ is inclusive.

Why shouldn't we care about χ if $\chi(a) = \chi(b)$?

Again, from the explicit formula

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The general theorem

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Assume GRH. Let $a_1, \dots, a_r$ be relatively prime to q. Suppose that the vectors

$$\big\{\big(\chi(a_1), \dots, \chi(a_r)\big) : \chi \pmod q \text{ is robust}\big\} \cup \{(1, \dots, 1)\}$$

span $\mathbb{C}^r$. Then the r-way prime number race among $\pi(x; q, a_1), \dots, \pi(x; q, a_r)$ is inclusive.

This implies the theorem on two-way races

As soon as $\chi(a) \ne \chi(b)$, the set $\big\{\big(\chi(a), \chi(b)\big), (1, 1)\big\}$ spans $\mathbb{C}^2$.

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As soon as $\chi(a) \ne \chi(b)$, the set $\big\{ \big(\chi(a), \chi(b)\big), (1,1) \big\}$ spans $\mathbb{C}^2$.

The general theorem

Theorem (M.–Ng, in progress)

Assume GRH. Let $a_1, \dots, a_r$ be relatively prime to q. Suppose that the vectors

$$\big\{ \big(\chi(a_1), \dots, \chi(a_r)\big) : \chi \pmod q \text{ is robust} \big\} \cup \{(1, \dots, 1)\}$$

span $\mathbb{C}^r$. Then the r-way prime number race among $\pi(x;q,a_1), \dots, \pi(x;q,a_r)$ is inclusive.

This implies the theorem on the full $\phi(q)$-way race

The set $\big\{ \big(\chi(a_1), \dots, \chi(a_{\phi(q)})\big) : \chi \pmod q \big\}$ is orthogonal (by orthogonality!), hence linearly independent, hence spans $\mathbb{C}^{\phi(q)}$.

Introducing random variables

Explicit formula for normalized error term (assuming GRH)

$$\left(\pi(x;q,a) - \frac{\pi(x)}{\phi(q)}\right)\frac{\log x}{\sqrt{x}} \sim -c_{q,a} - \sum_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} \bar\chi(a) \sum_{\substack{\gamma\in\mathbb{R} \\ L(1/2+i\gamma,\chi)=0}} \frac{x^{i\gamma}}{1/2+i\gamma}$$

For each $\gamma > 0$, let Z_γ denote a random variable distributed uniformly on the unit circle in $\mathbb{C}$; for $\gamma < 0$, let $Z_\gamma = \overline{Z_{-\gamma}}$. Then write down the random variable

$$E_{q,a} = -c_{q,a} + \sum_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} \sum_{\substack{\gamma\in\mathbb{R} \\ L(1/2+i\gamma,\chi)=0}} \frac{Z_\gamma}{|1/2+i\gamma|}$$

$$= -c_{q,a} + 2\,\mathrm{Re} \sum_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} \sum_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} \frac{Z_\gamma}{\sqrt{1/4+\gamma^2}}.$$

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Linear independence of ordinates, independence of random variables

As written, the definition of $E_{q,a}$ is incomplete, because we haven't said anything about the dependence of the $\{Z_\gamma\}$ on one another.

Theorem (restatement of Rubinstein/Sarnak, 1994)

Assuming GRH and LI, the limiting distribution of the normalized error term $\big(\pi(x;q,a) - \pi(x)/\phi(q)\big)(\log x)/\sqrt{x}$ is the same as the distribution function of the random variable $E_{q,a}$ when the $\{Z_\gamma : \gamma > 0\}$ are independent random variables.

This statement is formally proved by showing that Fourier transform of the limiting distribution of the normalized error term equals the characteristic function of the random variable $E_{q,a}$.

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Why the race is inclusive

The prime number race is inclusive if:

- the limiting distribution in $\mathbb{R}^r$ of (the normalized error-term version of) the vector $(\pi(x;q,a_1),\dots,\pi(x;q,a_r))$ visits the cone $t_1 > t_2 > \cdots > t_r$ and every similar cone obtained by permuting the variables
- the support of the distribution function of $(E_{q,a_1},\dots,E_{q,a_r})$ intersects every such "race cone"

In fact, Rubinstein and Sarnak show that the support of the limiting distribution of $(\pi(x;q,a_1),\dots,\pi(x;q,a_r))$ equals all of $\mathbb{R}^r$, using an analytic argument.

Alternatively, one could show straight from the definition of $E_{q,a}$ that the support of $(E_{q,a_1},\dots,E_{q,a_r})$ equals all of $\mathbb{R}^r$.

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Using self-sufficient zeros of robust characters

A self-sufficient random variable

$$S_{q,a} = 2\,\mathrm{Re} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0 \\ \chi \text{ robust}}} \sum_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0 \\ \gamma \text{ self-sufficient}}} \frac{Z_\gamma}{\sqrt{1/4+\gamma^2}}$$

where the Z_γ are independent, uniform on $\{|z|=1\}$

Heuristically, we would like to define:

$$N_{q,a} = -c_{q,a} + 2\,\mathrm{Re} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0 \\ \chi \text{ robust}}} \sum_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0 \\ \gamma \text{ not self-sufficient}}} \frac{Z_\gamma}{\sqrt{1/4+\gamma^2}} + 2\,\mathrm{Re} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0 \\ \chi \text{ not robust}}} \sum_{\substack{\gamma>0 \\ L(1/2+i\gamma,\chi)=0}} \frac{Z_\gamma}{\sqrt{1/4+\gamma^2}}$$

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Theorem (M.-Ng)

There exists a random variable $N_{q,a}$ such that $S_{q,a}$ and $N_{q,a}$ are independent, and $S_{q,a}+N_{q,a}$ has the same distribution function as the limiting distribution of the normalized error term $\big(\pi(x;q,a)-\pi(x)/\phi(q)\big)(\log x)/\sqrt{x}$.

Everywhere plus somewhere equals everywhere

Under our assumptions:

One can show from the definition of $S_{q,a}$ that the support of the vector $(S_{q,a_1}, \dots, S_{q,a_r})$ equals either all of $\mathbb{R}^r$ or else the entire hyperplane $t_1 + \cdots + t_r = 0$, which intersects every race cone.

- The spanning condition on $\{(\chi(a_1), \dots, \chi(a_r)) : \chi \text{ robust}\}$ ensures that no smaller subspace contains the support.
- The divergence of $\sum 1/\gamma$ in the definition of robustness rules out having bounded support in some direction.

We don't know much about the N_{q,a_j}, but:

No matter where the support of $(N_{q,a_1}, \dots, N_{q,a_r})$ is, the support of the sum $(S_{q,a_1} + N_{q,a_1}, \dots, S_{q,a_r} + N_{q,a_r})$ will contain some hyperplane $t_1 + \cdots + t_r = C$, which still intersects every race cone. Therefore the prime number race is inclusive.

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The end

The paper *Inclusive prime number races* is currently still in progress; these slides are available for downloading.

The paper

www.math.ubc.ca/~gerg/
index.shtml?abstract=IPNR

These slides

www.math.ubc.ca/~gerg/index.shtml?slides