Statistics of the multiplicative group

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joint work with Ben Chang, Jenna Downey, and Lee Troupe (in parallel)

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these slides can be found on my web page
www.math.ubc.ca/∼gerg/index.shtml?slides

Outline

What is the multiplicative group?

The finite ring $\mathbb{Z}/n\mathbb{Z}$ has:

- a cyclic additive group $C_n = (\mathbb{Z}/n\mathbb{Z})^+$ with n elements;
- an abelian multiplicative group $M_n = (\mathbb{Z}/n\mathbb{Z})^\times$ with $\phi(n)$ elements.

Overarching question

Which abelian group of $\phi(n)$ elements is M_n?

For example, M_n being cyclic is equivalent to n having a primitive root.

Two forms that answers to the question can take

- primary decomposition: $G \cong C_{p_1^{r_1}} \oplus \cdots \oplus C_{p_k^{r_k}}$, where the $p_j^{r_j}$ are prime powers (unique up to reordering)
- invariant factors: $G \cong C_{m_1} \oplus \cdots \oplus C_{m_\ell}$, where $m_1 \mid m_2 \mid \cdots \mid m_\ell$ (unique)

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Number theory within a computation of M_n

Example: M_n when $n = 11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$

$$\begin{aligned} M_{11!} &\cong M_{2^8} \times M_{3^4} \times M_{5^2} \times M_7 \times M_{11} \\ &\cong (C_2 \oplus C_{64}) \oplus C_{54} \oplus C_{20} \oplus C_6 \oplus C_{10} \\ &\cong (C_2 \oplus C_{64}) \oplus (C_2 \oplus C_{27}) \\ &\qquad \oplus (C_4 \oplus C_5) \oplus (C_2 \oplus C_3) \oplus (C_2 \oplus C_5) \\ &\cong C_2 \oplus C_2 \oplus C_2 \oplus C_2 \oplus (C_4 \oplus C_3 \oplus C_5) \oplus (C_{64} \oplus C_{27} \oplus C_5) \\ &\cong C_2 \oplus C_2 \oplus C_2 \oplus C_2 \oplus C_{60} \oplus C_{8,640} \end{aligned}$$

Largest invariant factor

$8{,}640 = \lambda(11!)$ (Carmichael lambda-function)

Number of invariant factors

- When n odd: exactly $\omega(n) = \#\{p \mid n\}$
- When n even: $\omega(n) - 1$ or $\omega(n)$ or $\omega(n) + 1$

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What's known about $\lambda(n)$

For purposes of comparison

$\phi(n)$ is never smaller than $(e^{-\gamma} + o(1))n/\log\log n$.

Theorem (Erdős/Pomerance/Schmutz, 1991)

For almost all integers n,

$$\lambda(n) = n/\exp\big((1+o(1))\log n \log\log n\big).$$

In other words, the normal order of $\log\big(n/\lambda(n)\big)$ is $\log n \log\log n$.

Drive-by question

What about the second-largest invariant factor, $\lambda_2(n)$?

- $\lambda(n) = \min\big\{k \ge 1 : a^k \in \langle 1\rangle \pmod n \text{ for all } (a,n)=1\big\}$.
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Distribution results: different strengths

By way of analogy: some historical results about the distribution of $\omega(n)$, the number of distinct prime factors of n.

- The average value of $\omega(n)$ is $\log\log n$.
  — requires an asymptotic formula for $\sum_{n\le x}\omega(n)$
- The normal order (typical size) of $\omega(n)$ is $\log\log n$.
  — requires estimate for variance $\sum_{n\le x}\bigl(\omega(n)-\log\log n\bigr)^2$
- Erdős–Kac theorem: $\omega(n)$ is asymptotically distributed like a normal random variable with mean $\log\log n$ and variance $\log\log n$. (More precise statement on next slide.)
  — requires asymptotic formulas for all central moments $\sum_{n\le x}\bigl(\omega(n)-\log\log n\bigr)^k$

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Erdős–Kac laws

Definition

A function $f(n)$ satisfies an Erdős–Kac law with mean $\mu(n)$ and variance $\sigma^2(n)$ if

$$\lim_{x\to\infty} \frac{1}{x} \#\left\{ n \le x : \frac{f(n)-\mu(n)}{\sigma(n)} < u \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\,dt$$

for every real number u.

Standard notation

$\omega(n)$ is the number of distinct prime factors of n.
$\Omega(n)$ is the number of prime factors of n counted with multiplicity.

Theorem (Erdős–Kac, 1940)

Both $\omega(n)$ and $\Omega(n)$ satisfy Erdős–Kac laws with mean $\log\log n$ and variance $\log\log n$.

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Other functions with Erdős–Kac laws

The paper of Erdős–Kac establishes these normal-distribution laws for a large class of additive functions: if $n = p_1^{r_1} \cdots p_k^{r_k}$, then $f(n) = f(p_1^{r_1}) + \cdots + f(p_k^{r_k})$. Examples of non-additive functions:

Liu (2007)

On GRH, $\omega(\#E(\mathbb{F}_p))$ satisfies an Erdős–Kac law with mean $\log\log p$ and variance $\log\log p$.

Erdős–Pomerance (1985)

$\omega(\phi(n))$ and $\Omega(\phi(n))$ satisfy Erdős– Kac laws with mean $\frac{1}{2}(\log\log n)^2$ and variance $\frac{1}{3}(\log\log n)^3$.

$\Omega(\phi(n))$ is not additive, but is "ϕ-additive": if $\phi(n) = p_1^{r_1} \cdots p_k^{r_k}$, then $\Omega(\phi(n)) = \Omega(p_1^{r_1}) + \cdots + \Omega(p_k^{r_k})$.

Other functions with Erdős–Kac laws

The paper of Erdős–Kac establishes these normal-distribution laws for a large class of additive functions: if $n = p_1^{r_1} \cdots p_k^{r_k}$, then $f(n) = f(p_1^{r_1}) + \cdots + f(p_k^{r_k})$. Examples of non-additive functions:

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A different question about the multiplicative group

Notation reminder

$M_n = \mathbb{Z}_n^\times$, an abelian group with $\phi(n)$ elements.

Question (Vukoslavcevic and Shparlinski, 2010)

How many subgroups does M_n have?

Notation (used throughout this section of the talk)

- $I(n)$ is the number of isomorphism classes of subgroups of M_n.
- $G(n)$ is the number of subsets of M_n that are subgroups (that is, subgroups not up to isomorphism).

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The number of subgroups has a similar property

Getting used to the notation

$I(n)$ is the number of isomorphism classes of subgroups of M_n.
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Every finite abelian group is the direct sum of its p-Sylow subgroups, so consequently:

If $G_p(n)$ denotes the number of subgroups of the p-Sylow subgroup of M_n, then $G(n) = \prod_{p \mid \#M_n} G_p(n) = \prod_{p \mid \phi(n)} G_p(n)$.

And similarly for $I(n)$.

In particular, both $I(n)$ and $G(n)$ are "ϕ-multiplicative" functions; so we might hope to get strong distributional information for the ϕ-additive functions $\log I(n)$ and $\log G(n)$.

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Erdős–Kac laws for the number of subgroups

Theorem (M.–Troupe, to appear in the Journal of the Australian Mathematical Society)

$\log I(n)$ satisfies an Erdős–Kac law with mean $\frac{\log 2}{2}(\log\log n)^2$ and variance $\frac{\log 2}{3}(\log\log n)^3$.

How did we prove this?

We showed that $\omega(\phi(n))\log 2 \le \log I(n) \le \Omega(\phi(n))\log 2$, and then quoted Erdős–Pomerance. □

Theorem (M.–Troupe)

$\log G(n)$ satisfies an Erdős–Kac law with mean $A(\log\log n)^2$ and variance $C(\log\log n)^3$, for certain constants A and C.

$\frac{\log 2}{2} \approx 0.34657$ while $A \approx 0.72109$, so typically $G(n) \approx I(n)^{2.08}$.

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We had to look at these constants, so you do too

Definition

$$A_0 = \frac{1}{4} \sum_p \frac{p^2 \log p}{(p-1)^3(p+1)}$$

$$A = \frac{\log 2}{2} + A_0 \approx 0.72109$$

$$B = \frac{1}{4} \sum_p \frac{p^3(p^4 - p^3 - p^3 - p - 1)(\log p)^2}{(p-1)^6(p+1)^2(p^2+p+1)}$$

$$C = \frac{(\log 2)^2}{3} + 2A_0 \log 2 + 4A_0^2 + B \approx 3.924$$

(The two sums are convergent sums over all primes p.)

How many subgroups can there be?

Theorem (M.–Troupe)

The order of magnitude of the maximal order of $\log I(n)$ *is* $\log n/\log\log n$. *More precisely,*

$$\frac{\log 2}{5}\frac{\log x}{\log\log x} \lesssim \max_{n\le x}\big(\log I(n)\big) \lesssim \pi\sqrt{\frac{2}{3}}\frac{\log x}{\log\log x}.$$

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Finite abelian groups and partitions

Facts about finite abelian p-groups

- Every finite abelian group of size p^m can be written uniquely as $C_{p^{\boldsymbol{\alpha}}} = C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}} \oplus \cdots \oplus C_{p^{\alpha_\ell}}$ for some partition $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of m (so $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_\ell$).
- So the number of isomorphism classes of subgroups of C_{p^α} is exactly the number of subpartitions $\beta \preceq \alpha$ ...

... which is somewhere between 2 and 2^m inclusive.

In other words:

$\log \#\{\text{subpartitions of } \alpha\}$ is between $\log 2$ and $m \log 2$.

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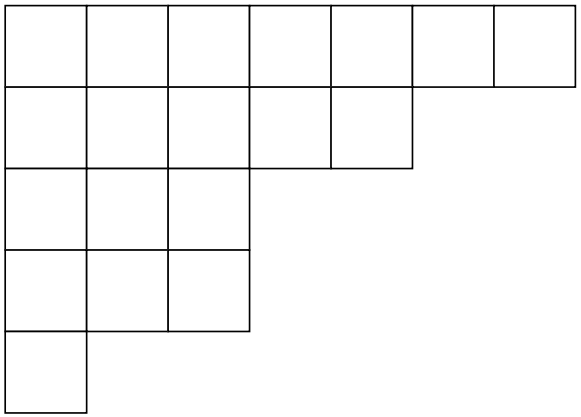

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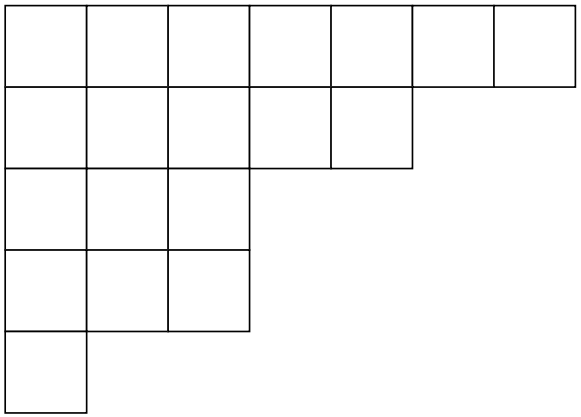

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Application to distribution of $I(n)$

$I(n)$ is the number of isomorphism classes of subgroups of M_n

More notation

Let $\phi(n) = \prod_{p|\phi(n)} p^{m(p)}$, so that $M_n \cong \bigoplus_{p|\phi(n)} C_{p^{\alpha(p)}}$ for some partitions $\alpha(p)$ of $m(p)$.

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"Anatomy of integers" techniques show: most primes dividing $\phi(n)$ do so only once.

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How many subgroups of each shape?

Notation: $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell)$, $C_{p^{\boldsymbol{\alpha}}} = C_{p^{\alpha_1}} \oplus \cdots \oplus C_{p^{\alpha_\ell}}$

Definition

Given a subpartition $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ and a prime p, define $N_p(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to be the number of subgroups inside $C_{p^{\boldsymbol{\alpha}}}$ that are isomorphic to $C_{p^{\boldsymbol{\beta}}}$.

Some classical exact formula (don't read it)

Let $\boldsymbol{a} = (a_1, a_2, \dots, a_{\alpha_1})$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_{\beta_1})$ be the conjugate partitions to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. Then

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where $\begin{bmatrix} k \\ \ell \end{bmatrix}_p = \prod_{j=1}^{\ell} \frac{p^{k-\ell+j}-1}{p^j - 1}$ is the Gaussian binomial coefficient.

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The difference between algebra and analysis

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is the number of subgroups inside $C_{p^{\boldsymbol{\alpha}}}$ isomorphic to $C_{p^{\boldsymbol{\beta}}}$.

It turns out that each factor is about $p^{(a_j-b_j)b_j}$, which is maximally $p^{a_j^2/4}$ when $b_j = a_j/2$, and is way smaller for noncentral values of b_j. So the total number of subgroups inside C_{p^α} is dominated by this special $\beta = \text{“}\frac{1}{2}\alpha\text{”}$.

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For any prime p and any partition α,

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$$N_p(\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{j=1}^{\alpha_1} p^{(a_j-b_j)b_{j+1}} \begin{bmatrix} a_j - b_{j+1} \\ b_j - b_{j+1} \end{bmatrix}_p$$

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If $M_n \cong \bigoplus_{p|\phi(n)} C_{p^{\alpha(p)}}$, then which partition is $\alpha(p)$?

Notation

Let $\omega_q(n)$ denote the number of distinct prime factors of n that are congruent to $1 \pmod q$.

Answer (exact for odd squarefree n, up to $O(1)$ in general)

$\alpha(p)$ is the conjugate partition to $\big(\omega_p(n), \omega_{p^2}(n), \dots\big)$.

Lemma

$$\log G_p(n) \approx \frac{\log p}{4} \sum_{j=1}^{\text{“}\infty\text{”}} \omega_{p^j}(n)^2 \text{ for any prime } p \text{ dividing } \phi(n).$$

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Sum the previous lemma over all primes

$$\log G(n) = \sum_{p \mid \phi(n)} \log G_p(n) \approx \sum_{\substack{p \mid \phi(n) \\ p^2 \nmid \phi(n)}} \log 2 + \sum_{p^2 \mid \phi(n)} \frac{\log p}{4} \sum_{j=1}^{\text{“}\infty\text{”}} \omega_{p^j}(n)^2.$$

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$$\log G(n) \approx \log 2 \cdot \omega(\phi(n)) + \frac{1}{4} \sum_{p^r} \omega_{p^r}(n)^2 \log p.$$

Each function here has a known normal order; plugging in gives

$$\log G(n) \approx \log 2 \cdot \frac{1}{2} (\log\log n)^2 + \frac{1}{4} \sum_{p^r} \left(\frac{\log\log n}{\phi(p^r)} \right)^2 \log p$$

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Final sketch

Getting beyond the normal order to an Erdős–Kac law requires computing all of the central moments of this approximation to $\log G(n)$. The correlations among the additive functions $\omega_q(n)$, and their correlations with $\omega(\phi(n))$, become important.

"Sieving and the Erdős–Kac theorem" (2007)

To compute the moments, we rely on a technique of Granville and Soundararajan to reduce the complexity of identifying the main terms of these moments.

Generalizing our method

Part of $\log G(n)$ is well approximated by a sum of squares of additive functions. Troupe and I (just submitted!) can obtain an Erdős–Kac law for any fixed nonnegative polynomial evaluated at values of appropriate additive functions—for example, Erdős–Kac laws for products of additive functions.

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Prohibited prime factors: related problems

Let q be a fixed odd prime.

Definition

The q-Sylow subgroup of a finite abelian group G is the largest subgroup of G whose cardinality is a power of q.

Question (Colin Weir, 2017)

If we fix our favorite prime q and our favorite group G, how often will we get G as the exact q-Sylow subgroup of M_n?

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$S_{q,G}(x) = \#\{n \le x\colon \text{the } q\text{-Sylow subgroup of } M_n \text{ equals } G\}$

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The trivial example

Another translation of algebra to number theory

$$\{n\colon \text{the } q\text{-Sylow subgroup of } M_n \text{ is trivial}\}$$
$$= \{n\colon q \text{ does not divide } \#M_n = \phi(n)\}$$
$$= \{n\colon \big(p \mid n \implies p \not\equiv 1 \pmod q\big) \text{ and } q^2 \nmid n\}.$$

Example (when $G = \{1\}$ is trivial)

$$S_{q,\{1\}}(x) = \#\{n \le x\colon \text{the } q\text{-Sylow subgroup of } M_n \text{ is trivial}\}$$
$$= \#\{n\colon \big(p \mid n \implies p \not\equiv 1 \pmod q\big) \text{ and } q^2 \nmid n\}:$$

we prohibit prime factors from a set of relative density $1/(q-1)$.

General philosophy

Prohibiting prime divisors from a set of primes of relative density δ divides the counting function by a factor of $(\log x)^\delta$. Correspondingly, we expect $S_{q,\{1\}}(x) \asymp x/(\log x)^{1/(q-1)}$.

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Theorems of this type go back to Landau; today it would be deemed a standard application of the Selberg–Delange method; this particular application (other than the extra $q^2 \nmid n$) was carried out by Ford/Luca/Moree (Math Comp., 2014).

Theorem

$$S_{q,\{1\}} \sim \frac{C_q}{\Gamma(1-\frac{1}{q-1})}\left(1-\frac{1}{q^2}\right)\frac{x}{(\log x)^{1/(q-1)}};$$

here

$$C_q = \left((1-\tfrac{1}{q})\prod_{\chi\neq\chi_0} L(1,\chi)\right)^{-1/(q-1)} \prod_{p\not\equiv 0,1 \pmod q}\left(1-\frac{1}{p^{k_p}}\right)^{-1/k_p},$$

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Suppose now that $G = C_{q^\alpha}$:

- Typically this q-Sylow subgroup of M_n arises from a single prime $p \mid n$ with $q^\alpha \| (p-1)$, and the other factor $\frac{n}{p}$ of the form discussed in the trivial example.
- For each possible p, we thus get a contribution of $\asymp \frac{x}{p}/(\log \frac{x}{p})^{1/(q-1)}$, or more simply, $\asymp \frac{x}{p}/(\log x)^{1/(q-1)}$.
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(Need to tread more carefully; we found a nice argument using partial summation, asymptotics for hypergeometric functions.)

In general, when $G = C_{q^{\alpha_1}} \oplus \cdots \oplus C_{q^{\alpha_\ell}}$:

Identify a prime $p \mid n$ with $q^{\alpha_\ell} \| (p-1)$; the cofactor $\frac{n}{p}$ will be an example with $G = C_{q^{\alpha_1}} \oplus \cdots \oplus C_{q^{\alpha_{\ell-1}}}$; then recurse. . . .

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Suppose now that $G = C_{q^\alpha}$:

- Typically this q-Sylow subgroup of M_n arises from a single prime $p \mid n$ with $q^\alpha \| (p-1)$, and the other factor $\frac{n}{p}$ of the form discussed in the trivial example.
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The result

Theorem (Downey–M., 2019)

If $G = C_{q^{\alpha_1}} \oplus \cdots \oplus C_{q^{\alpha_\ell}}$, *then*

$$S_{q,G} \sim \frac{C_q}{\Gamma(1-\frac{1}{q-1})} \left(\prod_{j=1}^{\infty} \frac{1}{n_j!} \right) \left(\frac{q^2-1}{q^{2+\alpha_1+\cdots+\alpha_\ell}} \right) \frac{x(\log\log x)^\ell}{(\log x)^{1/(q-1)}},$$

where $n_j = \#\{i : \alpha_i = j\}$*; as before,*

$$C_q = \left((1-\tfrac{1}{q}) \prod_{\chi \neq \chi_0} L(1,\chi) \right)^{-1/(q-1)} \prod_{p \not\equiv 0,1 \pmod q} \left(1 - \frac{1}{p^{k_p}}\right)^{-1/k_p},$$

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An M_n with a larger least invariant factor

Recall: $M_{11!} \cong C_2 \oplus C_2 \oplus C_2 \oplus C_2 \oplus C_{60} \oplus C_{8,640}$

Example: M_n when $n = 3^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$

$$\begin{aligned} M_n &\cong M_{3^3} \times M_{7^2} \times M_{13} \times M_{19} \times M_{31} \\ &\cong C_{18} \oplus C_{42} \oplus C_{12} \oplus C_{18} \oplus C_{30} \\ &\cong (C_2 \oplus C_9) \oplus (C_2 \oplus C_3 \oplus C_7) \\ &\qquad \oplus (C_4 \oplus C_3) \oplus (C_2 \oplus C_9) \oplus (C_2 \oplus C_3 \oplus C_5) \\ &\cong (C_2 \oplus C_3) \oplus (C_2 \oplus C_3) \oplus (C_2 \oplus C_3) \\ &\qquad \oplus (C_2 \oplus C_9) \oplus (C_4 \oplus C_9 \oplus C_5 \oplus C_7) \\ &\cong C_6 \oplus C_6 \oplus C_6 \oplus C_{18} \oplus C_{1,260}. \end{aligned}$$

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Counting "unusually large" smallest invariant factors

The least invariant factor of M_n equals 2 for almost all integers n. But we can be more precise:

Theorem (Chang–M., 2016+)

The number of integers $n \le x$ for which the least invariant factor of M_n does not equal 2 is

$$C \frac{x}{\sqrt{\log x}} + O\left(\frac{x}{(\log x)^{3/4-\varepsilon}} \right)$$

for a particular positive constant C (not the same one as before).

We also obtain the same result (with a different value of C) for integers for which the least primary-decomposition factor of M_n does not equal 2.

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From group theory to analytic number theory

Lemma

Fix an even number $k \geq 4$. The least invariant factor of M_n is a multiple of k if and only if all of the following conditions hold:

1. *for primes $p \nmid k$: if $p \mid n$ then we must have $p \equiv 1 \pmod{k}$;*
2. *$4 \nmid n$;*
3. *(some condition for odd primes $p \mid k$; for example, if $3 \mid k$, then either $3 \nmid n$ or $9 \mid n$)*

Definition

$$D_k(x) = \#\{n \leq x : k \text{ divides the least invariant factor of } M_n\}$$

By the lemma, $D_k(x)$ is very similar to

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General philosophy again

Prohibiting prime divisors from a set of primes of relative density δ divides the counting function by a factor of $(\log x)^\delta$. Correspondingly, we expect $D_k(x) \asymp x/(\log x)^{1-1/\phi(k)}$.

So we expect

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$$D_k(x) = \#\{n \le x\colon k \text{ divides the least invariant factor of } M_n\}$$
$$D_k(x) \text{ is similar to } \#\{n \le x\colon p \mid n \implies p \equiv 1 \pmod{k}\}$$

Trivially,

$$\#\{n \le x\colon \text{the least invariant factor of } M_n \text{ does not equal } 2\}$$
$$\le D_4(x) + D_6(x) + D_8(x) + D_{10}(x) + D_{12}(x) + \cdots.$$

General philosophy again

Prohibiting prime divisors from a set of primes of relative density δ divides the counting function by a factor of $(\log x)^\delta$. Correspondingly, we expect $D_k(x) \asymp x/(\log x)^{1-1/\phi(k)}$.

So we expect

$$\#\{n \le x\colon \text{the least invariant factor of } M_n \text{ does not equal } 2\}$$
$$= D_4(x) + D_6(x) + O\big(D_8(x) + D_{10}(x) + D_{12}(x) + \cdots\big).$$

Selberg–Delange with more uniformity

$D_k(x)$ is similar to $\#\{n \le x : p \mid n \implies p \equiv 1 \pmod{k}\}$

$\#\{n \le x$: the least invariant factor of M_n does not equal $2\}$

$$= D_4(x) + D_6(x) + O\big(D_8(x) + D_{10}(x) + D_{12}(x) + \cdots\big).$$

A straightforward application of the Selberg–Delange method (for example from Tenenbaum's book) gives

$$D_k(x) \sim C_k \frac{x}{(\log x)^{1-1/\phi(k)}} + O_k\left(\frac{x}{(\log x)^{2-1/\phi(k)}}\right)$$

(not the same C_k as before) . . . for any fixed k.

We have a sum over k in our error term; hence we need a version of Selberg–Delange with uniformity in k.

- uniformity of C_k: medium annoying
- uniformity of the O-constant: very annoying

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That's a weird constant

Theorem (Chang–M., 2016+)

The number of integers $n \le x$ for which the least invariant factor of M_n does not equal 2 is

$$C \frac{x}{\sqrt{\log x}} + O\left(\frac{x}{(\log x)^{3/4-\varepsilon}} \right)$$

where $C \approx 1.59747$ is given by

$$C = \frac{3}{2^{5/2}} \prod_{p \equiv 3 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{1/2} + \frac{7}{4 \cdot 3^{5/4}} \prod_{p \equiv 5 \pmod 6} \left(1 - \frac{1}{p^2}\right)^{1/2}.$$

Theorem (Chang–M., 2016+)

Let $m \ge 4$ be even. The number of integers $n \le x$ for which the least invariant factor of M_n equals m is (for some explicit C_m)

$$C_m \frac{x}{(\log x)^{1-1/\phi(m)}} + O_m\left(\frac{x}{(\log x)^{1-1/(2\phi(m))-\varepsilon}} \right).$$

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The end

These slides

www.math.ubc.ca/~gerg/index.shtml?slides

"The distribution of the number of subgroups of the multiplicative group", with Lee Troupe

www.math.ubc.ca/~gerg/index.shtml?abstract=DNSMG

"The distribution of sums and products of additive functions", with Lee Troupe

www.math.ubc.ca/~gerg/index.shtml?abstract=DSPAF

work in progress with Ben Chang and with Jenna Downey

Keep your eyes peeled on arXiv or my web page!