Linear independence of zeros of Dirichlet L-functions

Greg Martin
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joint work with Nathan Ng
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3rd Montreal–Toronto Workshop in Number Theory
University of Toronto
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in honour of John Friedlander's 70th birthday

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Outline

Zeros of Dirichlet L-functions: horizontal distribution

Classical fact: every Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ has infinitely many zeros $\rho = \beta + i\gamma$ whose real parts satisfy $0 < \beta < 1$ ("nontrivial zeros").

Conjecture GRH

Generalized Riemann hypothesis: every nontrivial zero actually satisfies $\beta = \frac{1}{2}$.

Notice that this conjecture actually addresses both:

- the analytic nature of the zeros' abscissae (the distribution function of β is a Dirac delta function at $\frac{1}{2}$);
- the algebraic nature of the zeros' abscissae (the β are all rational, for example).

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Zeros of Dirichlet L-functions: vertical distribution

We have some good ideas about the distribution of the imaginary parts as well:

- The number of zeros $\beta + i\gamma$ with $0 \le \gamma \le T$ is asymptotic to $\frac{T}{2\pi} \log \frac{qT}{2\pi}$; in fact we have an asymptotic formula for the number of zeros with $T \le \gamma \le T + y$ when y is almost bounded.
- We have conjectures for the distribution of gaps between ordinates and, more generally, for the n-level correlations of the sequence γ.

Note that these statements all concern the analytic nature of the zeros' ordinates.

Question

What about the algebraic nature of the zeros' ordinates γ?

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Linear independence conjecture for zeros of $\zeta(s)$

Let $Z_1 = \{\rho\colon \zeta(\rho) = 0,\ \operatorname{Re}\rho \ge \frac{1}{2},\ \operatorname{Im}\rho \ge 0\}$ (where non-simple zeros are listed several times according to their multiplicity, so that Z_1 is a multiset).

- We restrict to $\operatorname{Re}\rho \ge \frac{1}{2}$ and $\operatorname{Im}\rho \ge 0$ to avoid the zeros caused by the symmetry and functional equation of ζ.

Let S_1 be the multiset of imaginary parts of the elements of Z_1.

Conjecture LI_1

The ordinates of the zeros of $\zeta(s)$ are linearly independent over the rational numbers. More precisely,

$$S_1 \text{ is linearly independent over } \mathbb{Q}.$$

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Some history of LI_1

- Wintner used LI_1 to study the limiting (logarithmic) distribution of $\frac{\log x}{\sqrt{x}}(\pi(x) - \mathrm{li}(x))$, as did Montgomery (1979) and Monach (1980).
- Ingham (1942) showed that LI_1 implies:

$$\limsup_{x\to\infty} \left(x^{-1/2} \sum_{n\le x} \mu(n) \right) = +\infty.$$

  In particular, LI_1 implies that the Mertens conjecture $|M(x)| < \sqrt{x}$ is false.
- Odlyzko and te Riele (1986) unconditionally disproved the Mertens conjecture. Proof follows Ingham and makes use of numerical calculations where $\sum_{j=1}^{k} a_j\gamma_j$ is small ($\gamma_j \in S_1$).

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Analogously, for every Dirichlet character χ, define

$$Z_\chi = \{\rho\colon L(\rho,\chi)=0,\ \operatorname{Re}\rho \ge \tfrac12,\ \operatorname{Im}\rho \ge 0\}$$

and let S_χ be the multiset of the imaginary parts of the elements of Z_χ. Further, define

$$S_q = \bigcup_{\text{primitive } \chi \pmod q} S_\chi \quad \text{and} \quad S = \bigcup_{q=1}^{\infty} S_q.$$

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S is linearly independent over $\mathbb{Q}$.

- Conjecture LI appears in an article of Hooley (1977).
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Consequences of LI: Non-vanishing of $L(\frac{1}{2}, \chi)$

No linearly independent set can contain 0, so LI implies:

Conjecture

$L(\frac{1}{2}, \chi) \neq 0$ for all Dirichlet L-functions.

- For real characters, this is a conjecture of Chowla (1965).
- The proportion of real Dirichlet characters χ with $L(\frac{1}{2}, \chi) \neq 0$ is greater than $\frac{7}{8}$ (Soundararajan, 2000).

At least 34% of all even primitive Dirichlet characters χ with prime conductor satisfy $L(\frac{1}{2}, \chi) \neq 0$ (Bui, 2010).

- Builds on work of Iwaniec and Sarnak (1999) and Balasubramanian and V.K. Murty (1992).

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Consequences of LI: Simple zeros of $L(s,\chi)$

A linearly independent multiset can't in fact contain repeated elements, so LI implies:

Conjecture

All zeros of Dirichlet L-functions are simple, and no two Dirichlet L-functions share a zero.

- For any $L(s,\chi)$, at least $\frac{1}{3}$ of the zeros are simple and on the critical line (Bauer, 2000). For $\zeta(s)$, at least 41% are simple and critical (Bui, Conrey, and Young, 2010). Assuming GRH, at least $\frac{11}{12}$ of the zeros are simple (Ozluk, 1996).
- When all characters to all moduli are considered together, at least $\frac{1}{2}$ of the zeros of the $L(s,\chi)$ are simple (Conrey, Iwaniec, and Soundararajan, 2011).

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Vertical arithmetic progressions

Fix real numbers a, b and consider the arithmetic progression $\frac{1}{2} + i(a + kb)$ $(k = 1, 2, \dots)$ on the critical line.

Consequence of LI

Three terms in an arithmetic progression $a + kb$ are linearly dependent over $\mathbb{Q}$, so we expect to find at most two zeros of $L(s, \chi)$ in this arithmetic progression (and at most one zero if $a = 0$).

- Putnam (1954) proved that $\zeta(\frac{1}{2} + ikb, \chi) \neq 0$ for infinitely many values of k.
- Lapidus and van Frankenhuysen (2000) proved that $\gg T^{5/6}$ of the first T values $L(\frac{1}{2} + ikb, \chi)$ are nonzero; assuming GRH, they proved $\gg_\varepsilon T^{1-\varepsilon}$.

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More nonzero values

Theorem (M.–Ng, 2011)

Let χ be a Dirichlet character, and let a and b be real numbers with $b \neq 0$. Then

$$\#\{1 \le k \le T : L(\tfrac12 + i(a+kb), \chi) \neq 0\} \gg_{\chi,a,b} \frac{T}{\log T}.$$

- Our theorem strengthens Lapidus and van Frankenhuysen without requiring GRH, as well as extending the result to nonhomogeneous arithmetic progressions ($a \neq 0$).
- Our methods apply also to other vertical lines; however, zero-density results (Linnik, 1946) already show that $L(\sigma + i(a+kb), \chi) \neq 0$ for almost all $1 \le k \le T$ when $\sigma \neq \frac12$.
- We would like to have shown that a positive proportion of points in the arithmetic progression weren't zeros of $L(s,\chi)$, but for now this stronger statement remains open.

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The lowest nonzero value

Watkins (1998, unpublished) had determined a bound for the least k such that $L(\frac{1}{2} + ikb, \chi) \neq 0$. We can improve his bound:

Theorem (M.–Ng, 2011)

Let χ be a Dirichlet character modulo q, and let $0 \le a < b$ be real numbers. Then there exists a positive integer

$$k \ll_\varepsilon \left(q \max\{b^3, b^{-1}\}\right)^{1+\varepsilon}$$

such that $L(\frac{1}{2} + i(a + kb), \chi) \neq 0$.

- We obtain something a little more precise. For example, with $q = 1$, there exists a positive integer

$$k \ll 1 + b^3 \exp\left(17 \log b / \log\log b\right)$$

for which $\zeta(\frac{1}{2} + i(a + kb)) \neq 0$.

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such that $L(\frac{1}{2} + i(a+kb), \chi) \neq 0$.

- We obtain something a little more precise. For example, with $q = 1$, there exists a positive integer

$$k \ll 1 + b^3 \exp\left(17 \log b / \log\log b\right)$$

for which $\zeta(\frac{1}{2} + i(a+kb)) \neq 0$.

The lowest nonzero value

Watkins (1998, unpublished) had determined a bound for the least k such that $L(\frac{1}{2} + ikb, \chi) \neq 0$. We can improve his bound:

Theorem (M.–Ng, 2011)

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Outline of proof

Let $a = 2\pi\alpha$ and $b = 2\pi\beta$, and set

$$s_k = \tfrac{1}{2} + 2\pi i(\alpha + k\beta) \text{ for } k = 1, 2, \dots.$$

First and second mollified moments

Define

$$S_1(T) = \sum_{k=1}^{T} L(s_k, \chi) M(s_k) \quad \text{and} \quad S_2(T) = \sum_{k=1}^{T} |L(s_k, \chi) M(s_k)|^2,$$

where the mollifier is

$$M(s) = M_X(s) = \sum_{1 \le n \le X} \frac{\mu(n)\chi(n)}{n^s} \left(1 - \frac{\log n}{\log X}\right).$$

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Note that

$$S_1(T) = \sum_{k=1}^{T} L(s_k,\chi)M(s_k) = \sum_{\substack{1\le k\le T \\ L(s_k,\chi)\ne 0}} L(s_k,\chi)M(s_k)$$

Applying Cauchy–Schwarz:

$$S_1(T)^2 \le \Bigg(\sum_{\substack{1\le k\le T \\ L(s_k,\chi)\ne 0}} 1 \Bigg) \Bigg(\sum_{\substack{1\le k\le T \\ L(s_k,\chi)\ne 0}} |L(s_k,\chi)M(s_k)|^2 \Bigg)$$

After rearranging:

$$\#\{1 \le k \le T \colon L(s_k,\chi) \ne 0\} \ge \frac{S_1(T)^2}{S_2(T)}.$$

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Taking $X = T^{1/4}$ in the definition of $M(s)$, we have

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A linear form evaluated at zeros

Consequence of LI

Let $a_1, \dots, a_k$ be positive rational numbers. Whenever $\gamma_1, \cdots, \gamma_k$ are positive ordinates of zeros of $L(s, \chi)$, then

$$L\big(\tfrac{1}{2} + i(a_1\gamma_1 + \cdots + a_k\gamma_k), \chi\big) \neq 0$$

unless $a_1 + \cdots + a_k = 1$ and $\gamma_1 = \cdots = \gamma_k$.

Examples:

- When $a \in \mathbb{Q} \setminus \{\pm 1\}$, we expect $L(\frac{1}{2} + ia\gamma, \chi) \neq 0$. (In 2005, van Frankenhuysen verified that $\zeta(\frac{1}{2} + 2i\gamma) \neq 0$ for all $|\gamma| < 1.13 \times 10^6$).
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This will take a moment or two

An approach to this problem is to average $L(s,\chi)$ over copies of

$$I_\chi(T) = \{0 \le \gamma \le T \colon L(\tfrac12 + i\gamma, \chi) = 0\}.$$

(As before, $\#I_\chi(T) \sim \frac{T}{2\pi} \log \frac{qT}{2\pi}$.)

Try the same strategy again

If we can evaluate

$$S_1(T) = \sum_{\gamma_1,\dots,\gamma_k \in I_\chi(T)} L\big(\tfrac12 + i(a_1\gamma_1 + \cdots + a_k\gamma_k), \chi\big) \text{ and}$$

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then we can use Cauchy–Schwarz to understand the number of $(\gamma_1,\dots,\gamma_k)$ such that $L\big(\tfrac12 + i(a_1\gamma_1 + \cdots + a_k\gamma_k), \chi\big) \ne 0$.

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LI holds a lot of the time

Theorem (M.–Ng, 2011+)

Assume GRH. For fixed $0 < a_1, \dots, a_k < 1$, *we have*

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Prime number races

Definition

Let $a_1, \dots, a_r$ be distinct reduced residues $\pmod q$. We say that the prime number race among $a_1, \dots, a_r \pmod q$ is *inclusive* if, for every permutation $(\sigma_1, \dots, \sigma_r)$ of $(a_1, \dots, a_r)$, there are arbitrarily large real numbers x for which

$$\pi(x; q, \sigma_1) > \cdots > \pi(x; q, \sigma_r).$$

Rubinstein and Sarnak (1994) proved that all prime number races are inclusive—conditionally on GRH and LI.

Notation

$$I(\chi) = \{\gamma \ge 0 : L(\tfrac12 + i\gamma, \chi) = 0\} \text{ and } I(q) = \bigcup_{\chi \pmod q} I(\chi)$$

are multisets of ordinates of zeros of Dirichlet L-functions.

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Self-sufficient ordinates

Definition

We say that $\gamma \in I(q)$ is *self-sufficient* if γ cannot be written as a nontrivial finite $\mathbb{Q}$-linear combination of elements of $I(q) \setminus \{\gamma\}$.

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Weakening the LI assumption: example theorems

Theorem (M.–Ng, 2011+)

Assume GRH. If for every nonprincipal character $\chi \pmod q$,

$$\sum_{\gamma \in I^\spadesuit(\chi)} \frac{1}{\gamma}$$

diverges, then every prime number race $\pmod q$, including the full $\phi(q)$-way race, is inclusive.

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Assume GRH. Let $a, b \pmod q$ *be distinct reduced residues. If*

$$\sum_{\substack{\chi \pmod q \\ \chi(a) \neq \chi(b)}} \sum_{\gamma \in I^\spadesuit(\chi)} \frac{1}{\gamma}$$

diverges, then the two-way prime number race between a *and* $b \pmod q$ *is inclusive.*

Weakening the LI assumption: example theorems

Theorem (M.–Ng, 2011+)

Assume GRH. If for every nonprincipal character $\chi \pmod q$,

$$\sum_{\gamma \in I^{\spadesuit}(\chi)} \frac{1}{\gamma}$$

diverges, then every prime number race $\pmod q$, including the full $\phi(q)$-way race, is inclusive.

Theorem (M.–Ng, 2011+)

Assume GRH. Let $a, b \pmod q$ be distinct reduced residues. If

$$\sum_{\substack{\chi \pmod q \\ \chi(a) \ne \chi(b)}} \sum_{\gamma \in I^{\spadesuit}(\chi)} \frac{1}{\gamma}$$

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A simple challenge

To emphasize how little we know about the linear independence of zeros of Dirichlet L-functions, Silberman pointed out that the following problem is still open:

A simple challenge

To emphasize how little we know about the linear independence of zeros of Dirichlet L-functions, Silberman pointed out that the following problem is still open:

Prove that . . .

. . . there exists even one Dirichlet L-function (including the ζ-function) that has even one zero $\beta + i\gamma$ with γ irrational.

The end

My paper with Nathan on vertical arithmetic progressions

www.math.ubc.ca/~gerg/index.shtml?abstract=NVDVAP

Papers with Nathan in preparation

- Inclusive prime number races
- Nonzero values of Dirichlet L-functions at linear combinations of zeros

These slides

www.math.ubc.ca/~gerg/index.shtml?slides