Primitive sets

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joint work with Carl Pomerance and William D. Banks

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slides can be found on my web page
www.math.ubc.ca/∼gerg/index.shtml?slides

Outline

Primitive sets

Definition

A primitive set is a set $\mathcal{S} \subset \{2, 3, 4, \dots\}$ with no element dividing another: if m, n are distinct elements of $\mathcal{S}$, then $m \nmid n$.

Examples:

- $\{m, m+1, m+2, \dots, 2m-1\}$ for any $m \geq 2$
- the primes $\mathcal{P} = \{2, 3, 5, 7, 11, \dots\}$
- $\mathcal{P}_k = \{n \in \mathbb{N} \colon \Omega(n) = k\}$ for any $k \geq 2$, where $\Omega(n)$ is the number of prime factors of n counted with multiplicity. For example, $\mathcal{P}_2 = \{4, 6, 9, 10, 14, 15, 21, 22, 25, 26, \dots\}$.

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- $\mathcal{S} = \{2\} \cup \{3p \colon p \ge 3 \text{ prime}\} \cup \{5p_1p_2 \colon p_1 \ge p_2 \ge 5 \text{ prime}\}$
  $\cup \{7p_1p_2p_3 \colon p_1 \ge p_2 \ge p_3 \ge 7 \text{ prime}\} \cup \cdots$
- "Primitive abundant numbers": abundant numbers ($\sigma(n) > 2n$) without any abundant divisors. (Whence the name "primitive sets", I believe.)

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Density of primitive sets

Theorem (Erdős, 1935)

If $\mathcal{S}$ is a primitive set, then $\displaystyle\sum_{n\in\mathcal{S}} \frac{1}{n\log n}$ *converges.*

It seems like this would imply that every primitive set has density 0, but not quite. It certainly implies that every primitive set has lower density 0.

A counterintuitive set

On the other hand, Besicovitch gave a construction of primitive sets with upper density greater than $\frac{1}{2} - \delta$ for any $\delta > 0$.

In other words, if $S(x) = \#\{s \in \mathcal{S} : s \le x\}$, then $S(x) > (\frac{1}{2} - \delta)x$ for arbitrarily large x.

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A set that only works when it has to

Besicovitch's primitive sets

- Contained in $[x_1, 2x_1) \cup [x_2, 2x_2) \cup [x_3, 2x_3) \cup \cdots$ for a rapidly increasing sequence $\{x_1, x_2, x_3, \dots\}$
- Obtained from this union of integrals greedily
- $S(2x_j) > \frac{1}{2} - \delta$ for j sufficiently large
- Most of the time, the counting function $S(x)$ is very small (since $\{x_j\}$ grows so fast)

Question

How large can a primitive set's counting function be consistently?

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Consistently large primitive sets

Example

If $\mathcal{S} = \mathcal{P}$ is the set of primes, then $S(x) \sim \dfrac{x}{\log x}$.

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Theorem (Ahlswede/Khachatrian/Sárközy, 1999)

A primitive set $\mathcal{S}$ exists with $S(x) \gg \dfrac{x}{(\log\log x)(\log\log\log x)^{1+\varepsilon}}$.

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Partial summation

$\sum_{n\in\mathcal{S}} \dfrac{1}{n\log n}$ converges if and only if $\int_2^\infty \dfrac{S(x)}{x^2\log x}\,dx$ converges.

Consequently:

- By Erdős: if $\mathcal{S}$ is primitive, then $\int_2^\infty \dfrac{S(x)}{x^2\log x}\,dx$ converges.
- Impossible to have $S(x) \gg \dfrac{x}{(\log\log x)(\log\log\log x)}$, say.

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A sort of converse

Erdős plus partial summation

If $\mathcal{S}$ is primitive, then $\displaystyle\int_2^\infty \frac{S(x)}{x^2\log x}\,dx$ converges.

Theorem (M.–Pomerance, 2011)

If $F(x)$ is a "nice" function such that $\displaystyle\int_2^\infty \frac{F(x)}{x^2\log x}\,dx$ converges, then there exists a primitive set $\mathcal{S}$ with $S(x) \asymp F(x)$.

Corollary

For any $\varepsilon > 0$, there exists a primitive set $\mathcal{S}$ with

$$S(x) \gg \frac{x}{(\log\log x)(\log\log\log x)\cdots(\log_{2014} x)(\log_{2015} x)^{1+\varepsilon}}.$$

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A particular construction of primitive sets

Definition

Fix a sequence $p_1 < p_2 < \cdots$ of primes, and define

$$\mathcal{S}_k = \{n \in \mathbb{N} \colon \Omega(n) = k;\ p_k \mid n;\ p_1 \nmid n, \dots, p_{k-1} \nmid n\}.$$

Example

If $\{p_j\}$ is all the primes, then $\mathcal{S}_1 = \{2\}$, $\mathcal{S}_2 = \{3p \colon p \ge 3 \text{ prime}\}$, $\mathcal{S}_3 = \{5p_1p_2 \colon p_1 \ge p_2 \ge 5 \text{ prime}\}$, etc.

Then $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k$ is primitive.

Proof.

If $m, n \in \mathcal{S}$ are distinct and $m \mid n$, then $\Omega(m) < \Omega(n)$; but then $p_{\Omega(m)}$ divides m but not n. □

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Fix a sequence $p_1 < p_2 < \cdots$ of primes, with $\sum_{i=1}^\infty \frac{1}{p_i} < \frac{1}{2}$, whose growth rate is tied to $F(x)$. Define

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$$S_k(x) \ge \frac{1}{p_k} G\left(\frac{k-2}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-2}}{(k-2)!} \left(1 - \frac{k-3}{\log\log x} \sum_{j=1}^{k-1} \frac{1}{p_j}\right)$$

$$\gg \frac{1}{p_k} \frac{x}{\log x} \frac{(\log\log x)^{k-2}}{(k-2)!} \text{ for } k < \tfrac{3}{2} \log\log x.$$

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What's a "nice" function?

$F(x) = \frac{x}{\log_2 x \cdot \log_3 x \cdot L(\log_2 x)}$ with $L(x)$ positive and increasing, slowly varying (meaning $L(2x) \sim L(x)$), and $\int_2^\infty \frac{dt}{t \log t \cdot L(t)} < \infty$.

What is the sequence $\{p_k\}$?

The growth rate of $\{p_k\}$ is tied to $F(x)$ by setting p_k equal to the $\lfloor kL(k) \rfloor$th prime for k large enough (and then throwing away the first several primes to ensure $\sum_{i=1}^\infty \frac{1}{p_i} < \frac{1}{2}$).

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If $F(x)$ is a "nice" function such that $\int_2^\infty \frac{F(x)}{x^2 \log x}\,dx$ converges, then there exists a primitive set $\mathcal{S}$ with $S(x) \asymp F(x)$.

What's a "nice" function?

$F(x) = \frac{x}{\log_2 x \cdot \log_3 x \cdot L(\log_2 x)}$ with $L(x)$ positive and increasing, slowly varying (meaning $L(2x) \sim L(x)$), and $\int_2^\infty \frac{dt}{t \log t \cdot L(t)} < \infty$.

What is the sequence $\{p_k\}$?

The growth rate of $\{p_k\}$ is tied to $F(x)$ by setting p_k equal to the $\lfloor kL(k) \rfloor$th prime for k large enough (and then throwing away the first several primes to ensure $\sum_{i=1}^\infty \frac{1}{p_i} < \frac{1}{2}$).

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One constant to rule them all

Definition

$E(\mathcal{S}) = \sum_{n \in \mathcal{S}} \frac{1}{n \log n}$ for any $\mathcal{S} \subset \{2, 3, \dots\}$.

For $\mathcal{S}$ primitive, Erdős proved more than that $E(\mathcal{S})$ is finite; he proved that $E(\mathcal{S})$ is bounded by an absolute constant. (Erdős/Zhang, 1993: the constant 1.84 suffices.)

Conjecture (Erdős, 1988)

If $\mathcal{S}$ is primitive, then $E(\mathcal{S}) \le E(\mathcal{P}) = 1.63\dots$.

Known for certain classes of primitive $\mathcal{S}$ [Zhang 1991, 1993]:

- all elements $n \in \mathcal{S}$ satisfy $\Omega(n) \le 4$; or
- $\mathcal{S}$ is homogeneous: for $n \in \mathcal{S}$, the quantity $\Omega(n)$ depends only on smallest prime factor of n (e.g., earlier construction)

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Primitive sets with restricted primes

Definition

$$E(\mathcal{S}) = \sum_{n\in\mathcal{S}} \frac{1}{n\log n}$$

Notation: integers with restricted prime factors

For any $\mathcal{Q} \subset \mathcal{P}$, define $\mathbb{N}(\mathcal{Q}) = \{n \ge 2\colon \text{if } p \mid n, \text{ then } p \in \mathcal{Q}\}$.

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Changing the statistic

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$$E(\mathcal{S}) = \sum_{n\in\mathcal{S}} \frac{1}{n\log n} \quad \text{and} \quad E_t(\mathcal{S}) = \sum_{n\in\mathcal{S}} \frac{1}{n^t}$$

Observation / first-year calculus

$$\frac{1}{n\log n} = \int_1^\infty \frac{dt}{n^t}; \text{ therefore, } E(\mathcal{S}) = \int_1^\infty E_t(\mathcal{S})\,dt$$

If we want to show that $E(\mathcal{S}) \le E(\mathcal{Q})$ for every primitive $\mathcal{S} \subset \mathbb{N}(\mathcal{Q})$, it suffices to show that $E_t(\mathcal{S}) \le E_t(\mathcal{Q})$ for all $t > 1$.

- False if $\mathcal{Q}$ is too big (like $\mathcal{Q} = \mathcal{P}$)
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Theorem (Banks–M., 2013)

If $\displaystyle\sum_{p \in \mathcal{Q}} \frac{1}{p} \le 1 + \sqrt{1 - \sum_{p \in \mathcal{Q}} \frac{1}{p^2}}$, *then the conjecture holds.*

Corollary

If $\displaystyle\sum_{p \in \mathcal{Q}} \frac{1}{p} \le 1.74$, *then the conjecture holds.*

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Notation

$$\mathcal{T} = \{\text{all twin primes}\}, \quad \mathcal{T}_3 = \mathcal{T} \setminus \{3\}$$

Corollary

If $\mathcal{S}$ is a primitive subset of $\mathbb{N}(\mathcal{T}_3)$, then $E(\mathcal{S}) \le E(\mathcal{T}_3)$.

Brun's constant

Define B to be the sum of the reciprocals of the twin primes:
$B = (\frac{1}{3} + \frac{1}{5}) + (\frac{1}{5} + \frac{1}{7}) + (\frac{1}{11} + \frac{1}{13}) + (\frac{1}{17} + \frac{1}{19}) + (\frac{1}{29} + \frac{1}{31}) + \cdots$.

- True value believed to be $1.90216\cdots$
- Best bound known: $B < 2.347$ (Crandall/Pomerance)

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Out of all the primitive subsets of $\mathbb{N}(\mathcal{T}_3)$, we can identify the one that maximizes $\sum \frac{1}{n \log n}$ . . . even though we can't say whether that optimal set is finite or infinite!

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Suppose that $B < 2.09596$. If $\mathcal{S}$ is a primitive subset of $\mathbb{N}(\mathcal{T})$, then $E(\mathcal{S}) \le E(\mathcal{T})$.

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The end

The two papers described in this talk, as well as these slides, are available for downloading.

Primitive sets with large counting functions (with C.P.)

www.math.ubc.ca/~gerg/
index.shtml?abstract=PSLCF

Optimal primitive sets with restricted primes (with B.B.)

www.math.ubc.ca/~gerg/
index.shtml?abstract=OPSRP

These slides

www.math.ubc.ca/~gerg/index.shtml?slides