Dense Egyptian fractions

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Outline

Egyptian fractions

Definition

Let r be a positive rational number. An Egyptian fraction for r is a sum of reciprocals of distinct positive integers that equals r.

Example

$$1 = 1/2 + 1/3 + 1/6$$

Theorem (Fibonacci 1202, Sylvester 1880, . . .)

Every positive rational number has an Egyptian fraction representation. (Proof: greedy algorithm.)

Note: we'll restrict to $r = 1$ for most of the remainder of the talk; but everything holds true for any positive rational number r.

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Demoralizing Egyptian scribes

Question

How many terms can an Egyptian fraction for 1 have?

Cheap answer

Arbitrarily many, by the splitting trick:

$$\begin{aligned} 1 &= 1/2 + 1/3 + 1/6 \\ &= 1/2 + 1/3 + 1/7 + 1/(6 \times 7) \\ &= 1/2 + 1/3 + 1/7 + 1/43 + 1/(42 \times 43) = \dots \end{aligned}$$

But the denominators become enormous.

Better question

How many terms can an Egyptian fraction for 1 have, if the denominators are bounded by x?

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- $\mathcal{S}$ has 454 elements, all bounded by 1000

What's best possible?

Suppose that there are t denominators, all bounded by x, in an Egyptian fraction for 1. Then

$$1 = \sum_{j=1}^{t} \frac{1}{n_j} \ge \sum_{n=x-t+1}^{x} \frac{1}{n} \sim \log \frac{x}{x-t}.$$

So $e \gtrsim \frac{x}{x-t}$, giving an upper bound for the number of terms:

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Lemma ("No tiny multiples of huge primes")

If a prime p divides a denominator in an Egyptian fraction for 1 whose denominators are at most x, then $p \lesssim x/\log x$.

Proof

- If $pd_1, \dots, pd_j$ are all the denominators that are divisible by p, then $\frac{1}{pd_1} + \cdots + \frac{1}{pd_j}$ can't have p dividing the denominator when reduced to lowest terms.
- Its numerator $\operatorname{lcm}[d_1, \dots, d_j](\frac{1}{d_1} + \cdots + \frac{1}{d_j})$ is a multiple of p.
- If $M = \max\{d_1, \dots, d_j\}$, then
$$p \lesssim \operatorname{lcm}[1, \dots, M] \log M < e^{(1+o(1))M}.$$
- Therefore $\log p \lesssim M \le \frac{x}{p}$.

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Note: most places in this talk, when I say "prime" I really should be saying "prime power".

Using this lemma, it's easy to show that the number t of terms in an Egyptian fraction for 1 whose denominators are at most x satisfies

$$t \lesssim \left(1 - \frac{1}{e}\right)x - \delta \frac{x \log\log x}{\log x} \qquad \text{for some } \delta > 0.$$

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Theorem (M., 2000)

Given $x \geq 6$, there is an Egyptian fraction for 1 with $(1 - \frac{1}{e})x + O(x \log\log x / \log x)$ terms and every denominator bounded by x.

Method of proof (Croot; M.)

- Start with a large set $\mathcal{S}$ of integers not exceeding x so that $\frac{A}{B} = \sum_{n \in \mathcal{S}} \frac{1}{n}$ is approximately 1.
- Considering the primes q dividing B one by one, delete or add a few terms of $\mathcal{S}$ so that q doesn't divide the new denominator B. Make the deleted/added elements large, so that their small reciprocals don't affect the sum much.
- Sincerely hope that everything works out in the end.

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A desired congruence

Definition

Given an Egyptian fraction $\frac{A}{B} = \sum_{n \in \mathcal{S}} \frac{1}{n}$ and a prime q dividing B, define $a \equiv A(B/q)^{-1} \pmod{q}$.

- When deleting elements from $\mathcal{S}$: want to find a set $\mathcal{K}$ such that $q\mathcal{K} \subset S$ and $\sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q}$. Then the denominator of $\sum_{n \in \mathcal{S} \setminus q\mathcal{K}} \frac{1}{n} = \frac{A}{B} - \sum_{m \in \mathcal{K}} \frac{1}{qm}$ is no longer divisible by q.
- When adding elements to $\mathcal{S}$: want to find a set $\mathcal{K}$ such that $q\mathcal{K} \cap S = \emptyset$ and $\sum_{m \in \mathcal{K}} m^{-1} \equiv -a \pmod{q}$.
- To keep all new terms distinct, make sure the prime factors of the elements of $\mathcal{K}$ are always less than q.

Notation: $q\mathcal{K} = \{qm \colon m \in \mathcal{K}\}$

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Adapting Croot's technique

Proposition (suitable for large primes q)

Given a prime q, let $\log q < B < q$. Let $\mathcal{M}$ be a set of at least $B^{2/3}(\log q)^2$ integers not exceeding B, each of which is of the form $p_1 p_2$. Then for any integer a, there exists a subset $\mathcal{K}$ of $\mathcal{M}$ such that $\sum_{m\in\mathcal{K}} m^{-1} \equiv a \pmod q$.

Proof: The number of such subsets equals (with $e(x) = e^{2\pi i x}$)

$$\sum_{\mathcal{K}\subset\mathcal{M}} \frac{1}{q}\sum_{h=0}^{q-1} e\left(\frac{h}{q}\left(\sum_{m\in\mathcal{K}} m^{-1} - a\right)\right) = \frac{1}{q}\sum_{h=0}^{q-1} e\left(-\frac{ha}{q}\right) \prod_{m\in\mathcal{M}} \left(1 + e\left(\frac{hm^{-1}}{q}\right)\right).$$

A pigeonhole argument (on the divisors of some auxiliary integer A, which is where the form p_1p_2 is used) shows that for $h \neq 0$, lots of the $hm^{-1} \pmod q$ must be reasonably far from 0, which gives cancellation in the product.

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Proof: The number of such subsets equals (with $e(x) = e^{2\pi i x}$)

$$\sum_{\mathcal{K}\subset\mathcal{M}} \frac{1}{q}\sum_{h=0}^{q-1} e\left(\frac{h}{q}\left(\sum_{m\in\mathcal{K}} m^{-1} - a\right)\right) = \frac{1}{q}\sum_{h=0}^{q-1} e\left(-\frac{ha}{q}\right) \prod_{m\in\mathcal{M}} \left(1 + e\left(\frac{hm^{-1}}{q}\right)\right).$$

A pigeonhole argument (on the divisors of some auxiliary integer A, which is where the form p_1p_2 is used) shows that for $h \neq 0$, lots of the $hm^{-1} \pmod{q}$ must be reasonably far from 0, which gives cancellation in the product.

Adapting Croot's technique

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Small prime powers, explicitly

For the small prime powers $q_1 = 2$, $q_2 = 3$, $q_3 = 4$, ..., we add to $\mathcal{S}$ the single denominator $n_j = \operatorname{lcm}[1, \dots, q_j]/b$, where $b \in [1, q_j - 1]$ is chosen to make the earlier congruence hold.

Denominators are small enough, but not too small

- The n_j are less than x when $q_j < (1-\varepsilon)\log x$, say.
- Since n_j is at least $\operatorname{lcm}[1, \dots, q_j]/(q_j - 1)$, the sum of their reciprocals is (as Croot observed) less than the telescoping sum

$$\sum_{j=1}^{\infty} \frac{q_j - 1}{\operatorname{lcm}[1, \dots, q_j]} = \sum_{j=1}^{\infty} \left(\frac{1}{\operatorname{lcm}[1, \dots, q_{j-1}]} - \frac{1}{\operatorname{lcm}[1, \dots, q_j]} \right) = 1.$$

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The construction

To start: Let $\mathcal{S}$ be the set of all integers between $\frac{x}{e}$ and x that are not divisible by a prime larger than $x/(\log x)^{22}$.

- Cardinality of $\mathcal{S}$ is $(1-\frac{1}{e})x + O(x\log\log x/\log x)$
- $\sum_{n\in\mathcal{S}} \frac{1}{n} \sim 1 - 22\log\log x/\log x$

Large q: Delete a few elements from $\mathcal{S}$ for every large prime, by the earlier proposition.

- In all, delete $O(x/\log x)$ elements from the original $\mathcal{S}$
- $\sum_{n\in\mathcal{S}} \frac{1}{n} \lesssim 1 - 22\log\log x/\log x$

Small q: Finally, add at most 1 element to $\mathcal{S}$ for every small prime, as in the previous slide.

- Final cardinality of $\mathcal{S}$ is $\gtrsim (1-\frac{1}{e})x + O(x\log\log x/\log x)$
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The largest denominator

“Impossible integers”

Which integers *can’t* be the largest denominator in an Egyptian fraction for 1?

We’ve already seen “no tiny multiples of huge primes”; so the number of these impossible integers up to x is

$$\gtrsim \frac{x \log\log x}{\log x}.$$

Erdős and Graham asked:

Does the set of impossible integers have positive density, or even density 1?

It turns out the answer is **no**.

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Theorem (M., 2000)

The number of integers up to x that cannot be the largest denominator in an Egyptian fraction for 1 is $\ll x \log\log x / \log x$.

Proof.

Let m be any integer such that $p \mid m$ implies $p < m(\log m)^{-22}$. The previous construction works for the rational number $r = 1 - \frac{1}{m}$, since the initial set $\mathcal{S}$ of all integers between $\frac{m}{e}$ and $m - 1$ that are not divisible by a prime larger than $m/(\log m)^{22}$ contains all prime factors of the denominator of r. □

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The next Erdős–Graham question

Which integers cannot be the second-largest denominator in an Egyptian fraction for 1? Positive density?

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Given a large integer m, choose an integer $M \equiv -1 \pmod{m}$ such that $p \mid M$ implies $p < m(\log m)^{-22}$. Then apply the previous construction to $r = 1 - \frac{1}{m} - \frac{1}{Mm} = 1 - \frac{(M+1)/m}{M}$. □

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- The splitting trick immediately implies: for any $j \geq 2$, all but finitely many positive integers can be the jth-largest denominator in an Egyptian fraction for 1.

Issues to settle computationally

All of the implicit constants in the above theorems are effectively computable; so in principle, we know enough to settle the following questions:

Conjecture 1

If $m \geq 5$, then m can be the second-largest denominator in an Egyptian fraction for 1. (Note that $m = 2$ and $m = 4$ cannot.)

Conjecture 2

If $m \geq 2$ and $j \geq 3$, then m can be the jth-largest denominator in an Egyptian fraction for 1. (Our methods establish this for $j \geq j_0$, where j_0 is some effectively computable constant.)

Note: By splitting trickery, Conjecture 1 implies Conjecture 2.

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The end

Relevant papers of mine

- *Dense Egyptian fractions*
  www.math.ubc.ca/~gerg/index.shtml?abstract=DEF
- *Denser Egyptian fractions*
  www.math.ubc.ca/~gerg/index.shtml?abstract=DrEF

These slides

www.math.ubc.ca/~gerg/index.shtml?slides