Dimensions of spaces of newforms

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slides can be found on my web page
www.math.ubc.ca/∼gerg/index.shtml?slides

Outline

Prelude: Dirichlet characters

Definition

Let $C(n)$ be the group of Dirichlet characters mod n. We know that the cardinality of $C(n)$ is $\phi(n)$.

For every $d \mid n$, we have an injective map $i_{d,n}$ from $C(d)$ to $C(n)$: each character $\chi \in C(d)$ induces a character $i_{d,n}(\chi) = \chi\chi_0 \in C(n)$, where χ_0 is the principal character mod n.

Definition (primitive characters $(\bmod\ n)$)

Define $C_{prim}(n) = C(n) \setminus \bigcup_{\substack{d|n \\ d\neq n}} i_{d,n}(C(d))$.

Question

What is the cardinality of $C_{prim}(n)$?

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Everybody is primitive somewhere

$$C_{prim}(n) = C(n) \setminus \bigcup_{\substack{d|n \\ d \neq n}} i_{d,n}(C(d))$$

It turns out that the sets $i_{d,n}(C_{prim}(d))$ are disjoint: every character mod n is induced by a unique primitive character modulo one divisor of n. Thus we have the disjoint union

$$C(n) = C_{prim}(n) \cup \left(\bigcup_{\substack{d|n \\ d \neq n}} i_{d,n}(C_{prim}(d)) \right) = \bigcup_{d|n} i_{d,n}(C_{prim}(d)).$$

Define $\phi_{prim}(n)$ to be the cardinality of $C_{prim}(n)$, the number of primitive characters mod n. Then the above disjoint union gives

$$\phi(n) = \sum_{d|n} \phi_{prim}(d).$$

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Formula for the number of primitive characters

Möbius inversion to the rescue

$\phi(n) = \sum_{d|n} \phi_{prim}(d)$ is equivalent to $\phi_{prim}(n) = \sum_{d|n} \phi(d)\mu(n/d)$.

Explicit formula

ϕ_{prim} is the multiplicative function satisfying $\phi_{prim}(p^\alpha) = \phi(p^\alpha) - \phi(p^{\alpha-1})$, that is,

$$\phi_{prim}(p) = p - 2, \quad \phi_{prim}(p^\alpha) = p^{\alpha-2}(p-1)^2 \text{ for } \alpha \geq 2.$$

Notation: Dirichlet convolution

$$f * g(n) = \sum_{d|n} f(d)g(n/d)$$

Example: $\phi_{prim} = \phi * \mu$, while $\phi = \phi_{prim} * 1$ (μ is the inverse of 1).

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Cusp forms on $\Gamma_0(N)$

Notation

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \colon c \equiv 0 \pmod{N} \right\}$$

Definition (weight-k cusp forms on $\Gamma_0(N)$)

Let $S_k(\Gamma_0(N))$ denote the $\mathbb{C}$-vector space of functions f that are holomorphic on the upper half-plane $\Im z > 0$, and "holomorphic and zero at cusps", that satisfy

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

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Dimension of space of cusp forms

Notation

Let $g_0(k,N)$ denote the dimension of $S_k(\Gamma_0(N))$.

Proposition

For any even integer $k \geq 2$ and any integer $N \geq 1$,

$$g_0(k,N) = \frac{k-1}{12} N s_0(N) - \frac{1}{2}\nu_\infty(N) + c_2(k)\nu_2(N) + c_3(k)\nu_3(N) + \delta\left(\frac{k}{2}\right).$$

- s_0, ν_∞, ν_2, and ν_3 are certain multiplicative functions related to $\Gamma_0(N)$.

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- s_0 is the multiplicative function satisfying $s_0(p^\alpha) = 1 + \frac{1}{p}$ for all $\alpha \ge 1$.
- $Ns_0(N)$ is the index of $\overline{\Gamma}_0(N)$ in $\overline{SL_2}(\mathbb{Z})$, where $\overline{G}$ denotes the quotient of the group G by its center.

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- ν_∞ is the multiplicative function satisfying:
  - $\nu_\infty(p^\alpha) = 2p^{(\alpha-1)/2}$ if α is odd;
  - $\nu_\infty(p^\alpha) = p^{\alpha/2} + p^{\alpha/2-1}$ if α is even.
- $\nu_\infty(N)$ counts the number of (inequivalent) cusps of $\Gamma_0(N)$.

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- ν_2 is the multiplicative function satisfying:
  - $\nu_2(2) = 1$, and $\nu_2(2^\alpha) = 0$ for $\alpha \ge 2$;
  - if $p \equiv 1 \pmod 4$ then $\nu_2(p^\alpha) = 2$ for $\alpha \ge 1$;
  - if $p \equiv 3 \pmod 4$ then $\nu_2(p^\alpha) = 0$ for $\alpha \ge 1$.
- $\nu_2(N)$ counts the number of (inequivalent) elliptic points of $\Gamma_0(N)$ of order 2.

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- ν_3 is the multiplicative function satisfying:
  - $\nu_3(3) = 1$, and $\nu_3(3^\alpha) = 0$ for $\alpha \ge 2$;
  - if $p \equiv 1 \pmod 3$ then $\nu_3(p^\alpha) = 2$ for $\alpha \ge 1$;
  - if $p \equiv 2 \pmod 3$ then $\nu_3(p^\alpha) = 0$ for $\alpha \ge 1$.
- $\nu_3(N)$ counts the number of (inequivalent) elliptic points of $\Gamma_0(N)$ of order 3.

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- $c_2(k) = \frac{1}{4} + \left\lfloor \frac{k}{4} \right\rfloor - \frac{k}{4}$, so $c_2(k) \in \left\{-\frac{1}{4}, \frac{1}{4}\right\}$ for k even
- $c_3(k) = \frac{1}{3} + \left\lfloor \frac{k}{3} \right\rfloor - \frac{k}{3}$, so $c_3(k) \in \left\{-\frac{1}{3}, 0, \frac{1}{3}\right\}$
- $\delta(m) = 1$ if $m = 1$, and $\delta(m) = 0$ otherwise

Where that dimension formula comes from

We assume $N \ge 2$ and $k \ge 4$ to simplify the exposition.

Notation

Let g_N denote the genus of the (compactified) quotient of the upper half-plane by $\Gamma_0(N)$.

Formula for the genus

$$g_N = \frac{Ns_0(N)}{12} - \frac{\nu_\infty(N)}{2} - \frac{\nu_2(N)}{4} - \frac{\nu_3(N)}{3} + 1$$

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- The dimension $g_0(k,N)$ of the space of weight-k cusp forms on $\Gamma_0(N)$ is calculated by the Riemann–Roch theorem:

$$g_0(k,N) = (k-1)(g_N - 1) + \left(\tfrac{k}{2} - 1\right)\nu_\infty(N) + \left\lfloor \tfrac{k}{4} \right\rfloor \nu_2(N) + \left\lfloor \tfrac{k}{3} \right\rfloor \nu_3(N).$$

- Collecting the multiples of $\nu_\infty(N)$, $\nu_2(N)$, and $\nu_3(N)$ yields

$$g_0(k,N) = \tfrac{k-1}{12} Ns_0(N) - \tfrac{1}{2}\nu_\infty(N) + \left(\tfrac{1}{4} - \tfrac{k}{4} + \left\lfloor \tfrac{k}{4} \right\rfloor\right)\nu_2(N) + \left(\tfrac{1}{3} - \tfrac{k}{3} + \left\lfloor \tfrac{k}{3} \right\rfloor\right)\nu_3(N).$$

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Newforms

- If $f(z)$ is a cusp form on $\Gamma_0(d)$, then $f(mz)$ is a cusp form on $\Gamma_0(N)$ for any multiple N of dm.
- Thus for every triple (m,d,N) of positive integers with $dm \mid N$, we have an injection $i_{m,d,N} : S_k(\Gamma_0(d)) \to S_k(\Gamma_0(N))$.

Definition ($S_k^\#(\Gamma_0(N))$)

$$S_k^\#(\Gamma_0(N)) = \Bigg(\operatorname{span} \Big\langle \bigcup_{\substack{d \mid N \\ d \ne N}} \bigcup_{m \mid N/d} i_{m,d,N}\big(S_k(\Gamma_0(d))\big) \Big\rangle \Bigg)^{\perp},$$

where $\perp$ denotes the orthogonal complement with respect to the Petersson inner product in $S_k(\Gamma_0(N))$.

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- The cusp forms comprising $S_k^{\#}(\Gamma_0(N))$ are called newforms.

Proposition (Atkin–Lehner decomposition)

We can write $S_k(\Gamma_0(N))$ as a direct product of subspaces:

$$S_k(\Gamma_0(N)) = \bigoplus_{d|N} \bigoplus_{m|N/d} i_{m,d,N}\big(S_k^{\#}(\Gamma_0(d))\big).$$

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Relating dimensions

Atkin–Lehner decomposition

$$S_k(\Gamma_0(N)) = \bigoplus_{d|N} \bigoplus_{m|N/d} i_{m,d,N}\big(S_k^{\#}(\Gamma_0(d))\big)$$

- Recall that $g_0(k,N)$ denotes the dimension of $S_k(\Gamma_0(N))$.
- Let $g_0^{\#}(k,N)$ denote the dimension of $S_k^{\#}(\Gamma_0(N))$.
- Let $\tau(m)$ denote the number of positive divisors of m.

Corollary

$$g_0(k,N) = \sum_{d|N} \sum_{m|N/d} g_0^{\#}(k,d) = \sum_{d|N} g_0^{\#}(k,d)\tau(N/d)$$

- Put another way: $g_0 = g_0^{\#} * \tau$ for any fixed k, where $*$ is the Dirichlet convolution of two arithmetic functions.

Relating dimensions

Atkin–Lehner decomposition

$$S_k(\Gamma_0(N)) = \bigoplus_{d|N} \bigoplus_{m|N/d} i_{m,d,N}\big(S_k^{\#}(\Gamma_0(d))\big)$$

- Recall that $g_0(k,N)$ denotes the dimension of $S_k(\Gamma_0(N))$.
- Let $g_0^{\#}(k,N)$ denote the dimension of $S_k^{\#}(\Gamma_0(N))$.
- Let $\tau(m)$ denote the number of positive divisors of m.

Corollary

$$g_0(k,N) = \sum_{d|N} \sum_{m|N/d} g_0^{\#}(k,d) = \sum_{d|N} g_0^{\#}(k,d)\tau(N/d)$$

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Solving for $g_0^\#(k,N)$

Notation

Let $1(n) = 1$ denote the constant function, and let $\mu(n)$ denote the Möbius function. Note that $1 * \mu = \delta$.

Definition

Define λ to be the Dirichlet-convolution inverse of τ. Since $\tau = 1 * 1$, we have $\lambda = \mu * \mu$; equivalently, λ is the multiplicative function satisfying

$$\lambda(p) = -2, \quad \lambda(p^2) = 1, \quad \lambda(p^\alpha) = 0 \text{ for } \alpha \ge 3.$$

- Since $g_0 = g_0^\# * \tau$, it follows that $g_0^\# = g_0 * \lambda$, that is,

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Dimension of space of newforms

Theorem (M., 2005)

For any even integer $k \geq 2$ and any integer $N \geq 1$, the dimension $g_0^\#(k,N)$ of the space $S_k^\#(\Gamma_0(N))$ of newforms equals

$$\tfrac{k-1}{12} N s_0^\#(N) - \tfrac{1}{2}\nu_\infty^\#(N) + c_2(k)\nu_2^\#(N) + c_3(k)\nu_3^\#(N) + \delta\big(\tfrac{k}{2}\big)\mu(N).$$

- $s_0^\#$, $\nu_\infty^\#$, $\nu_2^\#$, and $\nu_3^\#$ are certain multiplicative functions.

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- $s_0^\#$ is the multiplicative function satisfying:
  - $s_0^\#(p) = 1 - \frac{1}{p}$;
  - $s_0^\#(p^2) = 1 - \frac{1}{p} - \frac{1}{p^2}$;
  - $s_0^\#(p^\alpha) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)$ if $\alpha \geq 3$.

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- $\nu_\infty^\#$ is the multiplicative function satisfying:
  - $\nu_\infty^\#(p^\alpha) = 0$ if α is odd;
  - $\nu_\infty^\#(p^2) = p - 2$;
  - $\nu_\infty^\#(p^\alpha) = p^{\alpha/2-2}(p-1)^2$ if $\alpha \geq 4$ is even.
- Note that $\nu_\infty^\#$ is supported on squares.

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- $\nu_2^\#$ is the multiplicative function satisfying:
  - $\nu_2^\#(2) = -1$, $\nu_2^\#(4) = -1$, $\nu_2^\#(8) = 1$, and $\nu_2^\#(2^\alpha) = 0$ for $\alpha \geq 4$;
  - if $p \equiv 1 \pmod 4$ then $\nu_2^\#(p) = 0$, $\nu_2^\#(p^2) = -1$, and $\nu_2^\#(p^\alpha) = 0$ for $\alpha \geq 3$;
  - if $p \equiv 3 \pmod 4$ then $\nu_2^\#(p) = -2$, $\nu_2^\#(p^2) = 1$, and $\nu_2^\#(p^\alpha) = 0$ for $\alpha \geq 3$.

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- $\nu_3^\#$ is the multiplicative function satisfying:
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Exact evaluations are easier

$$g_0^\#(k,N) = \tfrac{k-1}{12} N s_0^\#(N) - \tfrac12 \nu_\infty^\#(N) + c_2(k)\nu_2^\#(N) + c_3(k)\nu_3^\#(N) + \delta\big(\tfrac k2\big)\mu(N)$$

Having a closed-form formula instead of a recursive formula lets us better analyze its values, whether exactly or approximately.

- For example, the following corollary and theorem were useful in work of Bennett/Győry/Mignotte/Pintér (Compos. Math., 2006) on binomial Thue equations, and Bennett/Bruin/Győry/Hajdu (Proc. London Math. Soc., 2006) on products of terms in arithmetic progression.

Corollary

Let $M \ge 3$ be an odd, squarefree integer. Then

$$g_0^\#(k,32M) = (k-1)\phi(M).$$

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Lemma

- $0 \le N s_0^\#(N) \le \phi(N)$
- $0 \le \nu_\infty^\#(N) \le \sqrt{N}$
- $|\nu_2^\#(N)| \le 2^{\omega(N)}$
- $|\nu_3^\#(N)| \le 2^{\omega(N)}$

It follows that $g_0^\#(2,N) \le \frac{1}{12}\phi(N) + \frac{7}{12}2^{\omega(N)} + 1$.

Theorem (M., 2005)

$g_0^\#(2,N) \le (N+1)/12$, *with equality holding if and only if either* $N = 35$ *or* N *is a prime that is congruent to* $11 \pmod{12}$.

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Upper and lower bounds

Two constants

- Euler's constant $\gamma = \lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n} - \log x \right) \approx 0.577216$
- Define $A = \prod_p \left(1 - \frac{1}{p^2-p}\right) \approx 0.373956$

Theorem (M., 2005)

For all even integers $k \ge 2$ and all integers $N \ge 2$:

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- $\frac{A(k-1)}{12}\phi(N) + O(\sqrt{N}) < g_0^\#(k,N) < \frac{k-1}{12}\phi(N) + O(2^{\omega(N)})$
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Note: All of these bounds are best possible.

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Range of $g_0^\#(2, N)$

The lower bound for $g_0^\#(2, N)$ means that we can make exhaustive lists of levels N for which a given value is attained.

Example

The 40 solutions to $g_0^\#(2, N) = 100$ are:

$N =$ 1213, 1331, 2169, 2583, 2662, 2745, 3208, 3232, 3465, 3608, 4040, 4302, 4338, 4772, 4804, 4848, 5084, 5092, 5166, 5252, 5324, 5490, 5572, 5904, 6336, 6820, 6930, 7056, 7188, 7212, 7920, 8052, 8484, 8652, 8676, 8940, 9060, 10332, 10980, 13860.

Example

There are exactly 2,965 integers N for which $g_0^\#(2, N) \le 100$.

Conjecture (verified for $G \le 67{,}000$)

For every nonnegative integer G, there is at least one positive integer N such that $g_0^\#(2, N) = G$.

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Example

There are exactly 2,965 integers N for which $g_0^\#(2,N) \le 100$.

Conjecture (verified for $G \le 67{,}000$)

For every nonnegative integer G, there is at least one positive integer N such that $g_0^\#(2,N) = G$.

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The lower bound for $g_0^\#(2,N)$ means that we can make exhaustive lists of levels N for which a given value is attained.

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The analogous conjecture turns out to be false for $g_0(2,N)$ itself.

Example

The omitted values up to 1000 are:

$$g_0(2,N) \neq 150, 180, 210, 286, 304, 312, 336, 338, 348, 350, 480, 536, 570, 598, 606, 620, 666, 678, 706, 730, 756, 780, 798, 850, 876, 896, 906, 916, 970.$$

Csirik–Wetherell–Zieve calculations

- The first several thousand omitted values of $g_0(2,N)$ are even, but there are odd omitted values: the first is 49,267.
- The range of the function $g_0(2,N)$ actually has density zero in the positive integers.

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Average orders

Theorem (M., 2005)

- The average order of $g_0(k,N)$ is $(k-1)\frac{5}{4\pi^2}N$. In other words, $\sum_{N\le X} g_0(k,N) \sim \sum_{N\le X} (k-1)\frac{5}{4\pi^2}N$.
- Let $g_0^*(k,N)$ denote the number of nonisomorphic automorphic representations associated with $S_k(\Gamma_0(N))$. This number can be interpreted as the dimension of a particular subspace of $S_k(\Gamma_0(N))$ that contains $S_k^\#(\Gamma_0(N))$. Then the average order of $g_0^*(k,N)$ is $(k-1)\frac{15}{2\pi^4}N$.
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Gekeler's theorem

The number $g_0^*(k,N)$ of nonisomorphic automorphic representations associated with $S_k(\Gamma_0(N))$ is a similar linear combination of explicit multiplicative functions:

$$\frac{k-1}{12}Ns_0^*(N) - \frac{1}{2}\nu_\infty^*(N) + c_2(k)\nu_2^*(N) + c_3(k)\nu_3^*(N) + \delta\left(\frac{k}{2}\right)\delta(N).$$

Corollary

Let $k \ge 2$ be an even integer, and let $N \ge 2$ be a squarefree integer. Then $g_0^*(k,N) = \frac{k-1}{12}N - \frac{1}{2} + c_2(k)\left(\frac{-1}{N}\right) + c_3(k)\left(\frac{-3}{N}\right)$. In particular, $g_0^*(k,N)$ depends upon the residue class $N \pmod{12}$ but not upon the prime factorization of N.

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Cusp forms on $\Gamma_1(N)$

Notation

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \colon a \equiv d \equiv 1 \pmod{N},\ c \equiv 0 \pmod{N} \right\}$$

For $k \geq 2$ (not necessarily even), let $g_1(k,N)$ denote the dimension of the space of weight-k cusp forms on $\Gamma_1(N)$, and let $g_1^\#(k,N)$ denote the dimension of the space of weight-k newforms on $\Gamma_1(N)$.

Theorem (M., 2005)

For any integer $k \geq 2$:

- *The average order of $g_1(k,N)$ is $(k-1)N^2/24\zeta(3)$.*
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Lots of newforms

How many cusp forms on $\Gamma_1(N)$ are newforms?

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For all integers $k \ge 2$ and all integers $N \ge 1$ such that $g_1(k,N) \ne 0$,

$$\frac{g_1^\#(k,N)}{g_1(k,N)} > \frac{B\pi^2}{6} + O\bigg(\frac{1}{\log N \log\log N} + \frac{k}{N}\bigg),$$

where $B = \prod_p \big(1 - \frac{3}{p^2}\big) \approx 0.125487$.

Note that $\frac{B\pi^2}{6} \approx 0.206418$; we deduce that when N is large enough with respect to k, at least 20% of the space of weight-k cusp forms on $\Gamma_1(N)$ is taken up by newforms.

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- investigated $S_k(N, \psi)$, the space of cusp forms on $\Gamma_0(N)$ with character ψ
- characterized simultaneous Hecke eigenforms that are not newforms
- sufficient conditions for non-diagonalizability of Hecke operators on $S_k(N, \psi)$, from lower bounds on dimension of space of newforms $S_k^\#(N, \psi)$

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The end

The paper *Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$*, as well as these slides, are available for downloading:

The paper

`www.math.ubc.ca/~gerg/`
`index.shtml?abstract=DSCFN`

The slides

`www.math.ubc.ca/~gerg/index.shtml?slides`