Subgroups of the multiplicative group

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joint work with Lee Troupe

Analytic Number Theory
2017 CMS Winter Meeting
University of Waterloo
December 10, 2017

slides can be found on my web page
www.math.ubc.ca/∼gerg/index.shtml?slides

Outline

1. Background, and distribution of (ϕ-)additive functions
2. Results on the distribution of the number of subgroups of $\mathbb{Z}_n^\times$
3. Outline of the proofs

Our objects of study

Definition

The "multiplicative group" (or unit group) modulo n is $\mathbb{Z}_n^\times = (\mathbb{Z}/n\mathbb{Z})^\times$, the group of reduced residue classes under multiplication $(\mathrm{mod}\ n)$.

$\mathbb{Z}_n^\times$ is some finite abelian group with $\phi(n)$ elements (usually not cyclic). Questions about its structure often turn into number theory (example: its exponent is the Carmichael λ-function).

Overarching question (I heard it from Shparlinski)

How many subgroups does $\mathbb{Z}_n^\times$ usually have?

Notation (used throughout the talk)

$I(n)$ is the number of isomorphism classes of subgroups of $\mathbb{Z}_n^\times$.
$G(n)$ is the number of subsets of $\mathbb{Z}_n^\times$ that are subgroups (that is, subgroups not up to isomorphism).

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Distribution results: different strengths

By way of analogy: some historical results about the distribution of $\omega(n)$, the number of distinct prime factors of n.

- The average value of $\omega(n)$ is $\log\log n$.

  — requires an asymptotic formula for $\sum_{n\le x}\omega(n)$
- The normal order (typical size) of $\omega(n)$ is $\log\log n$.

  — requires estimate for variance $\sum_{n\le x}\big(\omega(n)-\log\log n\big)^2$
- Erdős–Kac theorem: $\omega(n)$ is asymptotically distributed like a normal random variable with mean $\log\log n$ and variance $\log\log n$. (More precise statement on next slide.)

  — requires asymptotic formulas for all central moments $\sum_{n\le x}\big(\omega(n)-\log\log n\big)^k$

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Erdős–Kac laws

Definition

A function $f(n)$ satisfies an Erdős–Kac law with mean $\mu(n)$ and variance $\sigma^2(n)$ if

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \le x\colon \frac{f(n)-\mu(n)}{\sigma(n)} < u\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}\,dt$$

for every real number u.

Standard notation

$\omega(n)$ is the number of distinct prime factors of n.
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Theorem (Erdős–Kac, 1940)

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Other functions with Erdős–Kac laws

The paper of Erdős–Kac establishes these normal-distribution laws for a large class of additive functions: if $n = p_1^{r_1} \cdots p_k^{r_k}$, then $f(n) = f(p_1^{r_1}) + \cdots + f(p_k^{r_k})$. Examples of non-additive functions:

Liu (2007)

On GRH, $\omega(\#E(\mathbb{F}_p))$ satisfies an Erdős–Kac law with mean $\log\log p$ and variance $\log\log p$.

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$\omega(\phi(n))$ and $\Omega(\phi(n))$ satisfy Erdős– Kac laws with mean $\frac{1}{2}(\log\log n)^2$ and variance $\frac{1}{3}(\log\log n)^3$.

$\Omega(\phi(n))$ is not additive, but is "ϕ-additive": if $\phi(n) = p_1^{r_1} \cdots p_k^{r_k}$, then $\Omega(\phi(n)) = \Omega(p_1^{r_1}) + \cdots + \Omega(p_k^{r_k})$.

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The number of subgroups has a similar property

Reminder of notation

$I(n)$ is the number of isomorphism classes of subgroups of $\mathbb{Z}_n^\times$.
$G(n)$ is the number of subsets of $\mathbb{Z}_n^\times$ that are subgroups.

Every finite abelian group is the direct sum of its p-Sylow subgroups, so consequently:

If $G_p(n)$ denotes the number of subgroups of the p-Sylow subgroup of $\mathbb{Z}_n^\times$, then $G(n) = \prod_{p \mid \#\mathbb{Z}_n^\times} G_p(n) = \prod_{p \mid \phi(n)} G_p(n)$.

And similarly for $I(n)$.

In particular, both $I(n)$ and $G(n)$ are “ϕ-multiplicative” functions; so we might hope to get strong distributional information for the ϕ-additive functions $\log I(n)$ and $\log G(n)$.

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Erdős–Kac laws for the number of subgroups

Theorem (M.-Troupe, submitted)

$\log I(n)$ *satisfies an Erdős–Kac law with mean* $\frac{\log 2}{2}(\log\log n)^2$ *and variance* $\frac{\log 2}{3}(\log\log n)^3$.

How did we prove this?

We showed that $\omega(\phi(n))\log 2 \le \log I(n) \le \Omega(\phi(n))\log 2$, and then quoted Erdős–Pomerance. $\square$

Theorem (M.-Troupe, submitted)

$\log G(n)$ *satisfies an Erdős–Kac law with mean* $A(\log\log n)^2$ *and variance* $C(\log\log n)^3$, *for certain constants* A *and* C.

$\frac{\log 2}{2} \approx 0.34657$ while $A \approx 0.72109$, so typically $G(n) \approx I(n)^{2.08}$.

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We had to look at these constants, so you do too

Definition

$$A_0 = \frac{1}{4} \sum_p \frac{p^2 \log p}{(p-1)^3 (p+1)}$$

$$A = \frac{\log 2}{2} + A_0 \approx 0.72109$$

$$B = \frac{1}{4} \sum_p \frac{p^3 (p^4 - p^3 - p^3 - p - 1)(\log p)^2}{(p-1)^6 (p+1)^2 (p^2 + p + 1)}$$

$$C = \frac{(\log 2)^2}{3} + 2A_0 \log 2 + 4A_0^2 + B \approx 3.924$$

(The two sums are convergent sums over all primes p.)

How many subgroups can there be?

Theorem (M.-Troupe, submitted)

The order of magnitude of the maximal order of $\log I(n)$ *is* $\log n/\log\log n$. *More precisely,*

$$\frac{\log 2}{5}\frac{\log x}{\log\log x} \lesssim \max_{n\le x}\big(\log I(n)\big) \lesssim \pi\sqrt{\frac{2}{3}}\frac{\log x}{\log\log x}.$$

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Consequence: $G(n)$ can be superpolynomially large

There are infinitely many integers n with $G(n) > n^{2017!}$ …

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Finite abelian groups and partitions

Facts about finite abelian p-groups

- Every finite abelian group of size p^m can be written uniquely as $\mathbb{Z}_{p^{\boldsymbol{\alpha}}} = \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_\ell}}$ for some partition $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ of m (so $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_\ell$).
- So the number of isomorphism classes of subgroups of $\mathbb{Z}_{p^\alpha}$ is exactly the number of subpartitions $\beta \preceq \alpha$ …

… which is somewhere between 2 and 2^m inclusive.

In other words:

$\log \#\{$subpartitions of $\alpha\}$ is between $\log 2$ and $m \log 2$.

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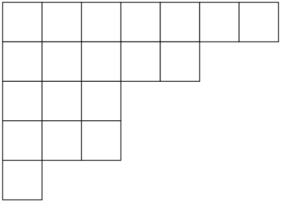

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Finite abelian groups and partitions

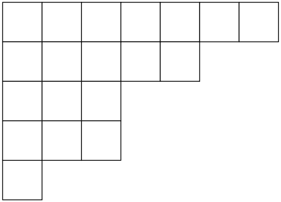
Facts about finite abelian p-groups

- Every finite abelian group of size p^m can be written uniquely as $\mathbb{Z}_{p^{\boldsymbol\alpha}} = \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_\ell}}$ for some partition $\boldsymbol\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ of m (so $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_\ell$).
- So the number of isomorphism classes of subgroups of $\mathbb{Z}_{p^{\boldsymbol\alpha}}$ is exactly the number of subpartitions $\boldsymbol\beta \preceq \boldsymbol\alpha$ …

… which is somewhere between 2 and 2^m inclusive.

In other words:

$\log \#\{$subpartitions of $\boldsymbol\alpha\}$ is between $\log 2$ and $m \log 2$.



Application to distribution of $I(n)$

$I(n)$ is the number of isomorphism classes of subgroups of $\mathbb{Z}_n^\times$

More notation

Let $\phi(n) = \prod_{p|\phi(n)} p^{m(p)}$, so that $\mathbb{Z}_n^\times \cong \bigoplus_{p|\phi(n)} \mathbb{Z}_{p^{\boldsymbol{\alpha}(p)}}$ for some partitions $\boldsymbol{\alpha}(p)$ of $m(p)$.

Then $\log I(n) = \sum_{p|\phi(n)} \log \#\{$subpartitions of $\alpha_p\}$ and hence

$$\sum_{p|\phi(n)} \log 2 \le \log I(n) \le \sum_{p|\phi(n)} m(p) \log 2$$

$$\omega(\phi(n)) \log 2 \le \log I(n) \le \Omega(\phi(n)) \log 2$$

Upper bound seems very wasteful, yet still good enough!

"Anatomy of integers" techniques show: most primes dividing $\phi(n)$ do so only once.

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How many subgroups of each shape?

Notation: $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell)$, $\mathbb{Z}_{p^{\boldsymbol{\alpha}}} = \mathbb{Z}_{p^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_\ell}}$

Definition

Given a subpartition $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ and a prime p, define $N_p(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to be the number of subgroups inside $\mathbb{Z}_{p^{\boldsymbol{\alpha}}}$ that are isomorphic to $\mathbb{Z}_{p^{\boldsymbol{\beta}}}$.

Some classical exact formula (don't read it)

Let $\boldsymbol{a} = (a_1, a_2, \dots, a_{\alpha_1})$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_{\beta_1})$ be the conjugate partitions to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. Then

$$N_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{j=1}^{\alpha_1} p^{(a_j - b_j)b_{j+1}} \begin{bmatrix} a_j - b_{j+1} \\ b_j - b_{j+1} \end{bmatrix}_p,$$

where $\begin{bmatrix} k \\ \ell \end{bmatrix}_p = \prod_{j=1}^{\ell} \frac{p^{k-\ell+j}-1}{p^j-1}$ is the Gaussian binomial coefficient.

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The difference between algebra and analysis

$$N_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{j=1}^{\alpha_1} p^{(a_j - b_j)b_{j+1}} \begin{bmatrix} a_j - b_{j+1} \\ b_j - b_{j+1} \end{bmatrix}_p$$

is the number of subgroups inside $\mathbb{Z}_{p^{\boldsymbol{\alpha}}}$ isomorphic to $\mathbb{Z}_{p^{\boldsymbol{\beta}}}$.

It turns out that each factor is about $p^{(a_j-b_j)b_j}$, which is maximally $p^{a_j^2/4}$ when $b_j = a_j/2$, and is way smaller for noncentral values of b_j. So the total number of subgroups inside $\mathbb{Z}_{p^\alpha}$ is dominated by this special $\beta =$ "$\frac{1}{2}\alpha$".

Lemma

For any prime p and any partition α,

$$\log \#\{\textit{subgroups of } \mathbb{Z}_{p^\alpha}\} = \frac{\log p}{4} \sum_{j=1}^{\alpha_1} a_j^2 + O(\alpha_1 \log p).$$

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If $\mathbb{Z}_n^\times \cong \bigoplus_{p\mid\phi(n)} \mathbb{Z}_{p^{\alpha(p)}}$, then which partition is $\alpha(p)$?

Notation

Let $\omega_q(n)$ denote the number of distinct prime factors of n that are congruent to $1 \pmod{q}$.

Answer (exact for odd squarefree n, up to $O(1)$ in general)

$\alpha(p)$ is the conjugate partition to $(\omega_p(n), \omega_{p^2}(n), \dots)$.

Lemma

$$\log G_p(n) \approx \frac{\log p}{4} \sum_{j=1}^{\text{“}\infty\text{”}} \omega_{p^j}(n)^2$$ *for any prime p dividing $\phi(n)$.*

Moreover, if $p \mid \phi(n)$ and $p^2 \nmid \phi(n)$, then $\log G_p(n) = \log 2$.

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Sum the previous lemma over all primes

$$\log G(n) = \sum_{p \mid \phi(n)} \log G_p(n) \approx \sum_{\substack{p \mid \phi(n) \\ p^2 \nmid \phi(n)}} \log 2 + \sum_{p^2 \mid \phi(n)} \frac{\log p}{4} \sum_{j=1}^{\text{``}\infty\text{''}} \omega_{p^j}(n)^2.$$

For most integers n, it's acceptable to extend both sums over all primes dividing $\phi(n)$ (the last sum should be suitably truncated):

$$\log G(n) \approx \log 2 \cdot \omega(\phi(n)) + \frac{1}{4} \sum_{p^r} \omega_{p^r}(n)^2 \log p.$$

Each function here has a known normal order; plugging in gives

$$\log G(n) \approx \log 2 \cdot \frac{1}{2} (\log\log n)^2 + \frac{1}{4} \sum_{p^r} \left(\frac{\log\log n}{\phi(p^r)} \right)^2 \log p$$

for almost all integers n. And the right-hand side is $A(\log\log n)^2$.

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Final sketch

Getting beyond the normal order to an Erdős–Kac law requires computing all of the central moments of this approximation to $\log G(n)$. The correlations among the additive functions $\omega_q(n)$, and their correlations with $\omega(\phi(n))$, become important.

"Sieving and the Erdős–Kac theorem" (2007)

To compute the moments, we rely on a technique of Granville and Soundararajan to reduce the complexity of identifying the main terms of these moments.

Generalizing our method

Part of $\log G(n)$ is well approximated by a sum of squares of additive functions. Troupe and I (work in progress) can obtain an Erdős–Kac law for any fixed (nonnegative) polynomial evaluated at values of (appropriate) additive functions—for example, Erdős–Kac laws for products of additive functions.

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The end

Our submitted paper "The distribution of the number of subgroups of the multiplicative group" and these slides are available for downloading.

The paper with Lee Troupe

`www.math.ubc.ca/~gerg/index.shtml?abstract=DNSMG`

These slides

`www.math.ubc.ca/~gerg/index.shtml?slides`