

Dimensions of spaces of newforms

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Outline

- 1 Cusp forms on $\Gamma_0(N)$
- 2 Newforms on $\Gamma_0(N)$
- 3 Consequences of the dimension formula
- 4 Related dimensions

Cusp forms on $\Gamma_0(N)$

Notation

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

Definition (weight- k cusp forms on $\Gamma_0(N)$)

Let $S_k(\Gamma_0(N))$ denote the \mathbb{C} -vector space of functions f that are holomorphic on the upper half-plane $\Im z > 0$, and “holomorphic and zero at cusps”, that satisfy

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

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Dimension of space of cusp forms

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Let $g_0(k, N)$ denote the dimension of $S_k(\Gamma_0(N))$.

Proposition

For any even integer $k \geq 2$ and any integer $N \geq 1$,

$$g_0(k, N) = \frac{k-1}{12} N s_0(N) - \frac{1}{2} \nu_\infty(N) + c_2(k) \nu_2(N) + c_3(k) \nu_3(N) + \delta\left(\frac{k}{2}\right).$$

- s_0 , ν_∞ , ν_2 , and ν_3 are certain multiplicative functions related to $\Gamma_0(N)$.

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- s_0 is the multiplicative function satisfying $s_0(p^\alpha) = 1 + \frac{1}{p}$ for all $\alpha \geq 1$.
- $Ns_0(N)$ is the index of $\bar{\Gamma}_0(N)$ in $\bar{SL}_2(\mathbb{Z})$, where \bar{G} denotes the quotient of the group G by its center.

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- ν_∞ is the multiplicative function satisfying:
 - $\nu_\infty(p^\alpha) = 2p^{(\alpha-1)/2}$ if α is odd;
 - $\nu_\infty(p^\alpha) = p^{\alpha/2} + p^{\alpha/2-1}$ if α is even.
- $\nu_\infty(N)$ counts the number of (inequivalent) cusps of $\Gamma_0(N)$.

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- ν_2 is the multiplicative function satisfying:
 - $\nu_2(2) = 1$, and $\nu_2(2^\alpha) = 0$ for $\alpha \geq 2$;
 - if $p \equiv 1 \pmod{4}$ then $\nu_2(p^\alpha) = 2$ for $\alpha \geq 1$;
 - if $p \equiv 3 \pmod{4}$ then $\nu_2(p^\alpha) = 0$ for $\alpha \geq 1$.
- $\nu_2(N)$ counts the number of (inequivalent) elliptic points of $\Gamma_0(N)$ of order 2.

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- ν_3 is the multiplicative function satisfying:
 - $\nu_3(3) = 1$, and $\nu_3(3^\alpha) = 0$ for $\alpha \geq 2$;
 - if $p \equiv 1 \pmod{3}$ then $\nu_3(p^\alpha) = 2$ for $\alpha \geq 1$;
 - if $p \equiv 2 \pmod{3}$ then $\nu_3(p^\alpha) = 0$ for $\alpha \geq 1$.
- $\nu_3(N)$ counts the number of (inequivalent) elliptic points of $\Gamma_0(N)$ of order 3.

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$$g_0(k, N) = \frac{k-1}{12}Ns_0(N) - \frac{1}{2}\nu_\infty(N) + c_2(k)\nu_2(N) + c_3(k)\nu_3(N) + \delta\left(\frac{k}{2}\right).$$

- $c_2(k) = \frac{1}{4} + \left\lfloor \frac{k}{4} \right\rfloor - \frac{k}{4}$, so $c_2(k) \in \left\{ -\frac{1}{4}, \frac{1}{4} \right\}$ for k even
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- $\delta(m) = 1$ if $m = 1$, and $\delta(m) = 0$ otherwise

Where that dimension formula comes from

We assume $N \geq 2$ and $k \geq 4$ to simplify the exposition.

Notation

Let g_N denote the genus of the (compactified) quotient of the upper half-plane by $\Gamma_0(N)$.

Formula for the genus

$$g_N = \frac{Ns_0(N)}{12} - \frac{\nu_\infty(N)}{2} - \frac{\nu_2(N)}{4} - \frac{\nu_3(N)}{3} + 1$$

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- The dimension $g_0(k, N)$ of the space of weight- k cusp forms on $\Gamma_0(N)$ is calculated by the Riemann–Roch theorem:

$$g_0(k, N) = (k-1)(g_N - 1) + \left(\frac{k}{2} - 1\right)\nu_\infty(N) \\ + \left\lfloor \frac{k}{4} \right\rfloor \nu_2(N) + \left\lfloor \frac{k}{3} \right\rfloor \nu_3(N).$$

- Collecting the multiples of $\nu_\infty(N)$, $\nu_2(N)$, and $\nu_3(N)$ yields

$$g_0(k, N) = \frac{k-1}{12}Ns_0(N) - \frac{1}{2}\nu_\infty(N) \\ + \left(\frac{1}{4} - \frac{k}{4} + \left\lfloor \frac{k}{4} \right\rfloor\right)\nu_2(N) + \left(\frac{1}{3} - \frac{k}{3} + \left\lfloor \frac{k}{3} \right\rfloor\right)\nu_3(N).$$

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Newforms

- If $f(z)$ is a cusp form on $\Gamma_0(d)$, then $f(mz)$ is a cusp form on $\Gamma_0(N)$ for any multiple N of dm .
- Thus for every triple (m, d, N) of positive integers with $dm \mid N$, we have an injection $i_{m,d,N} : S_k(\Gamma_0(d)) \rightarrow S_k(\Gamma_0(N))$.

Definition ($S_k^\#(\Gamma_0(N))$)

$$S_k^\#(\Gamma_0(N)) = \left(\text{span} \left\langle \bigcup_{\substack{d \mid N \\ d \neq N}} \bigcup_{m \mid N/d} i_{m,d,N}(S_k(\Gamma_0(d))) \right\rangle \right)^\perp,$$

where \perp denotes the orthogonal complement with respect to the Petersson inner product in $S_k(\Gamma_0(N))$.

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- The cusp forms comprising $S_k^\#(\Gamma_0(N))$ are called newforms.

Proposition (Atkin–Lehner decomposition)

We can write $S_k(\Gamma_0(N))$ as a direct product of subspaces:

$$S_k(\Gamma_0(N)) = \bigoplus_{d|N} \bigoplus_{m|N/d} i_{m,d,N}(S_k^\#(\Gamma_0(d))).$$

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Relating dimensions

Atkin–Lehner decomposition

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- Recall that $g_0(k, N)$ denotes the dimension of $S_k(\Gamma_0(N))$.
- Let $g_0^\#(k, N)$ denote the dimension of $S_k^\#(\Gamma_0(N))$.
- Let $\tau(m)$ denote the number of positive divisors of m .

Corollary

$$g_0(k, N) = \sum_{d|N} \sum_{m|N/d} g_0^\#(k, d) = \sum_{d|N} g_0^\#(k, d) \tau(N/d)$$

- Put another way: $g_0 = g_0^\# * \tau$ for any fixed k , where $*$ is the Dirichlet convolution of two arithmetic functions.

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Solving for $g_0^\#(k, N)$

Notation

Let $\mathbf{1}(n) = 1$ denote the constant function, and let $\mu(n)$ denote the Möbius function. Note that $\mathbf{1} * \mu = \delta$.

Definition

Define λ to be the Dirichlet-convolution inverse of τ . Since $\tau = \mathbf{1} * \mathbf{1}$, we have $\lambda = \mu * \mu$; equivalently, λ is the multiplicative function satisfying

$$\lambda(p) = -2, \quad \lambda(p^2) = 1, \quad \lambda(p^\alpha) = 0 \text{ for } \alpha \geq 3.$$

- Since $g_0 = g_0^\# * \tau$, it follows that $g_0^\# = g_0 * \lambda$, that is,

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- $g_0^\#(k, N) = \sum_{d|N} g_0(k, d)\lambda(N/d)$, that is, $g_0^\# = g_0 * \lambda$
- But $g_0(k, N)$ is a linear combination of multiplicative functions of N , with coefficients depending on k :

$$g_0(k, N) = \frac{k-1}{12}Ns_0(N) - \frac{1}{2}\nu_\infty(N) + c_2(k)\nu_2(N) + c_3(k)\nu_3(N) + \delta\left(\frac{k}{2}\right)1(N).$$

Distribute the *

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- This too is a linear combination of multiplicative functions of N , with coefficients depending on k .

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- $g_0^\#(k, N) = \sum_{d|N} g_0(k, d)\lambda(N/d)$, that is, $g_0^\# = g_0 * \lambda$

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- $g_0^\#(k, N) = \sum_{d|N} g_0(k, d)\lambda(N/d)$, that is, $g_0^\# = g_0 * \lambda$
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Dimension of space of newforms

Theorem (M., 2005)

For any even integer $k \geq 2$ and any integer $N \geq 1$, the dimension $g_0^\#(k, N)$ of the space $S_k^\#(\Gamma_0(N))$ of newforms equals

$$\frac{k-1}{12} N s_0^\#(N) - \frac{1}{2} \nu_\infty^\#(N) + c_2(k) \nu_2^\#(N) + c_3(k) \nu_3^\#(N) + \delta\left(\frac{k}{2}\right) \mu(N).$$

- $s_0^\#$, $\nu_\infty^\#$, $\nu_2^\#$, and $\nu_3^\#$ are certain multiplicative functions.

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- $s_0^\#$ is the multiplicative function satisfying:
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 - $s_0^\#(p^2) = 1 - \frac{1}{p} - \frac{1}{p^2}$;
 - $s_0^\#(p^\alpha) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)$ if $\alpha \geq 3$.

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- $\nu_\infty^\#$ is the multiplicative function satisfying:
 - $\nu_\infty^\#(p^\alpha) = 0$ if α is odd;
 - $\nu_\infty^\#(p^2) = p - 2$;
 - $\nu_\infty^\#(p^\alpha) = p^{\alpha/2-2}(p-1)^2$ if $\alpha \geq 4$ is even.
- Note that $\nu_\infty^\#$ is supported on squares.

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Exact evaluations are easier

$$g_0^\#(k, N) = \frac{k-1}{12} N s_0^\#(N) - \frac{1}{2} \nu_\infty^\#(N) \\ + c_2(k) \nu_2^\#(N) + c_3(k) \nu_3^\#(N) + \delta\left(\frac{k}{2}\right) \mu(N)$$

Having a closed-form formula instead of a recursive formula lets us better analyze its values, whether exactly or approximately.

- For example, the following corollary and theorem were useful in 2006 work of Bennett/Győry/Mignotte on binomial Thue equations and Bennett/Bruin/Győry/Hajdu on products of terms in arithmetic progression.

Corollary

Let $M \geq 3$ be an odd, squarefree integer. Then

$$g_0^\#(k, 32M) = (k-1)\phi(M).$$

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Lemma

- $0 \leq N s_0^\#(N) \leq \phi(N)$
- $0 \leq \nu_\infty^\#(N) \leq \sqrt{N}$
- $|\nu_2^\#(N)| \leq 2^{\omega(N)}$
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It follows that $g_0^\#(2, N) \leq \frac{1}{12} \phi(N) + \frac{7}{12} 2^{\omega(N)} + 1$.

Theorem (M., 2005)

$g_0^\#(2, N) \leq (N + 1)/12$, with equality holding if and only if either $N = 35$ or N is a prime that is congruent to 11 (mod 12).

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Upper and lower bounds

Two constants

- Euler's constant $\gamma = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{1}{n} - \log x \right) \approx 0.577216$
- Define $A = \prod_p \left(1 - \frac{1}{p^2 - p} \right) \approx 0.373956$

Theorem (M., 2005)

For all even integers $k \geq 2$ and all integers $N \geq 2$:

- $\frac{k-1}{12}N + O(\sqrt{N} \log \log N) < g_0(k, N) < \frac{e^\gamma(k-1)}{2\pi^2}N \log \log N + O(N)$
- $\frac{A(k-1)}{12}\phi(N) + O(\sqrt{N}) < g_0^\#(k, N) < \frac{k-1}{12}\phi(N) + O(2^{\omega(N)})$
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Note: All of these bounds are best possible.

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Range of $g_0^\#(2, N)$

The lower bound for $g_0^\#(2, N)$ means that we can make exhaustive lists of levels N for which a given value is attained.

Example

The 40 solutions to $g_0^\#(2, N) = 100$ are:

$N = 1213, 1331, 2169, 2583, 2662, 2745, 3208, 3232, 3465, 3608, 4040, 4302, 4338, 4772, 4804, 4848, 5084, 5092, 5166, 5252, 5324, 5490, 5572, 5904, 6336, 6820, 6930, 7056, 7188, 7212, 7920, 8052, 8484, 8652, 8676, 8940, 9060, 10332, 10980, 13860.$

Example

There are exactly 2,965 integers N for which $g_0^\#(2, N) \leq 100$.

Conjecture

For every nonnegative integer G , there is at least one positive integer N such that $g_0^\#(2, N) = G$.

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The analogous conjecture turns out to be false for $g_0(2, N)$ itself.

Example

The omitted values up to 1000 are:

$$g_0(2, N) \neq 150, 180, 210, 286, 304, 312, 336, 338, 348, \\ 350, 480, 536, 570, 598, 606, 620, 666, 678, 706, \\ 730, 756, 780, 798, 850, 876, 896, 906, 916, 970.$$

Csirik–Wetherell–Zieve calculations

- The first several thousand omitted values of $g_0(2, N)$ are even, but there are odd omitted values: the first is 49,267.
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Average orders

Theorem (M., 2005)

- *The average order of $g_0(k, N)$ is $(k-1)\frac{5}{4\pi^2}N$. In other words, $\sum_{N \leq X} g_0(k, N) \sim \sum_{N \leq X} (k-1)\frac{5}{4\pi^2}N$.*
- *Let $g_0^*(k, N)$ denote the number of nonisomorphic automorphic representations associated with $S_k(\Gamma_0(N))$. This number can be interpreted as the dimension of a particular subspace of $S_k(\Gamma_0(N))$ that contains $S_k^\#(\Gamma_0(N))$. Then the average order of $g_0^*(k, N)$ is $(k-1)\frac{15}{2\pi^4}N$.*
- *The average order of $g_0^\#(k, N)$ is $(k-1)\frac{45}{\pi^6}N$.*

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- *Let $g_0^*(k, N)$ denote the number of nonisomorphic automorphic representations associated with $S_k(\Gamma_0(N))$. This number can be interpreted as the dimension of a particular subspace of $S_k(\Gamma_0(N))$ that contains $S_k^\#(\Gamma_0(N))$. Then the average order of $g_0^*(k, N)$ is $(k - 1) \frac{15}{2\pi^4} N$.*
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Gekeler's theorem

The number $g_0^*(k, N)$ of nonisomorphic automorphic representations associated with $S_k(\Gamma_0(N))$ is a similar linear combination of explicit multiplicative functions:

$$\frac{k-1}{12}Ns_0^*(N) - \frac{1}{2}\nu_\infty^*(N) + c_2(k)\nu_2^*(N) + c_3(k)\nu_3^*(N) + \delta\left(\frac{k}{2}\right)\delta(N).$$

Corollary

Let $k \geq 2$ be an even integer, and let $N \geq 2$ be a squarefree integer. Then $g_0^(k, N) = \frac{k-1}{12}N - \frac{1}{2} + c_2(k)\left(\frac{-1}{N}\right) + c_3(k)\left(\frac{-3}{N}\right)$. In particular, $g_0^*(k, N)$ depends upon the residue class $N \pmod{12}$ but not upon the prime factorization of N .*

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Cusp forms on $\Gamma_1(N)$

Notation

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \right\}$$

For $k \geq 2$ (not necessarily even), let $g_1(k, N)$ denote the dimension of the space of weight- k cusp forms on $\Gamma_1(N)$, and let $g_1^\#(k, N)$ denote the dimension of the space of weight- k newforms on $\Gamma_1(N)$.

Theorem (M., 2005)

For any integer $k \geq 2$:

- The average order of $g_1(k, N)$ is $(k-1)N^2/24\zeta(3)$.
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Lots of newforms

How many cusp forms on $\Gamma_1(N)$ are newforms?

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For all integers $k \geq 2$ and all integers $N \geq 1$ such that $g_1(k, N) \neq 0$,

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where $B = \prod_p \left(1 - \frac{3}{p^2}\right) \approx 0.125487$.

Note that $\frac{B\pi^2}{6} \approx 0.206418$; we deduce that when N is large enough with respect to k , at least 20% of the space of weight- k cusp forms on $\Gamma_1(N)$ is taken up by newforms.

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The end

The paper *Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$* , as well as these slides, are available for downloading:

The paper

[www.math.ubc.ca/~gerg/
index.shtml?abstract=DSCFN](http://www.math.ubc.ca/~gerg/index.shtml?abstract=DSCFN)

The slides

www.math.ubc.ca/~gerg/index.shtml?slides