Prime number races: An asymptotic formula for the densities

Greg Martin
University of British Columbia

CMS Winter Meeting
Windsor, Ontario
December 6, 2009

Outline

A little *too* abstract

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Given two reduced residue classes a and $b \pmod q$, let $\delta_{q;a,b}$ be the "probability", when x is "chosen randomly", that more primes up to x are congruent to $a \pmod q$ than are congruent to $b \pmod q$. . . . In joint work with Daniel Fiorilli of Univ. Montréal (thanks to whom this eternal manuscript-in-preparation has finally seen the light of day), we give an asymptotic series for $\delta_{q;a,b}$ that can be used to calculate it to arbitrary precision. . . .

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I lost the race. Again.

Where all the fuss started

In 1853, Chebyshev wrote a letter to Fuss saying the following:

> *"There is a notable difference in the splitting of the prime numbers between the two forms $4n+3$, $4n+1$: the first form contains a lot more than the second."*

Since then, "notable differences" have been observed among primes of various forms $qn+a$. Recall the notation

$$\pi(x;q,a) = \#\{\text{primes } p \le x \colon p \equiv a \pmod{q}\}.$$

The general pattern

$\pi(x;q,a)$ tends to be bigger when a is a nonsquare $\pmod{q}$, compared to when a is a square $\pmod{q}$.

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Defining delta

The central question

How often is $\pi(x; q, a)$ ahead of $\pi(x; q, b)$?

Definition

Define $\delta_{q;a,b}$ to be the logarithmic density of the set of real numbers $x \ge 1$ satisfying $\pi(x; q, a) > \pi(x; q, b)$. More explicity,

$$\delta_{q;a,b} = \lim_{T\to\infty} \left(\frac{1}{\log T} \int\limits_{\substack{1\le x\le T \\ \pi(x;q,a)>\pi(x;q,b)}} \frac{dx}{x} \right).$$

$\delta_{q;a,b}$ is the limiting "probability" that when a "random" real number x is chosen, there are more primes up to x that are congruent to $a \pmod q$ than congruent to $b \pmod q$.

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Two hypotheses

Rubinstein and Sarnak (1994) investigated these densities $\delta_{q;a,b}$ under the following two hypotheses:

- The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet L-functions have real part equal to $\frac{1}{2}$
- A linear independence hypothesis (LI): the nonnegative imaginary parts of these nontrivial zeros are linearly independent over the rationals

- Work of Ford and Konyagin shows that certain hypothetical violations of GRH do actually lead to pathological behavior in prime number races.
- LI is somewhat analogous to a “nonsingularity” hypothesis: with precise information about any linear dependences that might exist, we could probably still work out the answer. . . .

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Rubinstein and Sarnak's results

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Under these two hypotheses GRH and LI, Rubinstein and Sarnak proved (1994):

- $\delta_{q;a,b}$ always exists and is strictly between 0 and 1
- $\delta_{q;a,b} + \delta_{q;b,a} = 1$
- "Chebyshev's bias": $\delta_{q;a,b} > \frac{1}{2}$ if and only if a is a nonsquare $\pmod q$ and b is a square $\pmod q$
- if a and b are distinct squares $\pmod q$ or distinct nonsquares $\pmod q$, then $\delta_{q;a,b} = \delta_{q;b,a} = \frac{1}{2}$
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Comparisons of the densities $\delta_{q;a,b}$

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Feuerverger and M. (2000) generalized Rubinstein and Sarnak’s approach in several directions.

We calculated (assuming, as usual, GRH and LI) many examples of the densities $\delta_{q;a,b}$.

- The calculations required numerical evaluation of complicated integrals, which involved many explicitly computed zeros of Dirichlet L-functions.
- One significant discovery is that even with q fixed, the values of $\delta_{q;a,b}$ vary significantly as a and b vary over nonsquares and squares $(\mathrm{mod}\ q)$.

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Current goals

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- A more precise understanding of the sizes of $\delta_{q;a,b}$. Recalling that $\delta_{q;a,b}$ tends to $\frac{1}{2}$ as q tends to infinity, for example, we would like an asymptotic formula for $\delta_{q;a,b} - \frac{1}{2}$.
- A way to decide which $\delta_{q;a,b}$ are likely to be larger than others as a and b vary (with q fixed), based on elementary criteria rather than laborious numerical calculation.

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Asymptotic formula, version I

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

Theorem (Fiorilli and M., 2009+)

Assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$, then

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi\phi(q)\log q}} + O\left(\frac{\rho(q)\log\log q}{\phi(q)^{1/2}(\log q)^{3/2}}\right).$$

In particular, $\delta_{q;a,b} = \frac{1}{2} + O_\varepsilon\left(q^{-1/2+\varepsilon}\right)$ for any $\varepsilon > 0$.

$$\rho(q) = \text{the number of square roots of 1} \pmod q$$
$$= 2^{\#\text{number of odd prime factors of } q} \times \{1, 2, \text{or } 4\}$$

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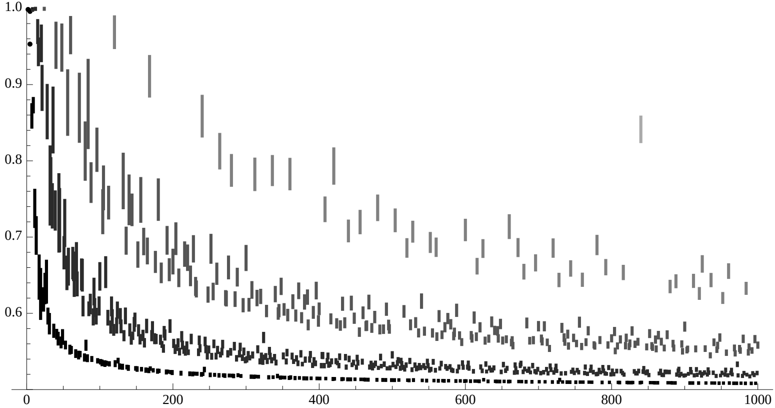
Graph of the densities

Figure: All densities $\delta_{q;a,b}$ with $q \le 1000$



Asymptotic formula, version II

$\delta_{q;a,b}$: the "probability" that $\pi(x;q,a) > \pi(x;q,b)$

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where $V(q;a,b)$ is the variance of a particular distribution, and

$$V(q;a,b) = 2 \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |\chi(b) - \chi(a)|^2 \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14 + \gamma^2}.$$

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where P_ℓ is a polynomial of degree ℓ whose coefficients, while depending on q, a, b, are bounded by a function of ℓ.

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Theorem (Fiorilli and M., 2009+)

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$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} \sum_{\ell=0}^{L} \frac{P_\ell\big(\rho(q)^2\big)}{V(q;a,b)^\ell} + O\big(q^{-L-3/2+\varepsilon}\big),$$

where P_ℓ is a polynomial of degree ℓ whose coefficients, while depending on q, a, b, are bounded by a function of ℓ.

Three terms depending on a and b

The variance, evaluated

$$V(q;a,b) = 2 \sum_{\substack{\chi \pmod q \\ \chi\ne\chi_0}} |\chi(b)-\chi(a)|^2 \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14+\gamma^2}$$
$$= 2\phi(q)\bigg(\log q - \sum_{p\mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi) + R_q(a-b) \bigg)$$
$$+ (2\log 2)\iota_q(-ab^{-1})\phi(q) + 2M(q;a,b).$$

There are three terms in this formula for the variance $V(q;a,b)$ that depend on a and b. Whenever any of the three is bigger than normal, the variance increases, causing the density $\delta_{q;a,b} = \frac12 + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\big(q^{-3/2+\varepsilon}\big)$ to decrease.

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Terms depending on a and b

The variance when the modulus q is prime and $b = 1$

$$V(q;a,1) = 2q\big(\log q - (\gamma_0 + \log 2\pi) + (2\log 2)\iota_q(a)\big) + 2M(q;a,1) + O(\log q).$$

- $\iota_q(a) = \begin{cases} 1, & \text{if } a \equiv -1 \pmod q, \\ 0, & \text{if } a \not\equiv -1 \pmod q \end{cases}$
- $M(q;a,1) = \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |1-\chi(a)|^2 \dfrac{L'(1,\chi)}{L(1,\chi)}$. Moreover, if $1 \le a, \tilde a < q$ are such that $\tilde a \equiv a^{-1} \pmod q$, then

$$M(q;a,1) = q\left(\frac{\Lambda(a)}{a} + \frac{\Lambda(\tilde a)}{\tilde a}\right) + O(\log q).$$

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Graph of normalized densities

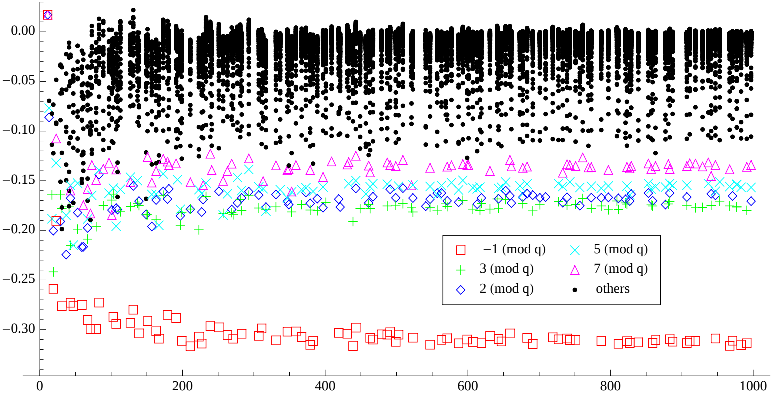

Figure: Densities $\delta_{q;a,1}$ for primes q, after a normalization to display them at the same scale

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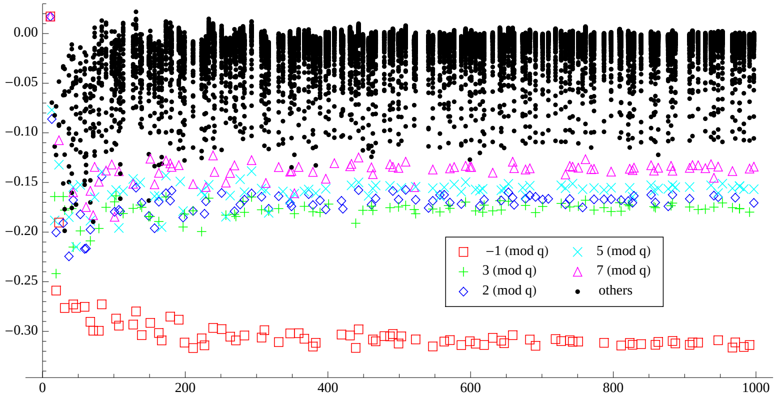

Note: $\frac{\Lambda(3)}{3} > \frac{\Lambda(2)}{2} > \frac{\Lambda(5)}{5} > \frac{\Lambda(7)}{7} > \cdots$

Top Ten List

Top 10 Most Unfair Races

Modulus q	Winner a	Loser b	Proportion $\delta_{q;a,b}$
24	5	1	99.9987%
24	11	1	99.9982%
12	11	1	99.9976%
24	23	1	99.9888%
24	7	1	99.9833%
24	19	1	99.9718%
8	3	1	99.9568%
12	5	1	99.9206%
24	17	1	99.9124%
3	2	1	99.9064%

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The end

The survey article *Prime number races*, with Andrew Granville

`www.math.ubc.ca/~gerg/index.shtml?abstract=PNR`

My research on prime number races

`www.math.ubc.ca/~gerg/`
`index.shtml?abstract=ISRPNRAFD`

These slides

`www.math.ubc.ca/~gerg/index.shtml?slides`