Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Write down a Riemann sum whose limit is the area under the graph of

$$y = \ln(x^3 + 5) \text{ from } -1 \le x \le 3.$$

(You don't have to evaluate the sum.)

The area under this graph is given by the definite integral $\int_{-1}^{3} \ln(x^3+5)\,dx$. Our integrand is $f(x) = \ln(x^3+5)$; we have $a = -1$ and $b = 3$, so $\Delta x = \frac{3-(-1)}{n} = \frac{4}{n}$ and $x_i = a + i\Delta x = -1 + \frac{4i}{n}$. We choose to use right endpoints, so that our sample points x_i^* are simply x_i. Therefore the Riemann sum in question is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \ln\left(\left(-1+\frac{4i}{n}\right)^3 + 5\right)\frac{4}{n}.$$

Grading scheme:

- 1 pt for Δx
- 1 pt for x_i (or other correct sample points)
- 1 pt for the correct form of sum. (The question asked for the sum without any limit, but no deduction for including the limit.)

Maximum 2 pts for an answer with no work shown.

1b. **[3 pts]** Write down a definite integral that equals

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\pi\left(1+\sqrt{\tfrac{2i}{n}}\right) - \pi\left(1+\tfrac{2i}{n}\right)\right)\frac{2}{n}.$$

(You don't have to evaluate the integral.)

We decide that Δx should equal $\frac{2}{n}$, and so we take $a = 0$ and $b = 2$ to make this happen. (Other choices are possible.) Then $x_i = a + i\Delta x = \frac{2i}{n}$. We see that we can write the summand as $\pi(1+\sqrt{x_i}) - \pi(1+x_i) = f(x_i)$ where $f(x) = \pi(1+\sqrt{x}) - \pi(1+x)$. Therefore

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \left(\pi\left(1+\sqrt{\tfrac{2i}{n}}\right) - \pi\left(1+\tfrac{2i}{n}\right)\right)\frac{2}{n} &= \lim_{n\to\infty} \sum_{i=1}^{n} \big(\pi(1+\sqrt{x_i}) - \pi(1+x_i)\big)\Delta x \\ &= \int_0^2 \big(\pi(1+\sqrt{x}) - \pi(1+x)\big)\,dx.\end{aligned}$$

Grading scheme:

- 1 pt for Δx
- 1 pt for an integral over an interval of length 2.
- 1 pt for correct integrand—depends on interval chosen (for example, a correct answer is $\int_1^3 \pi(1+\sqrt{x-1}-x)\,dx$, but not $\int_1^3 \pi(\sqrt{x}-x)\,dx$)

Maximum 2 pts for an answer with no work shown.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** Use the *Midpoint Rule*, with 3 rectangles, to approximate the area under the graph of $y = 1 + \cos x$ from $2 \le x \le 11$. (You don't have to simplify the resulting expression.)

Since we are using $n = 3$ rectangles, we have $\Delta x = \frac{11-2}{3} = 3$, and $x_i = a + i\Delta x = 2 + 3i$, that is, $\{x_0, x_1, x_2, x_3\} = \{2, 5, 8, 11\}$. The midpoints are

$$\frac{x_{i-1} + x_i}{2} = \frac{2 + 3(i-1) + 2 + 3i}{2} = \frac{1}{2} + 3i \text{ for } 1 \le i \le 3,$$

that is, the points $\frac{7}{2}$, $\frac{13}{2}$, and $\frac{19}{2}$. Therefore the Midpoint Rule's approximation to the area under the graph of $y = 1 + \cos x$ is simply

$$\big(1 + \cos \tfrac{7}{2}\big) \cdot 3 + \big(1 + \cos \tfrac{13}{2}\big) \cdot 3 + \big(1 + \cos \tfrac{19}{2}\big) \cdot 3.$$

Grading scheme:
- 1 pt for Δx
- 1 pt for correct midpoints
- 1 pt for correct Midpoint Rule expression

Maximum 1 pt for an answer with no work shown.

2b. **[3 pts]** Oil leaked from a pipeline at the rate of $r(t)$ liters per hour; the table below gives some values of $r(t)$ at 2-hour intervals. The leakage rate decreased as time passed. Find the best *upper estimate* possible for the total amount of oil that leaked out in the first 10 hours. Simplify your answer to a single number (with correct units).

t (hours)	0	2	4	6	8	10
$r(t)$ (liters per hour)	8.7	7.6	6.8	6.2	5.7	5.3

Since the function $r(t)$ is decreasing, an upper estimate can be found by taking a *left* endpoint Riemann sum. The time interval goes from $t = 0$ to $t = 10$, split into $n = 5$ subintervals of length $\Delta t = \frac{10-0}{5} = 2$. Therefore the upper estimate is

$$\sum_{i=1}^{5} r(t_{i-1})\Delta t = \big(r(t_0) + r(t_1) + r(t_2) + r(t_3) + r(t_4)\big) \text{ liters/hour} \cdot 2 \text{ hours}$$

$$= (8.7 + 7.6 + 6.8 + 6.2 + 5.7) \cdot 2 = 70 \text{ liters}.$$

Grading scheme:
- 1 pt for correct, simplified final answer with units
- 2 pts for correct left endpoint expression (1 pt for partially correct)

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** Define $g(x) = \displaystyle\int_{3x^3}^{4x^4} \frac{u-1}{u^2+1}\,du$. Find $g'(x)$. (You don't have to simplify.)

Define $h(t) = \int_0^t \frac{u-1}{u^2+1}\,du$. Then by the properties of integrals,

$$g(x) = \int_0^{4x^4} \frac{u-1}{u^2+1}\,du - \int_0^{3x^3} \frac{u-1}{u^2+1}\,du = h(4x^4) - h(3x^3).$$

The Fundamental Theorem of Calculus part 1 tells us that $h'(t)$ is exactly $\frac{t-1}{t^2+1}$. Therefore by the Chain Rule,

$$\begin{aligned} g'(x) &= h'(4x^4) \cdot \frac{d}{dx}(4x^4) - h'(3x^3) \cdot \frac{d}{dx}(3x^3) \\ &= \frac{4x^4-1}{(4x^4)^2+1} \cdot 16x^3 - \frac{3x^3-1}{(3x^3)^2+1} \cdot 9x^2. \end{aligned}$$

Grading scheme:

- 1 pt for writing as a difference of terms involving $4x^4$ and $3x^3$
- 1 pt for FTC (plugging $4x^4$ and $3x^3$ into the integrand)
- 1 pt for Chain Rule (multiplying by $16x^3$ and $9x^2$)

Maximum 2 pts for an answer with no work shown.

3b. **[3 pts]** Evaluate $\displaystyle\frac{1}{\pi}\int_0^1 \left(\frac{1}{t^2+1} + \frac{1}{\sqrt{1-t^2}}\right) dt$ as a fraction in lowest terms.

An antiderivative of $\frac{1}{t^2+1}$ is $\tan^{-1} t$, and an antiderivative of $\frac{1}{\sqrt{1-t^2}}$ is $\sin^{-1} t$. Therefore by the Fundamental Theorem of Calculus,

$$\frac{1}{\pi}\int_0^1 \left(\frac{1}{t^2+1} + \frac{1}{\sqrt{1-t^2}}\right) dt = \frac{1}{\pi}\big(\tan^{-1} t + \sin^{-1} t\big)\Big]_0^1.$$

Evaluating the inverse trigonometric functions means finding angles θ such that $\tan\theta$ equals 0 and 1 (such angles are 0 and $\frac{\pi}{4}$), and such that $\sin\theta$ equals 0 and 1 (such angles are 0 and $\frac{\pi}{2}$). Therefore

$$\frac{1}{\pi}\big(\tan^{-1} t + \sin^{-1} t\big)\Big]_0^1 = \frac{1}{\pi}\left(\left(\frac{\pi}{4} + \frac{\pi}{2}\right) - \big(0+0\big)\right) = \frac{3}{4}.$$

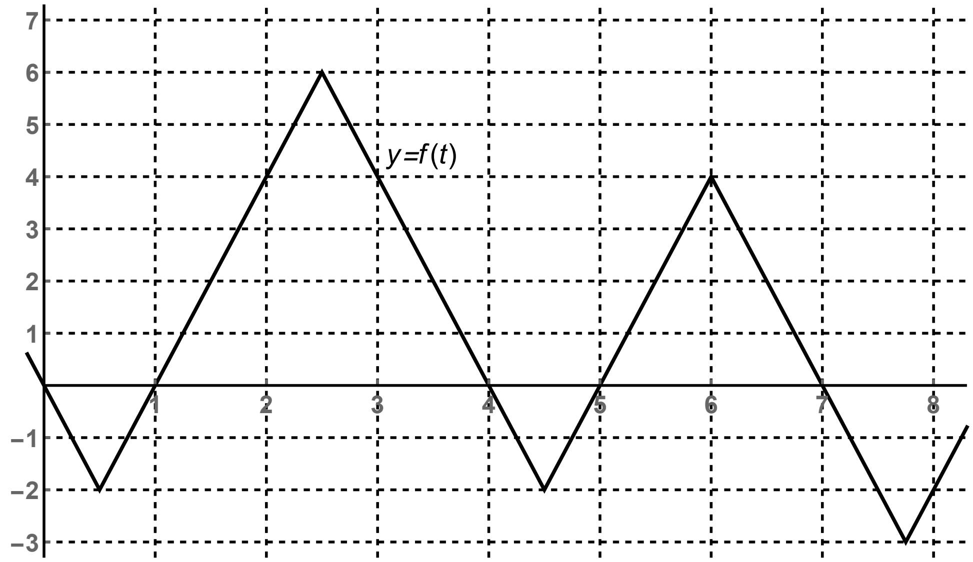
Grading scheme:

- 1 pt for correct antiderivative
- 1 pt for correct evaluation of either $\tan^{-1} 1$ or $\sin^{-1} 1$
- 1 pt for simplifying to a single fraction (some leniency for errors from previous step)

1 pt for an unsuccessful attempt using FTC.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** Define $F(x) = \int_0^x f(t)\,dt$, where $f(t)$ is given by the following graph. In the interval $0 \le x \le 8$, determine the value of x for which $F(x)$ is largest, and calculate $F(x)$ for that value of x.



The maximum value of $F(x)$ on the interval $[0, 8]$ must occur either at an endpoint or at a critical point of $F(x)$. By the Fundamental Theorem of Calculus, $F(x)$ is differentiable everywhere and its derivative is exactly $f(x)$. So the critical points of $F(x)$ are the zeros of its derivative $f(x)$; from the graph, we see that these occur at $x = 0, 1, 4, 5, 7$. The critical points at $x = 1$ and $x = 5$ are local minima of $F(x)$: the derivative $f(x)$ is negative to the left of these critical points and positive to the right. So the only values where $F(x)$ could have its maximum are $x = 0, 4, 7, 8$. The "area-so-far" function $F(x)$ can be calculated from the graph as the net area: we obtain $F(0) = 0$, $F(4) = 8$, $F(7) = 11$, and $F(8) < F(7)$. (For example, $F(7) = -1 + 9 - 1 + 4$, the signed sum of the areas of the four triangles.) Therefore, the maximum value $F(7) = 11$ occurs at $x = 7$.

Grading scheme:

- 1 pt for correctly saying that $x = 7$ gives the maximum value
- 1 pt for correct evaluation of $F(7) = 11$
- 1 pt for some work indicating how the value of F was correctly computed

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

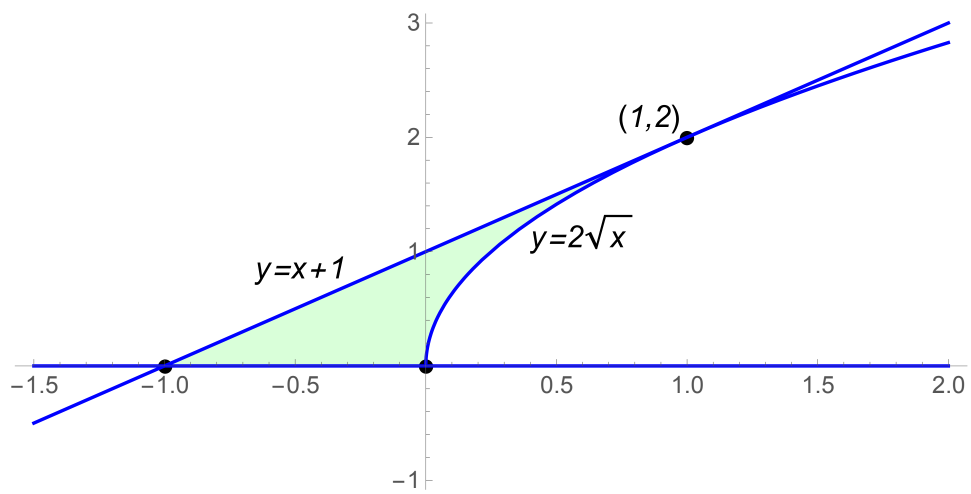
5. Let D be the region bounded by the x-axis and the graphs of $y = 2\sqrt{x}$ and $y = x + 1$.

(a) **[3 pts]** Sketch the region D, labeling the functions and the points of intersection.

The intersections of the two graphs with the x-axis are easy to calculate: we simply set $y = 0$ and solve for x. To find the intersections of $y = x + 1$ and $y = 2\sqrt{x}$, we solve $x + 1 = 2\sqrt{x}$ by squaring both sides:

$$\begin{aligned}(x+1)^2 &= (2\sqrt{x})^2\\ x^2+2x+1 &= 4x\\ x^2-2x+1 &= 0\\ (x-1)^2 &= 0\end{aligned}$$

which shows that x must equal 1, and so $y = 1 + 1 = 2$.



Grading scheme:

- 1 pt for finding the intersection point $(1, 2)$
- 1 pt for labeling graphs of the two functions and having their correct x-intercepts
- 1 pt for clear indication of the correct region D

Maximum 1 pt for an answer that included the vertical segment from $(0, 0)$ to $(0, 1)$ on the region's boundary.

(continued on next page)

(b) **[5 pts]** Calculate the area of the region D.

The total area is $\frac{1}{2} + \frac{1}{6} = 2 - \frac{4}{3} = \frac{2}{3}$.

[If the wrong region was obtained in part (a), it was still possible to get up to 5 pts in this part by starting from that region.]

Solution 1: we split the region into left and right halves with the y-axis. The area of the left half equals

$$\int_{-1}^{0} (x+1)\,dx = \left(\frac{x^2}{2} + x\right)\bigg]_{-1}^{0} = 0 - \left(\frac{(-1)^2}{2} + (-1)\right) = \frac{1}{2}$$

(or we can just compute the area of the triangle), while the area of the right half equals

$$\int_0^1 (x + 1 - 2\sqrt{x})\,dx = \left(\frac{x^2}{2} + x - \frac{4}{3}x^{3/2}\right)\bigg]_0^1 = \left(\frac{1}{2} + 1 - \frac{4}{3}\right) - 0 = \frac{1}{6}.$$

Grading scheme:

- 1 pt for correctly splitting into two integrals
- 2 pts for correct evaluation of the first integral (1 pt for partially correct)
- 2 pts for correct evaluation of the second integral (1 pt for partially correct)

Solution 2: we write the region as a large triangle with a curved piece cut out of its bottom right corner. The triangle has area

$$\int_{-1}^{1} (x+1)\,dx = \left(\frac{x^2}{2} + x\right)\bigg]_{-1}^{1} = \left(\frac{1^2}{2} + 1\right) - \left(\frac{(-1)^2}{2} + (-1)\right) = 2$$

(or we can just compute the area of the triangle), while the area of the curved piece equals

$$\int_0^1 (2\sqrt{x})\,dx = \left(\frac{4}{3}x^{3/2}\right)\bigg]_0^1 = \frac{4}{3}.$$

Grading scheme:

- 1 pt for correctly splitting into two integrals
- 2 pts for correct evaluation of the first integral (1 pt for partially correct)
- 2 pts for correct evaluation of the second integral (1 pt for partially correct)

Solution 3: we treat x as a function of y. The two curves become $x = y - 1$ and $x = (\frac{y}{2})^2 = \frac{y^2}{4}$, and the region stretches from $y = 0$ to $y = 2$. Therefore the area is

$$\int_0^2 \left(\frac{y^2}{4} - (y-1)\right) dy = \left(\frac{y^3}{12} - \frac{y^2}{2} + y\right)\bigg]_0^2 = \left(\frac{8}{12} - \frac{4}{2} + 2\right) - 0 = \frac{2}{3}.$$

Grading scheme:

- 2 pts for correctly rewriting functions as functions of y
- 1 pt for correct limits of integration
- 2 pts for correct evaluation of the integral (1 pt for partially correct)

Maximum 3 pts for an attempt that used the function $2\sqrt{x}$ to the left of the y-axis.

6a. **[4 pts]** A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm? (You don't have to simplify your answer, but use correct units.)

We first find the spring constant k from Hooke's law $F = kx$. The displacement from its natural length is $x = 0.15 - 0.10 = 0.05$ m, and so $40 = 0.05k$ or $k = 800$ N/m. Now the work it takes to stretch the spring from displacement x to displacement $x + \Delta x$ is force · distance = $kx \cdot \Delta x = 800x\Delta x$. Therefore the total work to stretch the spring from 15 cm to 18 cm—that is, from a displacement of 0.05 m to 0.08 m—is

$$\int_{0.05}^{0.08} 800x\,dx = 400x^2\Big]_{0.05}^{0.08} = 400(0.08^2 - 0.05^2) \text{ J}.$$

Alternatively, we could choose x to be the actual length of the spring, not its displacement from its natural length; in this case, the computation would look like

$$\int_{0.15}^{0.18} 800(x - 0.1)\,dx = 400(0.18)^2 - 80(0.18) - 400(0.15)^2 + 80(0.15).$$

One may also stay in cm throughout, if the final units are correct with the number given.

Grading scheme:

- 1 pt for correct value of k
- 1 pt for correct integrand
- 1 pt for correct limits of integration, corresponding to integrand
- 1 pt for correct evaluation of integral, including consistent units

Maximum 1 pt for an attempt that didn't use integration.

(continued on next page)

6b. **[4 pts]** Suppose $f(x)$ is continuous and that $\int_{-8}^{8} f(x)\,dx = 7$. Calculate

$$\int_{-2}^{2} \left(5x^2 f(x^3) + \tan\left(\tfrac{x^5}{55}\right)\right) dx.$$

(Hint: what kind of function is $\tan\left(\frac{x^5}{55}\right)$?)

By the properties of integrals,

$$\int_{-2}^{2} \left(5x^2 f(x^3) + \tan\left(\tfrac{x^5}{55}\right)\right) dx = \int_{-2}^{2} 5x^2 f(x^3)\,dx + \int_{-2}^{2} \tan\left(\tfrac{x^5}{55}\right) dx.$$

Moreover, this last integrand is an odd function (since $\tan\frac{(-x)^5}{55} = \tan(-\frac{x^5}{55}) = -\tan\frac{x^5}{55}$), and the integral is over the symmetric interval $[-2, 2]$; therefore the last integral equals 0. Finally, using the substitution $u = x^3$, so that $du = 3x^2\,dx$ and so $\frac{5}{3}\,du = 5x^2\,dx$, we see that

$$\int_{-2}^{2} 5x^2 f(x^3)\,dx = \int_{(-2)^3}^{2^3} \frac{5}{3} f(u)\,du = \frac{5}{3}\int_{-8}^{8} f(u)\,du = \frac{5}{3}\cdot 7 = \frac{35}{3}.$$

(Note that the two integrals $\int_{-8}^{8} f(u)\,du$ and $\int_{-8}^{8} f(x)\,dx$ are automatically equal—the name of the dummy variable in a definite integral doesn't affect its value.)

Grading scheme:

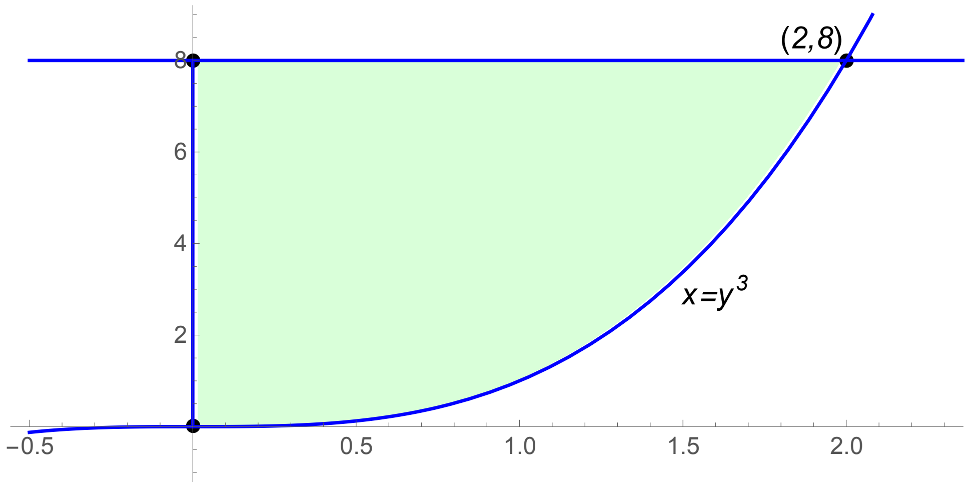
- 1 pt for correctly getting 0 for the second half of the integral
- 1 pt for a reasonable explanation (more than simply asserting "odd function") why 0 is correct for the second half of the integral
- 2 pts for correct evaluation of first half of integral (1 pt for partially correct answers)

7. A water tank has the shape obtained by rotating the portion of the graph of $y = x^3$ with $0 \le x \le 2$ about the y-axis. *Note: During the midterm, we added the clarification that the shape being rotated is* between the y-axis and *this graph portion.*

(a) **[4 pts]** Calculate the volume of the tank. (x and y are measured in metres.)

Since we are rotating about the y-axis, we need to treat x as a function of y. Solving for x in terms of y, we see that the shape being rotated can be described as the portion of the graph of $x = y^{1/3}$ with $0 \le y \le 8$. The volume is therefore

$$\int_0^8 \pi(y^{1/3})^2\,dy = \pi\int_0^8 y^{2/3}\,dy = \pi\frac{y^{5/3}}{5/3}\bigg]_0^8 = \frac{3\pi}{5}(8^{5/3}-0) = \frac{3\pi}{5}\cdot 32\text{ m}^3.$$



Grading scheme:

- 1 pt for solving for x in terms of y
- 1 pt for the correct limits of integration
- 1 pt for the correct integrand. (Mistaking the region to be to the right of $x = y^{1/3}$ was a common way to not get this point.)
- 1 pt for a correct evaluation of the integral.

Maximum of 2 pts for a solution that rotated around the x-axis.

(continued on next page)

(b) **[4 pts]** Suppose the tank is initially filled with water. Write down a definite integral representing the work W needed to pump all the water out of the top of the tank. You do *not* need to evaluate the integral. (The density of water is $1000\ \mathrm{kg}/\mathrm{m}^3$. You may use the rough approximation $g \approx 10\ \mathrm{m}/\mathrm{s}^2$.)

The water at height y (a circular disk of thickness Δy) has volume $\pi(y^{1/3})^2\Delta y = \pi y^{2/3}\Delta y$ cubic metres. Its mass is therefore $1000\pi y^{2/3}\Delta y$ kilograms, and the force required to raise it is roughly $1000\pi y^{2/3}\Delta y$ kg $\cdot$ $10\ \mathrm{m}/\mathrm{s}^2 = 10000\pi y^{2/3}\Delta y$ N. The water starting at height y has to be raised to the top of the tank, which is at $y = 8$; the distance it must be raised is therefore $8 - y$. The work required to lift that circular disk of water is then $10000\pi y^{2/3}(8 - y)\Delta y$ J. The total work to pump all the water out of the top of the tank is therefore

$$\int_0^8 10000\pi y^{2/3}(8 - y)\,dy \text{ J}.$$

Grading scheme:

- 1 pt for correct limits of integration
- 1 pt for correct volume function $\pi y^{2/3}\,dy$
- 1 pt for correct distance $8 - y$
- 1 pt for correct constants for water density and gravity (using 9.8 or 9.81 instead of $10\ \mathrm{m}/\mathrm{s}^2$ is fine)

Maximum 2 pts for an answer that had the volume from part (a), or some other constant, in place of the volume function $\pi y^{2/3}\,dy$.