Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Four weights with mass $m_1 = 2$, $m_2 = 3$, $m_3 = 5$, and $m_4 = 6$ are placed at the points $P_1 = (0,0)$, $P_2 = (-3,2)$, $P_3 = (1,-2)$, and $P_4 = (4,2)$, respectively. Calculate the centre of mass of the system of four weights. Simplify your answer fully.

The centre of mass is given by

$$\frac{2(0,0)+3(-3,2)+5(1,-2)+6(4,2)}{2+3+5+6}$$
$$= \frac{(0,0)+(-9,6)+(5,-10)+(24,12)}{16} = \frac{(20,8)}{16} = \left(\frac{5}{4}, \frac{1}{2}\right).$$

Grading scheme: points taken off for arithmetic errors, or for mistakes setting up the problem (not dividing by the total mass, switching the two coordinates, etc.)

1b. **[3 pts]** Evaluate $\displaystyle\int \frac{2x+3}{x^3+x}\,dx$.

The denominator factors as $x^3 + x = x(x^2+1)$, so we look for partial fractions of the form

$$\frac{2x+3}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1},$$

or equivalently $2x+3 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A$. We conclude that $A = 3$ and $C = 2$, which makes $B = -3$. Thus

$$\int \frac{2x+3}{x^3+x}\,dx = \int \left(\frac{3}{x} + \frac{-3x+2}{x^2+1}\right) dx$$
$$= 3\int \frac{1}{x}\,dx - \frac{3}{2}\int \frac{2x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx$$
$$= 3\ln|x| - \frac{3}{2}\ln|x^2+1| + 2\arctan x + C.$$

(The middle integral was done by the substitution $u = x^2+1$. Since x^2+1 is always positive, we can write (x^2+1) instead of $|x^2+1|$ if we want.)

Grading scheme:

- 1 pt for setting up the correct partial fraction form
- 1 pt for correctly solving for the constants A, B, C
- 1 pt for correctly integrating whatever partial fraction was arrived at. (No points deducted for forgetting absolute value symbols or the $+C$.)

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** Write the general partial fraction form for

$$\frac{3x^2+2x+1}{(x^2-1)^3(x^3-x^2+x)}.$$

Do *not* find numerical values for the constants $A, B, \ldots$.

The denominator factors as $(x^2-1)^3(x^3-x^2+x) = (x-1)^3(x+1)^3x(x^2-x+1)$, and x^2-x+1 is irreducible. Thus the general partial fraction form is

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+1}+\frac{E}{(x+1)^2}+\frac{F}{(x+1)^3}+\frac{G}{x}+\frac{Hx+I}{x^2-x+1}.$$

(Of course the terms can come in any order, and the constants can be given any names.)

Grading scheme:

- 1 pt for correctly giving the two terms with denominators x and x^2-x+1
- 1 pt for having the set of other six denominators correct
- 1 pt for having only constants, not linear polynomials, in the other numerators

2b. **[3 pts]** Determine whether the sequence limit

$$\lim_{n\to\infty}\sqrt{\frac{(-1)^n+4\sin(7n)+9n}{n}}$$

converges or diverges; if it converges, find its value.

Since $(-1)^n$ is always between -1 and 1 and $4\sin(7n)$ is always between -4 and 4, the fraction inside the square root is always between $9-\frac{5}{n}$ and $9+\frac{5}{n}$. Since the two functions $9-\frac{5}{x}$ and $9+\frac{5}{x}$ both converge to 9 as $x\to\infty$, the two sequences $9-\frac{5}{n}$ and $9+\frac{5}{n}$ also converge to 9 as $n\to\infty$; by the Squeeze Theorem,

$$\lim_{n\to\infty}\frac{(-1)^n+4\sin(7n)+9n}{n}=9$$

as well. Finally, since the function $\sqrt{t}$ is continuous at $t=9$, we conclude that the given sequence does converge:

$$\lim_{n\to\infty}\sqrt{\frac{(-1)^n+4\sin(7n)+9n}{n}}=\sqrt{9}=3.$$

(Alternatively, one can do the Squeeze Theorem argument directly on functions with square roots in them.)

Grading scheme: marks deducted for forgetting the square root in the final answer, for not justifying why certain limits (such as $\lim_{n\to\infty}\frac{(-1)^n}{n}$) tend to 0 or why the Squeeze Theorem can be applied, or for using a comparison test to get the actual value of the limit.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** Evaluate $\int \left(\sec^4 x \cdot \sqrt{\tan x}\right) dx$.

Since the power of $\sec x$ is even, we set $u = \tan x$, so that $du = \sec^2 x\, dx$. We get

$$\begin{aligned}\int \left(\sec^4 x \cdot \sqrt{\tan x}\right) dx &= \int \left((\tan^2 x + 1) \cdot \sqrt{\tan x}\right) \sec^2 x\, dx \\ &= \int (u^2+1)\sqrt{u}\, du \\ &= \int (u^{5/2} + u^{1/2})\, du \\ &= \frac{u^{7/2}}{7/2} + \frac{u^{3/2}}{3/2} + C = \frac{2(\tan x)^{7/2}}{7} + \frac{2(\tan x)^{3/2}}{3} + C.\end{aligned}$$

Grading scheme:

- 1 pt for choosing the correct substitution $u = \tan x$
- 1 pt for correctly converting the integral to the u variable
- 1 pt for correctly evaluating the u integral and returning to the x variable. (No points deducted for forgetting the $+C$.)

(continued on next page)

3b. **[3 pts]** Find the average value of the function

$$f(x) = \sqrt{5 + 4x - x^2}$$

on the interval $2 \le x \le 2 + \dfrac{\sqrt{3}}{2}$. You can leave your answer in calculator-ready form, but evaluate any values of trigonometric functions.

Completing the square, we have $5+4x-x^2 = 9-(x-2)^2$, so we choose the substitution $x-2 = 3\sin\theta$, so that $dx = 3\cos\theta\, d\theta$. When $x = 2$, we have $\sin\theta = 0$ and so $\theta = 0$; when $x = 2 + \frac{3}{\sqrt{2}}$, we have $\sin\theta = \frac{1}{\sqrt{2}}$ and so $\theta = \frac{\pi}{4}$. The length of the interval is $(2+\frac{3}{\sqrt{2}}) - 2 = \frac{3}{\sqrt{2}}$, and so the average value is

$$\begin{aligned}
\frac{1}{3/\sqrt{2}} \int_2^{2+3/\sqrt{2}} \sqrt{5+4x-x^2}\, dx &= \frac{\sqrt{2}}{3} \int_0^{\pi/4} \sqrt{9 - 3\sin^2\theta}\cos\theta \cdot 3\, d\theta \\
&= \frac{\sqrt{2}}{3} \int_0^{\pi/4} 9\cos^2\theta\, d\theta \\
&= 3\sqrt{2} \int_0^{\pi/4} \frac{1}{2}\big(1 + \cos(2\theta)\big)\, d\theta \\
&= \frac{3\sqrt{2}}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg]_0^{\pi/4} \\
&= \frac{3\sqrt{2}}{2}\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{3\pi\sqrt{2}}{8} + \frac{3\sqrt{2}}{4}.
\end{aligned}$$

Grading scheme:

- 1 pt for having the correct expression for the average value
- 1 pt for using the correct substitution $x-2 = 3\sin\theta$ (in one step or with two successive substitutions)
- 1 pt for a correct evaluation of the integral

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** Exactly one of these integrals diverges. Find the one that diverges, and explain why it diverges. (You don't have to say anything about the other two integrals.)

(a) $\displaystyle\int_2^\infty \frac{1}{x^{3/2}}\,dx$

(b) $\displaystyle\int_0^3 \frac{1}{x^{2/3}}\,dx$

(c) $\displaystyle\int_1^{\pi/2} \sec x \tan x\,dx$

The divergent integral is (c): The integrand is discontinuous at $x = \pi/2$, so we calculate

$$\lim_{t\to\pi/2-} \int_1^t \sec x \tan x\,dx = \lim_{t\to\pi/2-} \big(\sec x\big]_1^t \big) = \lim_{t\to\pi/2-} \big(\sec t - \sec 1\big) = \infty,$$

since $\sec t$ has a vertical asymptote at $t = \pi/2$.

(You didn't have to say anything about (a) and (b), but just to check if you want, they converge to $\sqrt{2}$ and $3^{4/3}$, respectively.)

Grading scheme:

- 1 pt for writing down a correct definition (involving a limit) for at least one of the improper integrals
- 1 pt for correctly identifying (c) as the divergent integral
- 1 pt for correct justification that the integral in (c) is divergent

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5. Consider a lamina L, with density $\rho = 7$, whose shape is the region under the graph of $y = \sec^2 x$ from $x = 0$ to $x = \pi/3$. (For both parts of this problem, you can leave your answer in calculator-ready form, but evaluate any values of trigonometric functions.)

(a) **[4 pts]** Calculate M_y, the moment of the lamina L with respect to the y-axis.

Starting with integration by parts, we evaluate

$$\begin{aligned} M_y &= \rho \int_a^b x f(x)\,dx = 7\int_0^{\pi/3} x\sec^2 x\,dx \\ &= 7\left(x\tan x\Big]_0^{\pi/3} - \int_0^{\pi/3} \tan x\,dx \right) \\ &= 7\left(\frac{\pi}{3}\tan\frac{\pi}{3} - 0 - \ln|\sec x|\Big]_0^{\pi/3} \right) \\ &= 7\left(\frac{\pi}{3}\sqrt{3} - \ln\left|\sec\frac{\pi}{3}\right| + \ln|\sec 0| \right) \\ &= 7\left(\frac{\pi}{\sqrt{3}} - \ln 2 \right). \end{aligned}$$

Grading scheme:

- 1 pt for including the density ρ
- 1 pt for correctly integrating by parts
- 1 pt for giving the right antiderivative for $\tan x$
- 1 pt for correctly evaluating the values of trigonometric functions. (No other simplification was necessary.)

(continued on next page)

(b) **[4 pts]** You are given the following evaluations (you don't have to prove them):

$$\int_0^{\pi/3} \sec^2 x\,dx = \sqrt{3}, \qquad \int_0^{\pi/3} \sec^4 x\,dx = 2\sqrt{3}, \qquad \int_0^{\pi/3} \sec^6 x\,dx = \frac{24\sqrt{3}}{5}.$$

Using this information and your answer to part (a), find the centroid of the lamina L. (Be careful with the constants.)

The mass of the lamina is

$$M = \rho \int_a^b f(x)\,dx = 7\int_0^{\pi/3} \sec^2 x\,dx = 7\sqrt{3},$$

while the moment with respect to the x-axis is

$$M_x = \rho \int_a^b \frac{1}{2} f(x)^2\,dx = \frac{7}{2}\int_0^{\pi/3} \sec^4 x\,dx = \frac{7}{2}\cdot 2\sqrt{3} = 7\sqrt{3}.$$

The centroid is therefore

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{7(\pi/\sqrt{3} - \ln 2)}{7\sqrt{3}}, \frac{7\sqrt{3}}{7\sqrt{3}}\right) = \left(\frac{\pi - \sqrt{3}\ln 2}{3}, 1\right).$$

Grading scheme:

- 1 pt for a correct evaluation of the mass M of the lamina (including the density)
- 1 pt for a correct evaluation of M_x
- 2 pts for assembling the above information into the centroid. (If mistakes were made in calculating M_y, M_x, or M, but the mistaken values were used correctly in the formula for the centroid, then these points are still awarded.)

6. Define $f(x) = \dfrac{x^5}{10} - \dfrac{x^6}{360}$. We are interested in approximating the integral $\displaystyle\int_0^{12} f(x)\,dx$.

(a) **[3 pts]** Write down the Simpson's Rule approximation to this integral, with $n = 6$. Do not simplify your answer (but it should be in calculator-ready form, not containing "f").

Since $\Delta x = \frac{12-0}{6} = 2$, the answer is

$$\begin{aligned} S_6 &= \frac{\Delta x}{3}\big(f(0) + 4f(2) + 2f(4) + 4f(6) + 2f(8) + 4f(10) + f(12)\big) \\ &= \frac{2}{3}\bigg\{0 + 4\bigg(\frac{2^5}{10} - \frac{2^6}{360}\bigg) + 2\bigg(\frac{4^5}{10} - \frac{4^6}{360}\bigg) + 4\bigg(\frac{6^5}{10} - \frac{6^6}{360}\bigg) + 2\bigg(\frac{8^5}{10} - \frac{8^6}{360}\bigg) \\ &\qquad + 4\bigg(\frac{10^5}{10} - \frac{10^6}{360}\bigg) + \bigg(\frac{12^5}{10} - \frac{12^6}{360}\bigg)\bigg\}. \end{aligned}$$

Grading scheme:

- 1 pt for the factor $\frac{\Delta x}{3} = \frac{2}{3}$
- 1 pt for the correct sequence of coefficients $(1,)\,4, 2, 4, 2, 4, 1$
- 1 pt for correctly evaluating $f(x)$ at the points $x = 0, 2, 4, 6, 8, 10, 12$

(b) **[3 pts]** Calculate the fourth derivative $f^{(4)}(x)$. Find the smallest number K such that $|f^{(4)}(x)| \le K$ for all $0 \le x \le 12$.

The first four derivatives of f are $f'(x) = \frac{x^4}{2} - \frac{x^5}{60}$, $f''(x) = 2x^3 - \frac{x^4}{12}$, $f^{(3)}(x) = 6x^2 - \frac{x^3}{3}$, and $f^{(4)}(x) = 12x - x^2$. This fourth derivative is a downward-pointing parabola whose global maximum is at its vertex $(6, 36)$ (there are several ways to see this, including completing the square or using calculus to find where $12x - x^2$ is increasing/decreasing). Furthermore, $12x - x^2 = x(12 - x)$ is always nonnegative for $0 \le x \le 12$. Therefore $|f^{(4)}(x)| = |12x - x^2| = 12x - x^2 \le 36$ for $0 \le x \le 12$, so we take $K = 36$.

Grading scheme:

- 1 pt for correctly computing $f^{(4)}(x)$
- 1 pt for finding the right value of K (given whatever function was gotten for $f^{(4)}(x)$)
- 1 pt for correctly justifying why the value of K was correct (testing the critical point and the endpoints, or talking about where $f^{(4)}(x)$ is positive/negative, increasing/decreasing, or referring to an accurate graph)

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(c) **[2 pts]** The error bound formula for Simpson's Rule is

$$|E_S| \le \frac{K(b-a)^5}{180n^4}.$$

In this problem, $a = 0$, $b = 12$, and $n = 6$; with the value of K computed in part (b), this inequality becomes $|E_S| \le 76.8$. Given this information, what are the smallest and largest possible values for $\int_0^{12} f(x)\,dx$? Assume that your answer to part (a) equals 35574.1.

The definition of E_S is the difference between the integral $\int_0^{12} f(x)\,dx$ and the Simpson's Rule approximation 35574.1. Consequently, the inequality $|E_S| \le 76.8$ is saying that

$$\left| \int_0^{12} f(x)\,dx - 35574.1 \right| \le 76.8.$$

Therefore the integral $\int_0^{12} f(x)\,dx$ must be between $35574.1 - 76.8 = 35497.3$ and $35574.1 + 76.8 = 35650.9$.

Grading scheme:

- 1 pt for knowing what the range of possible values for the integral is, given the bound for $|E_S|$ and the value of the integral
- 1 pt for using the given upper bound for $|E_S|$ in the above step

7. The two parts of this problem concern differential equations, but are not otherwise related.

(a) **[3 pts]** Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{5^x y^4}{\ln y}.$$

(You may leave your answer in implicit form—you don't have to solve for y.)

Separating the variables, we get $\frac{\ln y}{y^4}\,dy = 5^x\,dx$, or

$$\int \frac{\ln y}{y^4}\,dy = \int 5^x\,dx = \frac{5^x}{\ln 5} + C.$$

For the remaining integral, we use integration by parts with $u = \ln y$ and $dv = \frac{1}{y^4}\,dy$:

$$\int \frac{\ln y}{y^4}\,dy = (\ln y)\left(-\frac{1}{3y^3}\right) - \int -\frac{1}{3y^3}\cdot\frac{1}{y}\,dy$$

$$= -\frac{\ln y}{3y^3} + \int \frac{1}{3y^4}\,dy = -\frac{\ln y}{3y^3} - \frac{1}{9y^3}.$$

Therefore the final solution is $-\frac{\ln y}{3y^3} - \frac{1}{9y^3} = \frac{5^x}{\ln 5} + C.$

Grading scheme:

- 1 pt for correctly separating the variables
- 1 pt for correctly integrating the y side
- 1 pt for correctly integrating the x side

One point deducted for forgetting the $+C$ here, since the question specifically asked for the general form.

(continued on next page)

(b) **[5 pts]** Find a function $f(x)$ such that the graph of $y = f(x)$ has the following properties:

(a) it goes through the point $(2, -4)$;

(b) at every point (x, y) on the graph, the slope of the graph is $\dfrac{3x^2+1}{y}$.

We need to solve the differential equation $\frac{dy}{dx} = \frac{3x^2+1}{y}$ with initial condition $y(2) = -4$. Separating variables gives $y\,dy = (3x^2+1)\,dx$, or:

$$\int y\,dy = \int (3x^2+1)\,dx$$

$$\frac{y^2}{2} = x^3 + x + C.$$

We go ahead and solve for C right away, by plugging in $x = 2$ and $y = -4$:

$$\frac{(-4)^2}{2} = 2^3 + 2 + C$$

$$C = \frac{16}{2} - 10 = -2.$$

Our equation is therefore

$$\frac{y^2}{2} = x^3 + x - 2$$

$$y = -\sqrt{2x^3 + 2x - 4}.$$

(Notice we take the negative square root to ensure that $y(2) = -4$.)

Grading scheme: in this problem, 1 pt was allocated to having the exact right answer. The other 4 pts were allocated to demonstrating the correct procedure, as follows

- 2 pts for separating the variables and integrating both sides
- 1 pt for finding the value of C using the initial value
- 1 pt for solving for y as a function of x

If the procedure was entirely correct, then 4 points should be earned even if multiple errors were made along the way.