# Math 101—Practice Final Examination

Section #: **211** April 2016 (sort #: 4000)

Duration: 150 minutes

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**Do not open this test until instructed to do so!** This exam should have 12 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam. Phones cannot be visible at any point during the exam.

**UBC rules governing examinations:**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

| **Problem** | **Out of** | **Score** | **Problem** | **Out of** | **Score** |
|---|---|---|---|---|---|
| 1 | 8 | | 7 | 7 | |
| 2 | 8 | | 8 | 7 | |
| 3 | 6 | | 9 | 7 | |
| 4 | 6 | | 10 | 7 | |
| 5 | 6 | | 11 | 7 | |
| 6 | 6 | | **Total** | 75 | |

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

1a. **[2 pts]** For the integral $\int_0^1 e^{-x}\,dx$, which of the following approximate integration methods will give an *underestimate* for any value of $n$? Put the letter(s) corresponding to the correct answer(s) into the box; there can be any number of correct answers. Answer: ☐

**L:** Riemann sum using left endpoints of subintervals
**R:** Riemann sum using right endpoints of subintervals
**S:** Simpson's Rule
**T:** Trapezoid Rule

1b. **[2 pts]** Which expression is a Riemann sum for $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$? Answer: ☐

**A:** $\displaystyle\sum_{i=1}^{n} \frac{i}{n}\sqrt{1-\left(\frac{i}{n}\right)^2}$

**B:** $\displaystyle\sum_{i=1}^{n} \frac{2}{n}\sqrt{1-\left(\frac{i}{2n}\right)^2}$

**C:** $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\sqrt{1-\left(\frac{i-1}{n}\right)^2}$

**D:** $\displaystyle\sum_{i=1}^{n} \frac{i}{n^2}\sqrt{1-\left(\frac{i}{n}\right)^2}$

1c. **[4 pts]** Suppose $f(x)$ and $g(x)$ are twice differentiable functions with the following values:

| | | | | |
|---|---|---|---|---|
| $f(0)=-1$ | $f(1)=3$ | $f(2)=4$ | $f(3)=9$ | $f'(1)=2$ |
| $g(0)=3$ | $g(1)=0$ | $g(2)=1$ | $g(3)=6$ | $g'(1)=-1$ |

First, evaluate $\displaystyle\int_0^1 f'(g(x))g'(x)\,dx.$ Answer: ☐

Then, evaluate $\displaystyle\int_0^1 xg''(x)\,dx.$ Answer: ☐

Simplify both answers completely.

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

2a. **[4 pts]** For each of the following series, choose the appropriate statement.

**CA:** Converges absolutely
**CC:** Converges conditionally
**D:** Diverges

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+7}$ Answer: ☐

(ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ Answer: ☐

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2}$ Answer: ☐

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ Answer: ☐

2b. **[2 pts]** Which of the following Maclaurin series equals $\dfrac{x^3}{x^2+1}$? Answer: ☐

$\mathbf{F}: x^2 - x^4 + x^6 - x^8 + \cdots$

$\mathbf{G}: x^2 - \frac{1}{2}x^4 + \frac{1}{4}x^6 - \frac{1}{8}x^8 + \cdots$

$\mathbf{H}: x^3 - x^5 + x^7 - x^9 + \cdots$

$\mathbf{J}: x^3 - \frac{1}{2}x^5 + \frac{1}{4}x^7 - \frac{1}{8}x^9 + \cdots$

$\mathbf{K}: x^2 + x^4 + x^6 + x^8 + \cdots$

$\mathbf{M}: x^2 + \frac{1}{2}x^4 + \frac{1}{4}x^6 + \frac{1}{8}x^8 + \cdots$

$\mathbf{P}: x^3 + x^5 + x^7 + x^9 + \cdots$

$\mathbf{Q}: x^3 + \frac{1}{2}x^5 + \frac{1}{4}x^7 + \frac{1}{8}x^9 + \cdots$

2c. **[2 pts]** The power series $\displaystyle\sum_{n=0}^{\infty} A_n(x+2)^n$ converges at $x=-4$ and diverges at $x=1$. What are the possible values of its radius of convergence $R$? Write your answer either in interval notation, or in the form $a \le R \le b$ for some numbers $a$ and $b$. Answer: ☐

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

3a. **[3 pts]** Find $\int \sin^2 x \cos^3 x \, dx$.

3b. **[3 pts]** Find the area enclosed by the graphs of $y = x^2$ and $y = \sqrt{x}$. Simplify your answer completely.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

4a. **[3 pts]** Find the average value of the function $\dfrac{x^3 + \log x}{x}$ for $1 \le x \le 3$.

4b. **[3 pts]** Find the solution of the differential equation

$$y' = \frac{xe^{x^2} + 2}{y}$$

with initial condition $y(0) = -2$. Write your answer in the form $y = g(x)$ for some function $g(x)$.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

5a. **[3 pts]** Find $\displaystyle\int \frac{16x+16}{x(x-4)(x^2+4)}\,dx$.

5b. **[3 pts]** Find the sum of the series $\displaystyle\sum_{k=2}^{\infty}\left(\frac{5}{k}+\frac{6}{3^k}-\frac{5}{k+1}\right)$. Simplify your answer completely.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

6a. **[3 pts]** Define $F(x) = \int_{\cos x}^{\sin x} \log(3+2t)\,dt$. Find a formula for the derivative $F'(x)$.

6b. **[3 pts]** Determine the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k2^k}$.

Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

7. The two parts of this problem are unrelated.

(a) **[3 pts]** Calculate the volume of the solid obtained by rotating the region enclosed by $y = x^2$ and $y = 2x$ about the $y$-axis.

(b) **[4 pts]** A circular wading pool has a diameter of 2 m and sides $20$ cm tall. If it is half-full of water, how much work (in joules) is required to pump all of the water out over the side? (The density of water is $1000\,\mathrm{kg/m^3}$. Use $g = 9.8\,\mathrm{m/s^2}$ for the acceleration due to gravity.)

8. For both parts of this problem, you may use the formula $\frac{d}{dx}\big(\log(1+\sin x)\big) = \sec x - \tan x$ (you don't have to show this).

(a) **[5 pts]** Let $R$ be the region between the curves $y = \sec x$ and $y = \tan x$, for $x$ between $0$ and $\frac{\pi}{6}$. Find the $y$-coordinate of the centroid of $R$.

(b) **[2 pts]** Let $S$ be the region between the curves $y = \sec x$ and $y = |\tan x|$, for $x$ between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$. Using your answer to part (a), find the centroid of $S$.

9. The two parts of this problem are unrelated to each other.

(a) **[4 pts]** Given that $\int_0^\pi 10e^{-3x}\sin x\,dx = 1 + e^{-3\pi}$ (you don't have to show this), evaluate $\int_0^\pi 10e^{-3x}\cos x\,dx$.

(b) **[3 pts]** Does the improper integral

$$F(x) = \int_{\pi/3}^{\pi/2} \frac{\sin t}{\sqrt[3]{\cos t}}\,dt$$

converge or diverge?

10. The Maclaurin series of the function $\dfrac{1}{\sqrt{1-x^2}}$ is $1+\dfrac{x^2}{2}+\dfrac{3x^4}{8}+\dfrac{5x^6}{16}+\cdots$ (you don't have to show this).

(a) **[3 pts]** Give the first four non-zero terms of the Maclaurin series for $\arcsin x$.

(b) **[4 pts]** If $f(x)=\arcsin x$, evaluate $f^{(5)}(0)$.

11.

(a) **[3 pts]** Show that the alternating series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots$ converges to a value that is less than $0.9$.

(b) **[4 pts]** Show that the series $1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots$ converges absolutely.