Math 220, Section 203
Solutions to Study Questions for Second Midterm
March 17, 2003

For problems I-IV, determine whether the statement is true or false. If true, provide a proof; if false, provide a counterexample.

I. (D'Angelo and West, p. 288, #14.8) Let $\langle x \rangle$ be a sequence of real numbers.

(a) If $\langle x \rangle$ is unbounded, then $\langle x \rangle$ has no limit.
(b) If $\langle x \rangle$ is not monotone, then $\langle x \rangle$ has no limit.

(a) This is true—in fact it is exactly the contrapositive of problem VII, which we prove below.
(b) This is false. The sequence $(-1)^n/n$ is not monotone, but it converges to 0.

II. (D'Angelo and West, p. 288, #14.9) Suppose that $x_n \to L$.

(a) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \varepsilon$.
(b) There exists $n \in \mathbb{N}$ such that for all $\varepsilon > 0$, $|x_{n+1} - x_n| < \varepsilon$.
(c) There exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| < \varepsilon$.
(d) For all $n \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $|x_{n+1} - x_n| < \varepsilon$.

(a) This is true. Every convergent sequence is a Cauchy sequence, and so given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for every $m, n > N$. Just take $n = N + 1$ and $m = n + 1 = N + 2$.
(b) This is false in general. The only way for a nonnegative quantity to be less than ε for every $\varepsilon > 0$ is if that quantity equals 0. So the assertion boils down to "There exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| = 0$", and this is definitely not necessarily true for every convergent sequence (only if it happens to have two consecutive terms that are equal).
(c) This is true. Every convergent sequence is bounded (problem VII below), and so there exists a real number B such that $|x_n| \le B$ for all $n \in \mathbb{N}$. Then by the triangle inequality, $|x_{n+1} - x_n| \le |x_{n+1}| + |-x_n| \le 2B$, so choosing ε greater than $2B$ is sufficient.
(d) This is true for the silly reason that we get to pick ε after n is chosen. Just choose $\varepsilon = |x_{n+1} - x_n|$.

III. (D'Angelo and West, p. 288, #14.10) Let $\langle x \rangle$ be a sequence of real numbers.

(a) If $\langle x \rangle$ converges, then there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < 1/2^n$.
(b) If $|x_{n+1} - x_n| < 1/2^n$ for all $n \in \mathbb{N}$, then $\langle x \rangle$ converges.

(a) This is false. The sequence given by the formula $x_n = \frac{2}{n}$ is a counterexample: it converges to 0, but $|x_{n+1} - x_n|$ evaluates to $\frac{2}{n(n+1)}$, and it can be easily proved by induction that $\frac{2}{n(n+1)} \ge \frac{1}{2^n}$ for all $n \in \mathbb{N}$.
(b) This is true. Given $m > n$, the triangle inequality gives

$$|x_m - x_n| \le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|,$$

whereupon the inequality in the hypothesis gives

$$|x_m - x_n| < \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n} = \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}},$$

where we have used the formula for the sum of a finite geometric series. Now given any $\varepsilon > 0$, there exists an integer N such that $\frac{1}{2^{N-1}} < \varepsilon$ (solve the inequality using logarithms). Therefore, for any $m > n > N$, we have

$$|x_m - x_n| < \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} < \varepsilon.$$

This proves that $\langle x \rangle$ is a Cauchy sequence, which implies that $\langle x \rangle$ converges.

IV. (D'Angelo and West, p. 288, #14.12) If $a_n \to 0$ and $b_n \to 0$, then $\sum a_n b_n$ converges.

This is false: take $a_n = \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{\sqrt{n}}$, for example.

V. (D'Angelo and West, p. 291, #14.59) Let $\langle a \rangle$ be a convergent sequence of positive numbers. Prove that $\sum_{k=1}^{\infty} \frac{1}{ka_k}$ diverges.

Since $\langle a \rangle$ is convergent, it is bounded by problem VII below. Therefore, we can choose B such that $|a_n| \leq B$ for all $n \in \mathbb{N}$. (We may omit the absolute value signs, since we are given that the a_n are positive.) This implies that $\frac{1}{a_n} \geq \frac{1}{B}$ and that $\frac{1}{na_n} \geq \frac{1}{B}\frac{1}{n}$ for every $n \in \mathbb{N}$. Now the series $\sum \frac{1}{n}$ diverges, and so the series $\sum \frac{1}{B}\frac{1}{n}$ must also diverge (we're just multiplying by a constant: if one of the series converged, then by the arithmetic of limits the other would converge as well). Since $\frac{1}{na_n} \geq \frac{1}{B}\frac{1}{n}$ for every $kn \in \mathbb{N}$, the comparison test tells us that $\sum \frac{1}{na_n}$ also diverges.

VI. Let a and r be real numbers with $|r| < 1$. Prove that the series $\sum_{n=1}^{\infty} ar^n$ converges to the value $ar/(1-r)$.

Define $s_n = ar^1 + ar^2 + \cdots + ar^n$ to be the nth partial sum of the series in question; we need to prove that $\lim s_n = ar(1-r)$. We have a formula for this finite geometric series though, namely

$$s_n = ar^1 + ar^2 + \cdots + ar^n = ar\frac{1-r^n}{1-r}.$$

Now it's easy to show that when $|r| < 1$, we have $\lim r^n = 0$. (Miniproof when r is positive: since $r < 1$, the sequence r^n is decreasing, so if it's bounded, the limit equals the infimum. Proving that 0 is the infimum boils down to proving that r^n can get less than any positive ε; this inequality can be solved for n by taking logarithms.) Therefore by the arithmetic of limits,

$$\lim s_n = \lim\left(ar\frac{1-r^n}{1-r}\right) = (\lim ar)\frac{\lim 1 - \lim r^n}{\lim(1-r)} = ar\frac{1-0}{1-r} = \frac{ar}{1-r}$$

as desired.

VII. Prove that every convergent sequence is bounded.

Let $\langle s \rangle$ be a convergent sequence, and let $L = \lim s_n$. Choosing $\varepsilon = 1$, we see that there exists $N \in \mathbb{N}$ such that $|s_n - L| < 1$ for all $n > N$. This implies, by the triangle inequality, that

$$|s_n| = |(s_n - L) + L| \leq |s_n - L| + |L| < 1 + |L|$$

for all $n > N$. Therefore if we set

$$B = \max\{s_1, s_2, \dots, s_N, 1 + |L|\},$$

we have $|s_n| \leq B$ for all $n \in \mathbb{N}$. Therefore $\langle s \rangle$ is bounded.

VIII.

(a) *Define $s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$. Prove that $s_n \leq 2 - \frac{1}{n}$ for every $n \in \mathbb{N}$.*

(b) *Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.*

(c) *Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ converges.*

(a) We prove this by induction on n. The base case $n = 1$ is trivial, since $s_1 = 1$ and $2 - \frac{1}{1} = 1$ as well. For the induction step, suppose that $s_n \leq 2 - \frac{1}{n}$ for some $n \in \mathbb{N}$; we need to prove that $s_{n+1} \leq 2 - \frac{1}{n+1}$. Notice that since $n + 1 > n$, we have $\frac{1}{n+1} < \frac{1}{n}$ and so $\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$. Therefore

$$s_{n+1} = s_n + \frac{1}{(n+1)^2} \leq \left(2 - \frac{1}{n}\right) + \frac{1}{(n+1)^2} < \left(2 - \frac{1}{n}\right) + \frac{1}{n(n+1)}.$$

A little algebra now shows that $2 - \frac{1}{n} + \frac{1}{n(n+1)} = 2 - \frac{1}{n+1}$, as desired.

(b) With the partial sums s_n defined as in part (a), we need to prove that the sequence $\langle s \rangle$ converges. Notice that $\langle s \rangle$ is an increasing sequence, since $s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0$ for every $n \in \mathbb{N}$. Also, from part (a) we have $s_n \leq 2 - \frac{1}{n} < 2$. Therefore $|s_n| \leq 2$ for every $n \in \mathbb{N}$ (since each s_n is positive), which means that $\langle s \rangle$ is bounded. We conclude from the monotone convergence theorem that $\langle s \rangle$ converges.

(c) This is a simple application of the comparison test. Since $\pi > 2$, we have $n^{\pi} \geq n^2$ for every $n \geq 1$. Therefore $\frac{1}{n^{\pi}} \leq \frac{1}{n^2}$ for every $n \in \mathbb{N}$. Since we proved in part (b) that the series $\sum \frac{1}{n^2}$ converges, the comparison test tells us that the series $\sum \frac{1}{n^{\pi}}$ also converges.

IX.

(a) *Define a sequence $\langle a \rangle = \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \dots$; this sequence is given by the formula*

$$a_n = \begin{cases} \frac{1}{2}, & \text{if } 1 \leq n < 2, \\ \frac{1}{4}, & \text{if } 2 \leq n < 4, \\ \frac{1}{8}, & \text{if } 4 \leq n < 8, \\ \frac{1}{16}, & \text{if } 8 \leq n < 16, \\ & \vdots \\ \frac{1}{2^k}, & \text{if } 2^{k-1} \leq n < 2^k, \\ & \vdots \end{cases}$$

Prove that $a_1 + a_2 + \cdots + a_{2^k - 1} = \frac{k}{2}$ for every positive integer k.

(b) *Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.*

(c) *Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

(d) *Let $p \leq 1$ be a real number. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.*

(a) We proceed by induction on k. When $k = 1$, we have $2^k - 1 = 1$, and so what we need to prove is simply $a_1 = \frac{1}{2}$, which is straight from the definition. This finishes the base case.

For the inductive step, we assume for some $k \in \mathbb{N}$ that $a_1 + a_2 + \cdots + a_{2^k-1} = \frac{k}{2}$; we need to prove that $a_1 + a_2 + \cdots + a_{2^{k+1}-1} = \frac{k+1}{2}$. Notice that

$$a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1} = \tfrac{1}{2^{k+1}} + \tfrac{1}{2^{k+1}} + \cdots + \tfrac{1}{2^{k+1}} = \tfrac{1}{2^{k+1}} \cdot 2^k = \tfrac{1}{2},$$

since each of the terms is equal to $\frac{1}{2^{k+1}}$ by the definition of the sequence, and there are 2^k terms in the sum. Therefore, by the induction hypothesis together with this last calculation,

$$\begin{aligned} a_1 + a_2 + \cdots + a_{2^{k+1}-1} &= (a_1 + a_2 + \cdots + a_{2^k-1}) + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}) \\ &= \tfrac{k}{2} + \tfrac{1}{2} = \tfrac{k+1}{2} \end{aligned}$$

as desired.

(b) Let $s_n = a_1 + a_2 + \cdots + a_n$ be the nth partial sum of the series $\sum a_n$. We claim that $\langle s \rangle$ is unbounded. To see this, suppose for the sake of contradiction that $\langle s \rangle$ is actually bounded. If that were true, we could find an integer M such that $|s_n| \le M$ for every $n \in \mathbb{N}$. On the other hand, we could then look at $n = 2^{2M+1} - 1$. By part (a), we know that

$$s_n = s_{2^{2M+1}-1} = \tfrac{2M+1}{2} = M + \tfrac{1}{2} > M,$$

which contradicts the choice of M. Therefore $\langle s \rangle$ is in fact unbounded.

Now that we know that $\langle s \rangle$ is unbounded, problem VII (more precisely, its contrapositive) tells us that $\langle s \rangle$ diverges. Therefore $\sum a_n$ diverges as well by definition.

(c) We claim that $1/n > a_n$ for every $n \in \mathbb{N}$. To see this, choose $k \in \mathbb{N}$ such that $2^{k-1} \le n < 2^k$. (This value of k can be written as $\lfloor \log_2 n \rfloor + 1$, if you like, but that won't be important for this proof.) Then $a_n = 1/2^k$ by definition, and $1/2^k < 1/n$ since $n < 2^k$, which shows that $a_n < 1/n$. Now the divergence of $\sum 1/n$ follows from the comparison text, since we proved in part (b) that $\sum a_n$ diverges.

(d) Since $p \le 1$, we have $1 - p \ge 0$. Therefore $n^{1-p} \ge 1$ for all $n \in \mathbb{N}$. This is the same as $n^1/n^p \ge 1$, and dividing both sides by the positive number n yields the inequality $1/n^p \ge 1/n$ for every $n \in \mathbb{N}$. We already know from part (c) that the series $\sum 1/n$ diverges. Therefore, by the comparison test, the series $\sum 1/n^p$ also diverges.

Notice how much we used theorems and known results to help us with these more complicated problems. The fact that convergent sequences are always bounded, for example, showed up about five times on this study sheet alone.