Math 220, Section 203—Homework #6
due in class Thursday, March 13, 2003

Remember that all of your solutions must be written in complete sentences that are easy to read and in logically correct order.

I. D'Angelo and West, p. 95, #4.11

II. Let a and b be any real numbers, and define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + ax + b$. Prove that f is neither injective nor surjective. (Hint: complete the square.)

III. For each of the parts below, either write down a specific function with the specified properties (no proof necessary), or explain why such a function does not exist.
   (a) $a : \mathbb{Z} \to \mathbb{Z}$, a is injective but not surjective
   (b) $b : \mathbb{Z} \to \mathbb{Z}$, b is surjective but not injective
   (c) $c : \mathbb{Z} \to \mathbb{Z}$, c is a bijection
   (d) $d : [5] \to [3]$, d is an injection
   (e) $e : [5] \to [3]$, e is not a surjection
   (f) $f : [4] \to [4]$, f is injective but not surjective

IV. Let $f : [0, 1] \to \mathbb{R}$ be a decreasing function.
   (a) Prove that f is bounded.
   (b) Prove that f is injective.
   (c) Prove that f is not surjective.

V. Define the following set S, a subset of $\mathbb{R}^2$:
$$S = \{(m, 0) \colon m \in \mathbb{Z}\} \cup \{(0, n) \colon n \in \mathbb{Z}\}.$$
(So the points of S are where we put the "tick marks" on the x- and y-axes when we draw a graph.) Prove that S is countably infinite.

VI. Suppose that F is a finite set with $F \cap \mathbb{N} = \emptyset$. Prove that $F \cup \mathbb{N}$ is countably infinite, by finding a bijection from $\mathbb{N}$ to $F \cup \mathbb{N}$.

VII. Prove the following statement: if m and n be natural numbers with $m < n$, and if S is a set with m elements and T is a set with n elements, then every function $f : S \to T$ is not surjective. (In other words, a function from a "small" set to a "big" set can't attain all the possible values in the target. Hint: prove this statement by induction on m. The proof should be analogous to our proof of the Pigeonhole Principle, from class.)