Math 220, Section 203—Homework #7
due in class Thursday, March 27, 2003

Remember that all of your solutions must be written in complete sentences that are easy to read and in logically correct order.

I. Determine, with proof, the image of the function $f : (-2, 2) \to \mathbb{R}$ defined by the formula $f(x) = |x - 1| - |x + 1| + x$.

II. (a) Let $H = \{x \in \mathbb{R}\colon \frac{1}{x} \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Prove that $\sup(H) = 1$ and $\inf(H) = 0$.
(b) Let $J = (-\infty, 3)$. For each of $\sup(J)$ and $\inf(J)$, either determine (with proof) the value or prove that the value does not exist.

III. This problem concerns the two sets $A = (0, 1) \cap \mathbb{Q}$ and $B = (0, \sqrt{2}) \cap \mathbb{Q}$.
(a) What is wrong with the following "proof" that A and B have the same cardinality? "Define a function $f : A \to B$ by $f(x) = x + \frac{1}{3}$. This function is injective because it is increasing, and it is surjective because given any y, we can input $x = y - \frac{1}{3}$ into the function and obtain the value y. Therefore f is a bijection, so A and B have the same cardinality."
(b) What is wrong with the following "proof" that A and B have the same cardinality? "Define a function $f : A \to B$ by $f(x) = x\sqrt{2}$. This function is injective because it is increasing, and it is surjective because given any y, we can input $x = y/\sqrt{2}$ into the function and obtain the value y. Therefore f is a bijection, so A and B have the same cardinality."
(c) Give a correct proof that A and B have the same cardinality. (Hint: there is a way that doesn't involve finding a specific bijection between the two sets.)

IV. We have seen that some sets do not have a supremum. Prove that for every *finite* set F, the supremum $\sup(F)$ does exist and is always an element of F. (Hint: one way to prove this is by induction on the number of elements in F.)

V. Let f and g be two functions, and define $h = g \circ f$ to be their composition. You may use functions from $\mathbb{R}$ to $\mathbb{R}$, or functions from $[0, 1]$ to $[0, 1]$, or functions from $\mathbb{Z}$ to $\mathbb{Z}$—whatever you like, as long as the domain and target are the same subset of the real numbers.
(a) Give an example of two functions f and g where $h = g \circ f$ is surjective but f is not surjective.
(b) Prove that if h is surjective, then g is surjective.