A few extra study questions.

1. Prove that the set X of functions from $\{0,1\}$ to $\mathbb{N}$ is countable.
2. Prove that the set Y of functions from $\mathbb{N}$ to $\{0,1\}$ is uncountable.
3. Does
$$\sum_{n=1}^{\infty} \frac{n^2+n}{n^2+1}$$
converge?
4. Does
$$\sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n!} 5^{-n}$$
converge?
5. Let $f : \mathbb{R} \to \mathbb{R}$ be strictly decreasing. Define $g(x) = f(x^2)$. Show that g is not decreasing.

Solutions

1. A function $f : \{0,1\} \to \mathbb{N}$ defines a point $(f(0), f(1))$ in $\mathbb{N}^2$. This is a bijection from X to $\mathbb{N}^2$, which is countable as showed in class. So X is countable, being in bijection with a countable set.
2. Let $Y_0 \subset Y$ be the subset consisting of functions f which are not eventually equal to one, that is, $f(n) = 0$ for an infinite number of different $n \in \mathbb{N}$. Given $f \in Y_0$, define a real number $r \in [0,1)$ by its binary expansion $0.x_1x_2x_3\ldots$ where $x_1 = f(1), x_2 = f(2), \ldots$. This is a bijection from Y to the interval $[0,1) \subset \mathbb{R}$, and $[0,1)$ is uncountable. Therefore Y_0 is uncountable, and Y is uncountable since it contains an uncountable subset.
3. No, since the terms do not tend to zero:
$$\lim_{n\to\infty} \frac{n^2+n}{n^2+1} = 1.$$
Recall that if an infinite sum converges then the terms must go to zero.
4. Yes, by the ratio test:
$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!^2} 5^{-n-1} \Big/ \frac{(2n)!}{n!^2} 5^{-n} = \frac{(2n+2)(2n+1)}{5(n+1)(n+1)}$$
which has limit $4/5$ when $n \to \infty$. Since $4/5 < 1$ the original series converges.
5. $g(-1) - g(0) = f(1) - f(0) < 0$. So g is not decreasing.