Math 220, Section 201
Solutions for First Midterm
(February 2, 2005)

1. Show that $(P \vee Q) \Rightarrow (P \wedge Q)$ is **NOT** *a tautology.*

When P is true and Q is false, $P \vee Q$ is true but $P \wedge Q$ is false, and therefore $(P \vee Q) \Rightarrow (P \wedge Q)$ is false. (The other counterexample is when P is false and Q is true).

2. Express the following statements using quantifiers.

(a) For $x \in [0,1]$, the maximum of the function $f(x)$ is less than or equal to 2.
(b) For $x \in [0,1]$, the minimum of the function $f(x)$ is less than or equal to 3.

(a) $\big(\forall x \in [0,1]\big)\big(f(x) \leq 2\big)$
(b) $\big(\exists x \in [0,1]\big)\big(f(x) \leq 3\big)$

3. For real numbers x, let $R(x)$ be the statement "$x^2 - 2x \leq 0$" and $S(x)$ be the statement "$x^2 \leq 9$". Prove that $R(x) \Rightarrow S(x)$.

The statement $x^2 - 2x \leq 0$ is equivalent to $x^2 - 2x + 1 \leq 0 + 1$, or $(x-1)^2 \leq 1$. Since the square root function is an increasing function on the nonnegative real numbers, taking square roots of both sides preserves the inequality, yielding the equivalent statement $|x - 1| \leq 1$. This is the same as $-1 \leq x - 1 \leq 1$, or $0 \leq x \leq 2$. We have thus shown that $R(x)$ is equivalent to $0 \leq x \leq 2$.

Now assume that $R(x)$ is true; we need to prove $S(x)$. Since $R(x)$ is true, so is $0 \leq x \leq 2$. Since all these quantities are positive, squaring preserves inequalities (since the squaring function is also an increasing function on nonnegative real numbers), and so $0 \leq x^2 \leq 4$. In particular, $x^2 \leq 4 \leq 9$, which proves $S(x)$ as desired.

4. Say that a function f is **happy** *at x if there exist a and b with $a < x < b$ such that for all y, we have that $a < y < b$ implies that $f(y) \geq f(x)$. State what it means for a function to not be happy at x, (without using a superficial negative such as "it is false that").*

A function f is happy at x if

$$\big(\exists a, b\big)\big((a < x < b) \wedge (\forall y)(a < y < b \Rightarrow f(y) \geq f(x))\big).$$

Therefore the function f being not happy at x means

$$\big(\forall a, b\big)\big((a \geq x \vee x \geq b) \vee (\exists y)(a < y < b \wedge f(y) < f(x))\big).$$

In words this is: the function f is not happy at x if for all a and b, either $a \geq x$ or $x \geq b$ or else there exists a y with $a < y < b$ and $f(y) < f(x)$.

Another statement which is logically equivalent to the above negation is

$$\big(\forall a, b\big)\big(a < x < b \Rightarrow (\exists y)(a < y < b \wedge f(y) < f(x)\big),$$

which in words is: for all a and b, if $a < x < b$ then there exists a y with $a < y < b$ such that $f(y) < f(x)$.

(Note that f is happy at x is the same as saying that f has a local minimum at x: f is smaller at x than at every other point in an interval around x.)

5. Prove the following statement: "If A, B, and C are sets such that $B \subseteq A$ and $C \subseteq (A \cup B)$, then $C \subseteq A$."

We must prove that, if $x \in C$, then $x \in A$. So suppose that $x \in C$. By the second hypothesis, $x \in A \cup B$, which means $x \in A$ or $x \in B$. If $x \in A$ then we have what we want to prove; on the other hand, if $x \in B$ then $x \in A$ by the first hypothesis $B \subseteq A$. So in both cases, $x \in A$ as desired.

(Remark: Many students tried to use the statement "$C \subseteq A \cup B$, therefore $C \subseteq A$ or $C \subseteq B$". This is false! A subset of a union of two sets is not necessarily a subset of one or the other set. For example, $[1,3]$ is a subset of $[0,2] \cup [2,4]$ but is not a subset of either $[0,2]$ nor $[2,4]$.)

Math 220, Section 202
Solutions for First Midterm
(February 7, 2005)

1. Say that a set of integers S is **satisfactory** *if there exists an integer k such that every element of S is greater than or equal to k, and also for every $m \in S$ and $n \in S$, if $m > n$ then the difference $m - n$ is also an element of S. State what it means for a set to* **not** *be satisfactory (without using a superficial negative such as "it is false that").*

The set S being satisfactory means

$$\big(\exists k \in \mathbb{Z}\big)\big((\forall x \in S)(x \geq k)\big) \wedge (\forall m, n \in S)(m > n \Rightarrow m - n \in S).$$

Therefore S not being satisfactory means

$$\big(\forall k \in \mathbb{Z}\big)\big((\exists x \in S)(x < k)\big) \vee (\exists m, n \in S)(m > n \wedge m - n \notin S).$$

In words, this is: either for every integer k, there exists an element of S that is less than k, or else there exists $m \in S$ and $n \in S$ such that $m > n$ but $m - n$ is not an element of S.

2. Show that $\big((P \vee Q) \Rightarrow Q\big) \Leftrightarrow (P \vee Q)$ is **NOT** *a tautology.*

If P is true and Q is false, then $P \vee Q$ is true, which makes $(P \vee Q) \Rightarrow Q$ false. Therefore $\big((P \vee Q) \Rightarrow Q\big) \Leftrightarrow (P \vee Q)$ is asserting that a false statement is equivalent to a true statement, which is false. (The other counterexample is when P and Q are both false.)

3. Express the following statements about the function $f\colon \mathbb{R} \to \mathbb{R}$ using quantifiers, without using the words "bounded" or "decreasing". (You don't have to prove the statements.)

- *(a)* **[5 pts]** *The function $f(x)$ is bounded above by 5 but not bounded below by 1.*
- *(b)* **[5 pts]** *The function $f(x)$ is decreasing. (If it helps, remember that "decreasing" means "strictly decreasing".)*

(a) $\big(\forall x \in \mathbb{R}\big)\big(f(x) \leq 5\big) \wedge \big(\exists x \in \mathbb{R}\big)\big(f(x) < 1\big)$
(b) $\big(\forall x, y \in \mathbb{R}\big)\big(x > y \Rightarrow f(x) < f(y)\big)$

4. Prove the following statement: "Suppose A and B are subsets of $\mathbb{R}$ and that x is a real number. If $x \in A \cup B$ but $x \notin A \cap B$, prove that either $x \in A - B$ or $x \in B - A$."

We are given that $x \notin A \cap B$, which means $\neg(x \in A \cap B)$ or $\neg(x \in A \wedge x \in B)$; by de Morgan's rule, this is the same as $x \notin A \vee x \notin B$. We are also given that $x \in A \cup B$, which means that $x \in A$ or $x \in B$.

Suppose first that $x \in A$. Then $x \notin A$ is false, and so $x \notin B$ must be true to make $x \notin A \vee x \notin B$ true. We now know that $x \in A \wedge x \notin B$, and this is the definition of $x \in A - B$. In particular, we have shown that $x \in A - B \vee x \in B - A$.

On the other hand, suppose that $x \in B$. Then $x \notin B$ is false, and so $x \notin A$ must be true. We now know that $x \in B \wedge x \notin A$, and this is the definition of $x \in B - A$. In particular, we have shown that $x \in A - B \vee x \in B - A$.

In both cases, we have shown that $x \in A - B \vee x \in B - A$, and hence we are done.

5. For real numbers x, let $R(x)$ be the statement "$x^2 + 3x \leq 0$" and $S(x)$ be the statement "$x^3 \leq 8$". Prove that for every real number x, we have $R(x) \Rightarrow S(x)$.

We show the two implications $R(x) \Rightarrow x \leq 0$ and $x \leq 0 \Rightarrow x^3 \leq 8$, which together imply $R(x) \Rightarrow S(x)$. For the first one, we prove the contrapositive: $x > 0 \Rightarrow \neg R(x)$. If $x > 0$, then both x^2 and $3x$ are positive as well, and so $x^2 + 3x$ is certainly positive, which is to say that $R(x)$ is false as desired. For the second implication, if $x \leq 0$, then $x^3 \leq 0$ as well (the cube of a nonpositive number is always nonpositive). Certainly $0 \leq 8$, and so $x^3 \leq 8$, which is $S(x)$.