Math 220, Sections 201/202
Solutions for Second Midterm
(March 9, 2005)

I. Write down the statement of the Strong Induction Principle.

To prove $(\forall n \in \mathbb{N})P(n)$, it suffices to prove $P(1)$ and

$$(\forall n \in \mathbb{N})\big((P(1) \wedge \ldots \wedge P(n-1)) \Rightarrow P(n)\big).$$

Also acceptable is

$$\Big(P(1) \wedge (\forall n \in \mathbb{N})\big((P(1) \wedge \ldots \wedge P(n-1)) \Rightarrow P(n)\big)\Big) \Rightarrow (\forall n \in \mathbb{N})P(n)$$

or an equivalent statement in words.

II. Let $f : A \to B$ and $g : B \to C$ be surjections. Prove that the composition $g \circ f$ is also a surjection.

Given $c \in C$, we need to prove that there is an $a \in A$ such that $g(f(a)) = c$. There exists a $b \in B$ such that $g(b) = c$, since g is surjective. For this b, there exists $a \in A$ such that $f(a) = b$ since f, is surjective. Therefore $g(f(a)) = g(b) = c$, proving that $g \circ f$ is surjective.

III. Find, with proof, a surjective function from $\mathbb{Q} \cap (0,1)$ to $\mathbb{N}$.

One possibility is to define $f(1/n) = n$ for natural numbers $n \geq 2$, and $f(x) = 1$ if $1/x \notin \mathbb{N}$. This is surjective, since for any natural number n greater than or equal to 2, the real number $1/n \in (0,1)$ maps to n, while $f(2/3) = 1$.

IV. Let S be the set of all real numbers x such that $x^2 \in \mathbb{Z}$. (For example, $\sqrt{2} \in S$ and $-3 \in S$, but $\frac{1}{2} \notin S$ and $\pi \notin S$.) Prove that S is countable.

Define $f : S \to \mathbb{Z}$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0. \end{cases}$$

Let us show that f is an injection. Since f maps nonnegative numbers to nonnegative numbers and negative numbers to negative numbers, we need only check that $f(x) = f(y) \Rightarrow x = y$ when x, y are both nonnegative or both negative. If $x^2 = y^2$ and $x, y \geq 0$, then $x = y$. Similarly if $-x^2 = -y^2$ and $x, y < 0$, then $x = y$.

On the other hand, f is surjective: given $n \in \mathbb{Z}$, we have $f(\sqrt{n}) = n$ if n is nonnegative or $f(-\sqrt{|n|}) = n$ if n is negative (it is important to note that $\sqrt{n}$ and $-\sqrt{|n|}$ truly are elements of S). Thus f is a bijection from S to $\mathbb{Z}$. Since we know $\mathbb{Z}$ is countable, S is therefore also countable.

V. Let T be the set $T = \{x \in \mathbb{Q} \colon x^2 < 3\}$. Find, with proof, $\inf T$.

The infimum is $-\sqrt{3}$. To see this we need to know two things: that it is a lower bound, and it is larger than any other lower bound. Well, $-\sqrt{3}$ is a lower bound, since if $y < -\sqrt{3}$ then $y^2 > 3$ and so $y \notin T$. This is the contrapositive of "if $y \in T$ then $y \geq -\sqrt{3}$".

If $-\sqrt{3} < x$, let's show that x is not a lower bound. If $x > 0$, then x is definitely not a lower bound since $0 \in T$. If on the other hand $-\sqrt{3} < x \leq 0$, there is a rational number y

with $-\sqrt{3} < y < x \leq 0$. By squaring, $y^2 < 3$, so that $y \in T$, which shows that x is not a lower bound for T.

VI. Prove that if you add together all the odd positive integers up to any point, you always get a perfect square. (For example, $1 + 3 + 5 + 7 + 9 + 11 = 36$.) You may use induction if you wish.

Let's prove that $\sum_{j=1}^{n}(2j-1) = n^2$. Denote this statement by $P(n)$. Then $P(1)$ is just the statement that $2 \cdot 1 - 1 = 1^2$, which is true. Assume $P(n-1)$ is true, that is, assume

$$\sum_{j=1}^{n-1}(2j-1) = (n-1)^2.$$

Adding $2n-1$ to both sides we have

$$\sum_{j=1}^{n}(2j-1) = \sum_{j=1}^{n-1}(2j-1) + (2n-1) = (n-1)^2 + 2n - 1 = (n^2 - 2n + 1) + 2n - 1 = n^2.$$