Math 220, Section 203 Study Questions for Second Midterm (in class Tuesday, March 18, 2003)

- I. Suppose that a_1, a_2, \ldots are negative real numbers. Prove, using induction on n, that for all $n \in \mathbb{N}$, the number $(-1)^n a_1 \times \cdots \times a_n$ is positive.
- II. D'Angelo and West, p. 75, #3.58(a). L-tilings are described in Solution 3.72 on pages 61–62 of D'Angelo and West.
- III. Imagine a robot that, whenever it sees a positive integer *n* written down, subtracts the number of digits of *n* from *n* and writes the answer down. For instance, if the robot saw the integer 108 written down, it would subtract 3 (since 108 has three digits) and write the number 105 down. Suppose you could tell the robot to keep doing this with all the numbers it writes down. For example, after writing down 105, the robot would write down 102 (= 105 3), then 99 (= 102 3), then 97 (= 99 2, since 99 has two digits), and so on. Prove that no matter what positive integer *n* the robot first sees, eventually the robot will see the the number 1 written down.
- IV. D'Angelo and West, p. 268, #13.6
- V. D'Angelo and West, p. 268, #13.7
- VI. D'Angelo and West, p. 270, #13.37
- VII. D'Angelo and West, p. 270, #13.39
- VIII. Write down four different bijections $f : \mathbb{N} \to \mathbb{N}$.
 - IX. (a) How many bijections are there from the set [3] to itself?
 - (b) How many injections are there from the set [3] to itself?
 - (c) How many functions are there from the set [3] to itself?
 - X. Let *A* be a countably infinite set, and let *B* be any set. Prove that $A \setminus B$ is either finite or countably infinite.
 - XI. Prove that all of the following sets have the same cardinality as one another: (a) $\mathbb R$
 - (b) (0,1)
 - (c) (s, t), for any real numbers s < t
 - (d) $(0,\infty)$

(Note that it is not enough to show that all four sets are uncountable!)

- XII. Suppose that *F* is a finite set with $F \cap \mathbb{N} = \emptyset$. Prove that $F \cup \mathbb{N}$ is countably infinite, by finding a bijection from \mathbb{N} to $F \cup \mathbb{N}$.
- XIII. Prove the following statement: if *m* and *n* are natural numbers with m < n, and if *S* is a set with *m* elements and *T* is a set with *n* elements, then every function $f : S \rightarrow T$ is not surjective. (In other words, a function from a "small" set to a "big" set can't attain all the possible values in the target. Hint: prove this statement by induction on *m*.)

- XIV. Determine, with proof, the image of the function $f : (-2, 2) \rightarrow \mathbb{R}$ defined by the formula f(x) = |x 1| |x + 1| + x.
- XV. Let $J = (-\infty, 3)$. For each of sup(J) and inf(J), either determine (with proof) the value or prove that the value does not exist.
- XVI. We have seen that some sets do not have a supremum. Prove that for every *finite* set F, the supremum $\sup(F)$ does exist and is always an element of F. (Hint: one way to prove this is by induction on the number of elements in F.)